Feliks Przytycki's contribution to the theory of Dynamical Systems – selected topics

1. Iteration of rational functions and interval maps: non-uniform hyperbolicity

Feliks Przytycki introduced a definition of a class of non-uniformly hyperbolic maps called *Topological Collet–Eckmann* (TCE) maps and worked out foundations of their theory. In particular, he proved the equivalence of several conditions describing TCE and existence and exponential mixing for geometric equilibrium (invariant measure in the class of conformal measure with exponent being hyperbolic Hausdorff dimension – generalizing absolutely continuous probability). A part of these results has been obtained in collaboration with Juan Rivera-Letelier.

Feliks Przytycki proved (jointly with Tomasz Nowicki) that for unimodal rational maps or interval maps (with negative Schwarzian derivative) the Collet–Eckmann property (positive Lyapunov exponent at the critical value) is a topological invariant. This was a well known conjecture.

2. Iteration of rational functions and interval maps: ergodic theory methods

Feliks Przytycki introduced and studied various definitions of the pressure $P(t)$ for the geometric singular potential $-t\log|f'|$. Several characterizations were proved by him (and his collaborators) to be equivalent, including the following variational definition: supremum over invariant measures of measure entropy plus expectation value of the potential.

Another part of results concerns stochastic properties and uniqueness of conformal and invariant equilibria. In particular, a real analyticity of the topological pressure $P(t)$ except at (at most) two phase transition parameters was proved, for all topologically transitive interval maps and large classes of rational maps. Similar tools gave a proof of the real-analyticity of Hausdorff dimension spectrum for characteristic Lyapunov exponent and local dimension. Some of the above results have been obtained in collaboration with Stanislav Smirnov and Juan Rivera-Letelier.

Feliks Przytycki studied properties of the Gibbs measures for an arbitrary rational map and Hölder continuous potentials. In particular, he proved uniqueness, mixing with the speed $\exp\sqrt{n}$ and Central Limit Theorem. Some of these results were obtained in collaboration with Manfred Denker and Mariusz Urbański.

Another conjecture proved by Feliks Przytycki is the following: for a rational map on the Julia set or a multimodal smooth interval map with non-flat singularities on the complement of an attractive basins, the Lyapunov exponent of any finite invariant measure is non-negative.

3. Properties of Julia sets

An important Feliks Przytycki's result on the Newton method of finding roots of polynomials says that the basins of attraction to these roots are simply connected. He also discovered the existence of "exotic" non-simply connected immediate basins of attraction for rational functions, with the number of critical points less than the degree of the map on the basin.

4. Boundary properties of basins of attraction and other limit sets. Przytycki's coding trees. Harmonic measure

The following result on harmonic measure was proved by Feliks Przytycki in the beginning of 1980's: the Hausdorff dimension of the harmonic measure on a simply connected basin of attraction is equal to 1. Further results led to the following dichotomy: the boundary is either analytic or fractal (partly a joint work with Mariusz Urbański and Anna Zdunik).

Feliks Przytycki is the author of a valuable technique of *geometric coding trees*, called also *Przytycki's coding trees* with a coding map replacing a Riemann map in the absence of a basin. Using this technique he proved, among many other results, a remarkable formula relating the growth rate of the derivative of the Riemann map on a basin of attraction along a typical ray and the Lyapunov exponents of two measures: the invariant measure for the "lifted map" (supported on the unit circle) and its image under the Riemann map (supported on the boundary of a basin of attraction). A more abstract version of this theorem leads to well known *Przytycki's formula* for the dimension of the maximal entropy measure on the Julia set of a quadratic polynomial.

5. Non-differentiable Weierstrass-like functions

In a joint paper with Mariusz Urbański, Feliks Przytycki proved a nice general result which guarantees that under some natural assumptions, the graph of a real function on an interval has Hausdorff dimension larger than 1. Moreover, several valuable results on the dimension of the graphs of the Weierstrass-like functions were proved in this paper.

6. Results related to the Entropy Conjecture

A famous result, joint with Michał Misiurewicz, states that for every C^1 endomorphism of the smooth compact manifold, the entropy is not smaller than the logarithm of the degree of the map. Another valuable result of Feliks Przytycki gave an upper bound for the entropy in terms of the growth of the logarithm of the integral (with respect of the volume measure) of the norm of the derivative (acting in the external algebra).

7. Foundations of theory of Anosov and Axiom A endomorphisms

Following the theory of Anosov and Smale diffeomorphisms, Feliks Przytycki developed a corresponding theory for non-invertible maps. He proved a rigid statement on the structural stability: for Axiom A endomorphisms stability is equivalent to the condition that on basic sets in the spectral decomposition the map is either invertible or expanding.

8. Stability of vector fields

Feliks Przytycki proved the conjecture of Newhouse, Palis and Takens about non-structural stability of saddle-node cycle bifurcation, up to density of hyperbolicity for interval maps (the latter has been proved recently by Oleg Kozlovski, Weixao Shen and Sebastian van Strien).

9. The book *Conformal fractals: ergodic theory methods*

The book, written jointly with Mariusz Urbański, is a perfect and deep study of the ergodic theory tools applied to holomorphic dynamical systems.

Feliks Przytycki – list of publications

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