ERGODIC METHODS IN DYNAMICS

April 23–27, 2012

Conference in honor of **Feliks Przytycki**
on the occasion of his 60th birthday

*** Chaos in ergodic systems ***

Tomasz Downarowicz
Institute of Mathematics and Computer Science
University of Technology of Wroclaw

T.D.: *Positive topological entropy implies chaos DC2*,
PAMS (to appear)

T.D. and Y. Lacroix: *Measure-theoretic chaos*,
ETDS (to appear)

SURVEY OF TOPOLOGICAL CHAOS

Standard definition of chaos has three parts:

(1) Define ?-*scrambled pair* (somehow)

(2) Define ?-*scrambled set* as one in which every off-diagonal pair is ?-scrambled

(3) Say that a system is ?-*chaotic* whenever it contains an uncountable ?-scrambled set

It remains to define scrambled pairs (in various senses).

The first notion of chaos is due to Li and Yorke.

DEFINITION 0.
A pair (x, y) is *Li–Yorke scrambled* if

$$\liminf_{n\to\infty} d(T^n x, T^n y) = 0 \quad \text{and} \quad \limsup_{n\to\infty} d(T^n x, T^n y) > 0.$$

This can be rephrased as follows:

There exist:
1. increasing sequence $\{n_i\}$ s.t. $d(T^{n_i}x, T^{n_i}y) \xrightarrow{i} 0$,
2. increasing sequence $\{m_i\}$ and $s > 0$ s.t. $\forall i\ d(T^{m_i}x, T^{m_i}y) \geq s$.

Positive topological entropy implies Li–Yorke chaos
(Blanchard–Glasner–Kolyada–Maass, 2002),
but the converse does not hold **even for interval maps**.

In 1994, B. Schweizer and J. Smital introduced *distributional chaos* which, **for interval maps**, is **equivalent** to positive topological entropy:

DEFINITION 1.
A pair (x, y) is *distributionally scrambled* if there exist:

1. increasing sequence $\{n_i\}$ *of upper density 1* s.t. $d(T^{n_i}x, T^{n_i}y) \xrightarrow{i} 0$,

2. increasing sequence $\{m_i\}$ *of upper density 1* and $s > 0$ s.t.
$\forall i \; d(T^{m_i}x, T^{m_i}y) \geq s$.

Unfortunately, on general spaces, distributional chaos does not imply and **is not implied by** positive topological entropy (Pikula, 2000).

In 2004 J. Smital and M. Stefankova refined distributional chaos into three versions: the original distributional chaos is called DC1, then there are weaker versions: DC2 and DC3. (We will skip discussing DC3.)

DEFINITION 2.
A pair (x, y) is *DC2-scrambled* if there exist:

1. increasing sequence $\{n_i\}$ of upper density 1 s.t. $d(T^{n_i}x, T^{n_i}y) \xrightarrow{i} 0$,

2. increasing sequence $\{m_i\}$ *of upper density* $\eta > 0$ and $s > 0$ s.t. $\forall i \; d(T^{m_i}x, T^{m_i}y) \geq s$.

On the interval DC2 $\iff$ DC1 ($\iff$ positive topological entropy).
Smital conjectured that positive entropy implies DC2.
The question remained open until 2011.

Last time Smital recalled the problem in June 2011 at a conference in Banska Bystrica **attended by Feliks Przytycki**.

It is easy to see that a pair (x, y) is DC2-scrambled if and only if

$$\liminf_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} d(T^i x, T^i y) = 0 \quad \text{and} \quad \limsup_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} d(T^i x, T^i y) > 0.$$

In this form DC2 is a natural analog of Li–Yorke chaos.
Moreover, it is expressed in terms of **ergodic averages**.
(DC1 does not admit such a formulation.)

Hence, DC2 is a natural starting point to create a measure-theoretic notion of chaos.

But before we do that, we introduce chaos intermediate between DC1 and DC2:

DEFINITION 1.5.
A pair (x, y) is *DC1.5-scrambled* if there exist:

1. increasing sequence $\{n_i\}$ *of upper density 1* s.t. $d(T^{n_i}x, T^{n_i}y) \xrightarrow{i} 0$,

2. *for every* $\eta < 1$: an increasing sequence $\{m_{\eta,i}\}$ of upper density η, and $s_\eta > 0$, s.t. $\forall i\ d(T^{m_{\eta,i}}x, T^{m_{\eta,i}}y) \geq s_\eta$.

Obvious:

$$\text{DC1} \implies \text{DC1.5} \implies \text{DC2} \implies \text{Li–Yorke chaos}$$

None of the reversed implications hold.

UNIFORM CHAOSES

Li–Yorke chaos is *uniform* if the constant s is common for all pairs in the scrambled set.

DC1 is *uniform* if the constant s is common for all pairs in the scrambled set.

DC2 is *uniform* if the constants s and η are common for all pairs in the scrambled set.

DC1.5 is *uniform* if the assignment $\eta \mapsto s_\eta$ is common for all pairs in the scrambled set.

All the defined above notions of chaos (and uniform chaos) are **topological invariants**.

MEASURE THEORETIC CHAOS

Let (X, Σ, μ, T) be an **ergodic** measure-preserving transformation of a standard probability space. Our goal is to define measure-theoretic chaos so it is analogous to DC2 (DC1.5) and is an **isomorphism invariant**.

In particular, the notion must not refer to any metric d and must be *persistent under removing null sets.*

Given a partition $\mathcal{P}$, we will write $x \overset{\mathcal{P}}{\sim} y$ if x and y belong to the same atom of $\mathcal{P}$.

Fix a *refining* sequence of finite partitions $\{\mathcal{P}_k\}$
(i.e., $\mathcal{P}_{k+1} \succcurlyeq \mathcal{P}_k \ \forall k$, and $\bigvee_k \mathcal{P}_k \overset{\mu}{=} \Sigma$)

DEFINITION 3 (3^+).
A pair (x, y) is $\{\mathcal{P}_k\}$-scrambled ($\{\mathcal{P}_k\}^+$-scrambled) if there exist:

1. increasing sequence $\{n_i\}$ of upper density 1 s.t.
$\forall k$ and large enough i, $T^{n_i}x \overset{\mathcal{P}_k}{\sim} T^{n_i}y$,

2. increasing sequence $\{m_i\}$ of upper density $\eta > 0$ and $k_0 > 0$ s.t.
$\forall i \ T^{m_i}x \overset{\mathcal{P}_{k_0}}{\not\sim} T^{m_i}y$.

(2^+. for every $\eta < 1$: increasing sequence $\{m_{\eta,i}\}$ of upper density η and $k_\eta > 0$ s.t. $\forall i \ T^{m_{\eta,i}}x \overset{\mathcal{P}_{k_\eta}}{\not\sim} T^{m_{\eta,i}}y$.)

DEFINITION 4 (4^+).
The system (X, Σ, μ, T) is measure-theoretically (measure-theoretically$^+$) chaotic if:

for every refining sequence of finite partitions $\{\mathcal{P}_k\}$ there exists an uncountable $\{\mathcal{P}_k\}$-scrambled ($\{\mathcal{P}_k\}^+$-scrambled) set.

The above chaos is *uniform* if the constants k_0 and η
(the assignment $\eta \mapsto k_\eta$)
are common for all pairs in the scrambled set.

MAIN FACTS

THEOREM 1.
Measure-theoretic and measure-theoretic$^+$ chaoses, and their uniform versions **are** isomorphism invariants.

THEOREM 2.
Let (X, T) be a topological dynamical system. Suppose there exists an ergodic invariant measure μ such that (X, Σ, μ, T) is measure-theoretically (measure-theoretically$^+$, uniformly measure-theoretically, uniformly measure-theoretically$^+$) chaotic.
Then (X, T) is DC2 (DC1.5, uniformly DC2, uniformly DC1.5) chaotic.

The converse implications fail, nonetheless, we have

THEOREM 3.
Let (X, T) be a topological dynamical system and let μ be an ergodic measure. The following conditions are equivalent:

(a) (X, Σ, μ, T) is measure-theoretically (measure-theoretically$^+$, uniformly measure-theoretically, uniformly measure-theoretically$^+$) chaotic,

(b) for every null set A there exists an uncountable DC2 (DC1.5, uniformly DC2, uniformly DC1.5) -scrambled set disjoint from A (topological chaos *persistent under removing null sets*).

UNNKOWN:

Is measure-theoretic$^+$ chaos essentially stronger than measure-theoretic chaos?

Equivalently, does DC2 *persistent under removing null sets* (for some ergodic measure) imply DC1.5?

MAIN MAIN FACT

THEOREM 4.
Let (X, Σ, μ, T) be an ergodic measure-theoretic dynamical system with **positive Kolmogorov-Sinai entropy**.

Then the system is **uniformly measure-theoretically^{+} chaotic**.

COROLLARY.
A topological dynamical system with positive topological entropy is **uniformly DC1.5 chaotic**.

In particular it is DC2 chaotic (Smital's conjecture is true).

(In particular it is Li–Yorke chaotic, so this results strengthens [B–G–K–M].)

TECHNICAL PART

We will consider an ergodic measure-theoretic dynamical system (X, Σ, μ, T).

LEMMA 1.
Fix arbitrarily **one** refining sequence $\{\mathcal{P}_k\}$ of finite partitions. The system is measure-theoretically (measure-theoretically$^+$, uniformly measure-theoretically, uniformly measure-theoretically$^+$) chaotic

if and only if

for every null set A there exists an uncountable $\{\mathcal{P}_k\}$-scrambled ($\{\mathcal{P}_k\}^+$-scrambled, uniformly $\{\mathcal{P}_k\}$-scrambled, uniformly $\{\mathcal{P}_k\}^+$-scrambled) set disjoint from A
(*persistence of chaos under removing null sets*),

if and only if

for every set B of positive measure there exists an uncountable $\{\mathcal{P}_k\}$-scrambled ($\{\mathcal{P}_k\}^+$-scrambled, uniformly $\{\mathcal{P}_k\}$-scrambled, uniformly $\{\mathcal{P}_k\}^+$-scrambled) set contained in B
(*ubiquitous presence of chaos*).

Proof. One implication is obvious. Assume the definition of chaos holds (for every refining sequence of partitions). Given a null set A define $\mathcal{P}_k^A = \mathcal{P}_k|_{A^c} \cup \{A\}$. The sequence $\{\mathcal{P}_k^A\}$ is refining. By assumption, there exists an uncountable $\{\mathcal{P}_k^A\}$-scrambled set and then at most one point of this set belongs A. The rest is an uncountable $\{\mathcal{P}_k\}$-scrambled set disjoint from A.

Conversely, assume the persistence for $\{\mathcal{P}_k\}$ and consider another refining sequence $\{\mathcal{Q}_k\}$ of finite partitions. Let A be the null set of points which do not fulfill the ergodic theorem for at least one set in the **countable** field $\mathcal{F}$ generated by all sets in $\{\mathcal{P}_k\}$ and in $\{\mathcal{Q}_k\}$. By assumption, there exists an uncountable $\{P_k\}$-scrambled set E disjoint from A. We will show that E is $\{\mathcal{Q}_k\}$-scrambled.

Let (x, y) be an off-diagonal pair in E. Let $\{n_i\}$, $\{m_i\}$, η and k_0 be the parameters of scrambling.

1. Fix some $\mathcal{Q}_k$. Given $\epsilon > 0$, there exists k' and a set X_ϵ of measure $1-\epsilon$ such that $\mathcal{P}_{k'}|_{X_\epsilon} \succcurlyeq \mathcal{Q}_k|_{X_\epsilon}$. Note that $X_\epsilon \in \mathcal{F}$. For i with upper density 1 we have

$$T^{n_i}x \overset{\mathcal{P}_{k'}}{\sim} T^{n_i}y. \tag{*}$$

while

$$T^n x \notin X_\epsilon \ \vee \ T^n y \notin X_\epsilon$$

happens with **density** at most 2ϵ (the ergodic theorem holds for x, y and X_ϵ). So for n_i's of upper density $1 - 2\epsilon$ we have (*) for points in X_ϵ, which implies

$$T^{n_i}x \overset{\mathcal{Q}_k}{\sim} T^{n_i}y.$$

Since ϵ is arbitrarily small, the above holds with upper density 1.

2. Let k_0' be such that $Q_{k_0'}|_{Y_\epsilon} \not\succcurlyeq \mathcal{P}_{k_0}|_{Y_\epsilon}$, where $\mu(Y_\epsilon) = 1 - \epsilon$ for $\epsilon < \frac{\eta}{2}$. Again, $Y_\epsilon \in \mathcal{F}$. For all i (with upper density η) we have

$$T^{m_i}x \overset{\mathcal{P}_{k_0}}{\not\sim} T^{m_i}y \tag{**}$$

while

$$T^m y \notin Y_\epsilon \ \vee \ T^m y \notin Y_\epsilon$$

happens with **density** at most $2\epsilon < \eta$. So, for m_i's with upper density $\eta' \geq \eta - 2\epsilon$ (which is **positive**) we have (**) for points in Y_ϵ, which implies that

$$T^{m_i}x \overset{\mathcal{Q}_{k_0'}}{\not\sim} T^{m_i}y.$$

This ends the proof. □

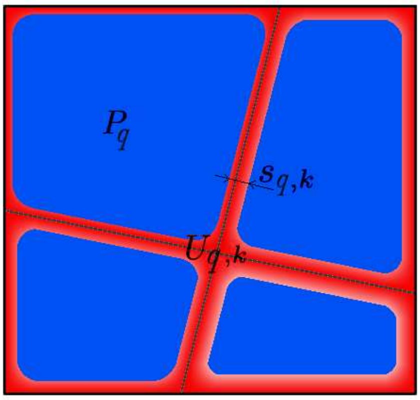
The proof for the other types pf chaos is (almost) identical.

Now the isomorphism invariance of the measure-theoretic notions of chaos (THEOREM 1) is obvious: an isomorphism is a map that **modulo null sets** is a bijection and translates refining sequences of partitions to refining sequences of partitions and preserves the dynamics.

The next easy proof is that of THEOREM 2 (measure-theoretic chaos in a topological system implies DC2).

Proof. Let μ be an ergodic invariant measure on X for which we have the measure-theoretic chaos. Let $\{\mathcal{P}_k\}$ be a refining sequence of partitions with the additional **topological** property $\mathrm{diam}(\mathcal{P}_k) \xrightarrow{k} 0$.

Given natural k and rational $q > 0$, in each atom P of $\mathcal{P}_k$ we can find a closed set P_q such that $\mu(P \setminus P_q) < \frac{q}{\#\mathcal{P}_k}$. Let $s_{q,k} > 0$ denote the smallest distance between points in different closed sets P_q. Let $U_{q,k}$ be the complement of the union of all sets P_q ($P \in \mathcal{P}_k$). Clearly, $\mu(U_{q,k}) < q$.



Let A be the null set of points which do not fulfill the ergodic theorem for at least one of the **countably many** sets $U_{q,k}$, and let E be a $\{\mathcal{P}_k\}$-scrambled set disjoint from A. We will show that E is DC2-scrambled. Let (x, y) be an off-diagonal pair in E.

1. There exists a sequence $\{n_i\}$ of upper density 1 s.t. $T^{n_i}x \overset{\mathcal{P}_k}{\sim} T^{n_i}y$, eventually, for each k. By the topological property of $\{\mathcal{P}_k\}$ it is clear that $d(T^{n_i}x, T^{n_i}y) \xrightarrow{i} 0$.

2. We have

$$(***) \qquad T^{m_i}x \overset{\mathcal{P}_{k_0}}{\not\sim} T^{m_i}y$$

for some k_0 and $\{m_i\}$ of upper density $\eta > 0$. Pick $q < \frac{\eta}{2}$. Then the set of times n s.t.

$$T^n y \in U_{q,k_0} \ \vee \ T^n y \in U_{q,k_0}$$

has density at most $2q$. So, for m_i's with upper density $\eta' \geq \eta - 2q$ (which is **positive**) we have (***) for points **not in** U_{q,k_0}. But then these points are in different closed sets P_q, so the distance between them is larger than $s = s_{q,k_0}$. This ends the proof. $\square$

SKETCH OF THE PROOF THAT
"ENTROPY IMPLIES CHAOS"

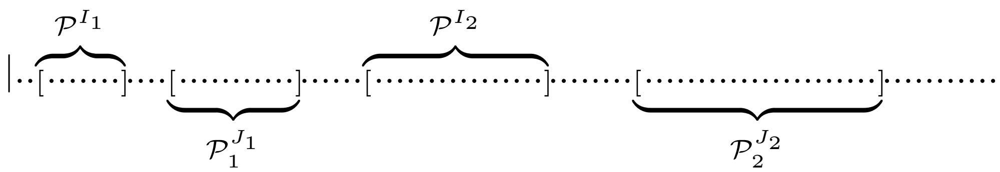
Let (X, Σ, μ, T) be an ergodic system with positive entropy $h_\mu(T)$ and let $\{\mathcal{P}_k\}$ be a refining sequence of partitions. There is k_0 such that $h_\mu(T, \mathcal{P}_{k_0}) > 0$. We will construct an uncountable uniformly $\{\mathcal{P}_k\}$-scrambled set (with common parameters k_0 and some $\eta > 0$). We will abbreviate $\mathcal{P}_{k_0}$ as $\mathcal{P}$ and $h_\mu(T, \mathcal{P}_{k_0})$ as h.

On the time axis $\mathbb{N}$ mark disjoint ordered intervals

$$I_1 = [a_1, b_1{-}1],\ J_1 = [c_1, d_1{-}1],\ I_2 = [a_2, b_2{-}1],\ J_2 = [c_2, d_2{-}1],\ \dots$$

so that $\frac{b_k}{a_k} \to \infty$ and $\frac{d_k}{c_k} \to \infty$. Then the union of the "odd" intervals I_k has upper density 1 and the union of the "even" intervals J_k also has upper density 1.

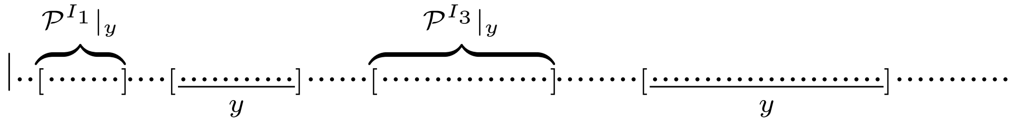
Assign to the **odd** intervals I_k the partitions $\mathcal{P}^{I_k}$ and assign to the **even** intervals J_k the partitions $\mathcal{P}_k^{J_k}$.



By the Shannon-McMillan-Breiman Theorem, the cardinality of each "odd" partition $\mathcal{P}^{I_k}$ is approximately $e^{b_k h}$.

By some standard arguments using partitions and entropy, if the lengths of the intervals and the gaps between them grow fast enough, all the partitions are "almost" independent. Arguing further, we deduce that

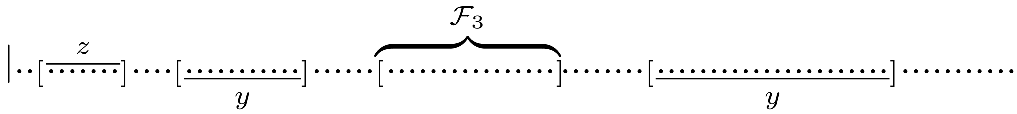
• On almost every atom y of the partition $\bigvee_{k=1}^{\infty} \mathcal{P}_k^{J_k}$ the partitions $\mathcal{P}^{I_k}|_y$ have cardinalities approximately $e^{b_k h}$ and are mutually "almost" independent of one-another. We now fix one some atom y.



It is a standard fact that the **Hamming ball** of radius η around a block of length n over an alphabet of cardinality l has cardinality around $e^{nH(\eta,1-\eta)\log l}$. So if η is small enough, this is less than $e^{n\frac{h}{2}}$.

Thus we can select within $\mathcal{P}^{I_k}|_y$ a family $\mathcal{F}_k$ of approximately $e^{b_k\frac{h}{2}}$ atoms, such that two any two of them are **more than some η apart** in the Hamming distance.

By the "almost independence", we can do so also in (almost every) atom z of the join of the preceding partitions $\mathcal{R}_{k-1} = \bigvee_{j=1}^{k-1} \mathcal{P}^{I_j}|_y$.



Now we can inductively create a "tree" $\mathbf{T}$ of atoms of the sequence $\mathcal{R}_k$, such that every atom of $\mathcal{R}_{k-1}$ selected to $\mathbf{T}$ splits into many (two is enough in fact) atoms of $\mathcal{R}_k$ selected to $\mathbf{T}$, two different atoms of $\mathcal{R}_{k-1}$ have no common "continuations" in $\mathcal{R}_k$ within $\mathbf{T}$, and any two branches which differ on an interval I_k differ there with Hamming distance at least η.

If we knew that all the branches have nonempty intersections, we would have obtained uncountably many atoms of $\mathcal{R}_y = \bigvee_{k=1}^{\infty} \mathcal{P}^{I_k}|_y$ such that any two of them differ on all but finitely many intervals I_k and every time with the Hamming distance η. Such atoms would be $\{\mathcal{P}_k\}$-scrambled, because

1. By belonging to the same atom y they satisfy the condition 1. of scrambling: along times of upper density 1 they are eventually (simultaneously) in the same atoms for every $\mathcal{P}_k$.

2. Two atoms are separated by the partition $\mathcal{P} = \mathcal{P}_{k_0}$ along times of upper density $\eta > 0$.

Obtaining nonempty intersections is done by topolgizing the standard space X and using regularity of the measure to make the branches of $\mathbf{T}$ insignificantly smaller (still splitting into many further branches) but **compact**.

Happy birthday, Felek!