The topological entropy of Banach spaces

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joint work with H. Bruin
The set of all continuous functions $f : [0, 1] \to \mathbb{R}$ equipped with the supremum norm.

**Theorem (Banach-Mazur)**

*The Banach space $C([0, 1])$ is universal, i.e., every real, separable Banach space $X$ is isometrically isomorphic to a closed subspace of $C([0, 1])$.***

**Theorem**

*Every infinite-dimensional closed subspace of $C([0, 1])$ must contain a function with infinite variation.*
Every infinite-dimensional closed subspace $E$ of $C([0,1])$ must contain a function which is not differentiable at some point of $[0,1]$. 

**Theorem**

*Every isometrically isomorphic copy of* $\ell_1$ *in* $C([0, 1])$ *contains a function which is non-differentiable at every point of a perfect subset of* $[0, 1]$. 

**Theorem**

*Every separable Banach space is isometrically isomorphic to a space of continuous nowhere differentiable functions.*

**Theorem**

Every separable Banach space is isometrically isomorphic to a space of continuous nowhere approximately differentiable and nowhere Hölder functions.
Let $C_b(X)$ denote the set of all bounded continuous functions $f : X \to \mathbb{R}$ equipped with the supremum norm. Clearly, $C_b(\mathbb{R})$ is a non-separable Banach space. Let $[a, b]$ be a closed finite subinterval of $\mathbb{R}$. We identify $f : [a, b] \to \mathbb{R}$ with its extension

$$(Exf)(x) = \begin{cases} 
  f(x) & \text{if } x \in [a, b]; \\
  f(b) & \text{if } x \geq b; \\
  f(a) & \text{if } x \leq a.
\end{cases}$$

Under this identification, $C([a, b]) \subset C_b(\mathbb{R})$. We will deal with the topological entropy of maps from $C_b(\mathbb{R})$ defined as $h_{top}(f) := h_{top}(f|_{f(\mathbb{R})})$. 
In the next lemma we denote by $F(X)$ a linear space of functions $f: X \to \mathbb{R}$.

**Lemma**

Given $n$ linearly independent functions in $F(X)$, there exist $n$ points $x_1, \ldots, x_n \in X$ such that the vectors

\[
\begin{pmatrix}
  f_1(x_1) \\
  f_1(x_2) \\
  \vdots \\
  f_1(x_n)
\end{pmatrix}, \quad
\begin{pmatrix}
  f_2(x_1) \\
  f_2(x_2) \\
  \vdots \\
  f_2(x_n)
\end{pmatrix}, \quad \ldots, \\
\begin{pmatrix}
  f_n(x_1) \\
  f_n(x_2) \\
  \vdots \\
  f_n(x_n)
\end{pmatrix}
\]

are linearly independent in $\mathbb{R}^n$. 
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Definition

For a given set $\mathcal{B} \subset C_b(\mathbb{R})$, let

$$h_{top}^+(\mathcal{B}) = \sup\{h_{top}(f) : f \in \mathcal{B}\},$$

$$h_{top}^-(\mathcal{B}) = \inf\{h_{top}(f) : f \in \mathcal{B}, f \text{ is non-zero}\}.$$

Proposition

If a linear space $\mathcal{B} \subset C_b(\mathbb{R})$ has dimension $n$, then

$$h_{top}^+(\mathcal{B}) \geq \log(n - 1).$$

In particular, $h_{top}^+(\mathcal{B}) = \infty$ if $\dim(\mathcal{B}) = \infty$. 
Example
There exists an isometrically isomorphic copy $E$ of $c_0$ in $C([0, 1])$ such that every $f \in E$ has a finite topological entropy.

Example
There is a universal Banach space $A \subset C([-1, 1])$ such that $h_{\text{top}}(f) = \infty$ for every non-zero $f$ from $A$. 
The maps $f \in C([0,1])$ and $\Psi(f) = g \in C([\frac{-4}{3}, \frac{4}{3}])$, $p_n = \left(\frac{1}{2}\right)^n$, $q_n = \left(\frac{2}{3}\right)^n$, $n \geq 0$.

Figure: The maps $f \in C([0,1])$ and $\Psi(f) = g \in C([\frac{-4}{3}, \frac{4}{3}])$, $p_n = \left(\frac{1}{2}\right)^n$, $q_n = \left(\frac{2}{3}\right)^n$, $n \geq 0$. 
Recall that \( f \in C^\alpha(\mathbb{R}) \) (\( f \) is \( \alpha \)-Hölder on \( \mathbb{R} \)) for some \( \alpha \in (0, 1) \) if
\[
\sup \left\{ \frac{|f(x) - f(y)|}{|x - y|^{\alpha}} : x, y \in \mathbb{R}, 0 < |x - y| \leq 1 \right\} < \infty.
\]

For some fixed \( \alpha \in (0, 1) \), if we choose \( q_n = p_n^\alpha \) and \( f \in C^\alpha([0, 1]) \), then \( \Psi(f) \) is \( \alpha \)-Hölder on \( \mathbb{R} \). Therefore \( \mathcal{A}^\alpha := \Psi(C^\alpha([0, 1])) \subset C_b^\alpha(\mathbb{R}) \) is a normed (infinite dimensional) linear space such that \( h_{top}(f) = \infty \) for every non-zero \( f \) from \( \mathcal{A}^\alpha \).
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**Theorem**

Let $A \subset C([0, 1])$ be isometrically isomorphic to $\ell_1$. Then $A$ contains a function with infinite topological entropy.

The following statement shows that the entropy can behave extremely rigidly on a one-dimensional subspace of $C_b(\mathbb{R})$.

**Theorem**

For any $t \in [0, \infty]$, there exists a function $f \in C_b(\mathbb{R})$ such that for $\mathcal{B} = \{\lambda f\}_{\lambda \in \mathbb{R}}$ satisfies $h_{\text{top}}^-(\mathcal{B}) = h_{\text{top}}^+(\mathcal{B}) = t$. 