BILLIARD CAUSTICS, FLOATING IN NEUTRAL EQUILIBRIUM, AND THE ISOPERIMETRIC INEQUALITY

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Abstract. This is an expanded version of my lecture at the conference "Ergodic Methods in Dynamics" honoring Feliks Przytycki, in Bedlewo, Poland, April 23-27, 2012. I discuss a few surprising connections between dynamics, geometry and mathematical physics.

Contents

1. Introduction

I first met Feliks at the London Mathematical Society Research Symposium on Dynamical Systems, in July 1988 in Durham, England. It was organized by Peter Walters, David Rand, and Antony Manning. The Symposium overlapped with another one, entitled "Spinors, Twistors and Complex Structure in General Relativity". The other Symposium had such distinguished participants as Sir Michael Atiyah, Stephen Hawking, and Sir Roger Penrose. Ours, on the other hand, besides Michal Misiurewicz, Feliks Przytycki and Maciej Wojtkowski, featured Adrian Douady, Michel Herman, Jaco Palis, Peter Perry, Steve Smale and Yakov Sinai. The curious reader may view the happy participants in the Dynamical Systems Symposium at the web site http://www.maths.dur.ac.uk/events/Meetings/LMS/pics/1988DS.jpg.

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The two groups intermingled in the symposium dining hall. At a lunch, my neighbor, undoubtedly a member of the parallel symposium, asked me extremely politely: "Are you a spinor, a twistor, or a *dynamical sister*"? Whatever my polite lunch mate meant, I am convinced that geometry is, in fact, a "dynamical sister". In this lecture I will illustrate this with examples.

The background in my lecture is the *billiard ball problem* on a compact, convex planar domain with a piecewise smooth boundary. It was championed by G.D. Birkhoff in the early 20th century. In his influential book [2] Birkhoff says, in particular, that "... in this problem the formal side, usually so formidable in dynamics, almost completely disappears, and only the interesting qualitative questions need to be considered". Indeed, it takes just a few lines to introduce the problem and to formulate basic questions. Surprisingly, many of them are open [13]. The billiard table can be viewed as a "playground of mathematician" [17], a source of open problems, and as a link between the dynamics and other mathematical branches, e.g., geometry, analysis, and number theory. To streamline the exposition, I have chosen to stay with smooth, convex billiard tables. Other types of billiard tables also yield basic open questions and provide links to geometry and analysis [9, 4, 22, 10].

I introduce the billiard flow and the billiard map in section 2. In order to illustrate the geometry aspect, I derive the classical isoperimetric inequality from the billiard ball problem in section 3. In section 4 I introduce the concept of caustics and discuss the famous (open) Birkhoff conjecture. Bialy's theorem settles an important special case of the Birkhoff conjecture. See Theorem 1. Since Wojtkowski's proof of Bialy's theorem uses only the billiard ball problem and the isoperimetric inequality, we now have a purely billiard proof of Bialy's theorem. Then I study billiard tables with constant angle caustics. Finally, I describe a relationship between billiard caustics and the concept of floating in neutral equilibrium arising in the mathematical fluid mechanics. Using the constant angle caustics, I obtain a description of noncircular cylinders that float in neutral equilibrium at every orientation. See Theorem 2. This answers an old question in fluid mechanics [6].

2. The billiard ball problem

The billiard ball problem, in its simplest version, deals with a bounded plane domain $\Omega \in \mathbb{R}^2$ with a piecewise smooth boundary $\partial\Omega$. We say that Ω is a *billiard table*. The *billiard flow* is the frictionless, unit speed motion of a material point (the billiard ball) on Ω. At $\partial\Omega$ the ball bounces off by the rule: "the angles of incidence and reflection are equal". This is the *billiard flow* on Ω. See Figure 1. Its phase space Ψ consists of unit vectors with base points in Ω. We use the notation $F^t : \Psi \to \Psi$ for the billiard flow.

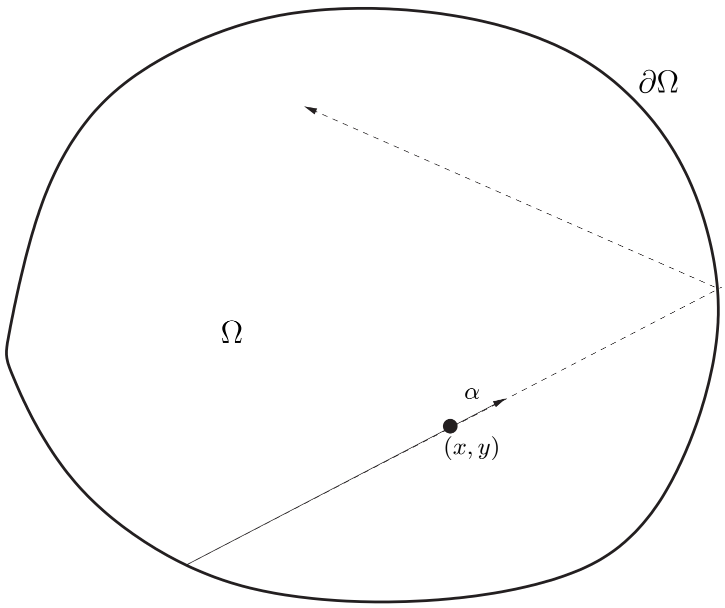

FIGURE 1. The billiard flow on Ω and its phase space.

The *billiard map* is the Poincare map for the natural cross-section $\Phi \subset \Psi$ consisting of unit vectors with base points in $\partial\Omega$. See Figure 2. Let (x, y) be euclidean coordinates in $\mathbb{R}^2$, and let $\{0 \leq \alpha < 2\pi\}$ be the set of *directions*. Let s be an arc length coordinate on $\partial\Omega$ and let $0 < \theta < \pi$ be the "shooting angle". We use the notation $f(s, \theta) = (s_1, \theta_1)$ for the billiard map. The *Liouville measure* on Ψ, invariant under the billiard flow, corresponds to the density $d\lambda = dxdyd\alpha$. The *induced measure* on Φ, which we also call the Liouville measure, has the density $d\mu = \sin\theta ds d\theta$. It is invariant under the billiard map. By a straightforward computation

$$\lambda(\Psi) = 2\pi|\Omega|, \ \mu(\Phi) = 2\,|\partial\Omega|. \tag{1}$$

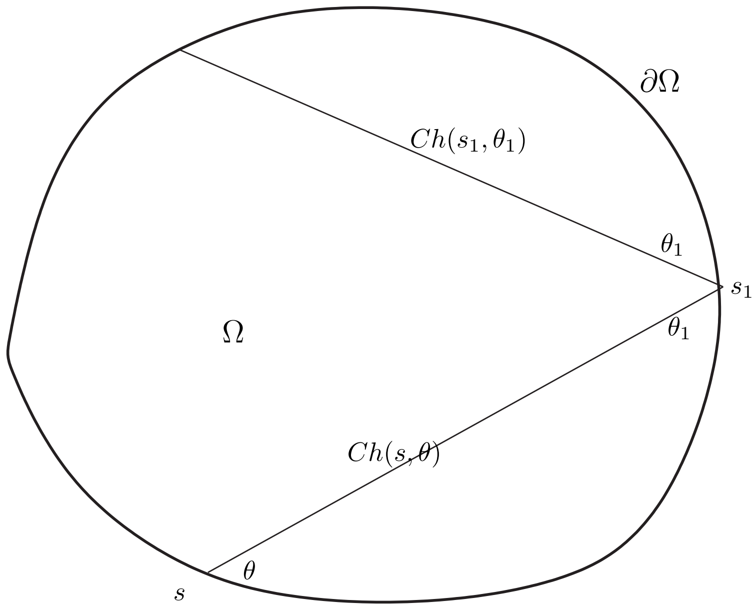

FIGURE 2. The canonical cross-section for the billiard flow on Ω, and the billiard map.

3. Isoperimetric inequality

The classical *isoperimetric inequality* [21] says i) that

$$|\Omega| \leq \frac{1}{4\pi}|\partial\Omega|^2 \tag{2}$$

and ii) that the equality holds iff Ω is a round disc. There are many proofs of these statements in the literature. We note that equation (2) has many generalizations, refinements, and extensions [21, 20, 3]. We will now derive the isoperimetric inequality from the billiard ball problem.

If Ω is not convex, then there is another bounded region, say Ω_1, such that $\Omega \subset \Omega_1$, the inclusion is strict, and $|\partial\Omega| = |\partial\Omega_1|$. Hence, we assume that Ω is convex. For $(s, \theta) \in \Phi$ let $Ch(s, \theta) \subset \Omega$ be the corresponding chord, and let $\ell(s, \theta) = |Ch(s, \theta)|$ be its length. See Figure 2. The billiard flow is the suspension flow over the billiard map the the roof function $\ell(\cdot)$, hence

$$\lambda(\Psi) = \int_\Phi \ell d\mu. \tag{3}$$

In view of equation (1), this yields

$$2\pi|\Omega| = \int_\Phi \ell(s,\theta)\sin\theta\, ds d\theta. \tag{4}$$

We will estimate the integral, say $I(\Omega)$, in the right hand side of equation (4). For $s \in \partial\Omega$ set $L(s) = \int_0^\pi \ell(s,\theta)\sin\theta d\theta$. Thus, $I(\Omega) = \int_{\partial\Omega} L(s)ds$. When $s \in \partial\Omega$ is fixed and θ varies from 0 to π, the chord $Ch(s,\theta)$ sweeps Ω. Let $A(s,\theta)$ be the area enclosed between $Ch(s,\theta)$ and $\partial\Omega$, to the right of $Ch(s,\theta)$. See Figure 3. With s fixed, $A(s,\theta)$ is a function of θ, and

$$dA(s,\theta) = \frac{1}{2}\ell^2(s,\theta)d\theta.$$

Thus, $\int_0^\pi \ell^2(s,\theta)d\theta = 2A(s,\pi) - 2A(s,0) = 2|\Omega|$. By the Cauchy-Schwarz-Buniakowski inequality

$$L(s)^2 \leq \int_0^\pi \ell^2(s,\theta)d\theta \int_0^\pi \sin^2\theta d\theta. \tag{5}$$

Since $\int_0^\pi \sin^2\theta d\theta = \frac{\pi}{2}$, equation (5) yields

$$L(s)^2 \leq \pi|\Omega|.$$

Thus

$$2\pi|\Omega| = I(\Omega) = \int_{\partial\Omega} L(s)ds \leq \sqrt{\pi}\sqrt{|\Omega|}|\partial\Omega|,$$

yielding

$$\sqrt{|\Omega|} \leq \frac{1}{2\sqrt{\pi}}|\partial\Omega|,$$

i.e., equation (2). The equality in inequality (2) takes place iff there is equality in inequality (5) for a.a. $s \in \partial\Omega$. By the Cauchy-Schwarz-Buniakowski theorem, this holds iff there exists a positive function $c(s)$ such that $\ell(s,\theta) = c(s)\sin\theta$ for a.a. $s \in \partial\Omega$. But the identity $\ell(s,\theta) = c\sin\theta$, even for a single point $s \in \partial\Omega$, implies, by elementary geometry, that Ω is a circle of radius $c/2$. This concludes our proof of the isoperimetric inequality.

4. Caustics, the Birkhoff conjecture, curves of constant width, and floating in neutral equilibrium

In this section the billiard table $\Omega \subset \mathbb{R}^2$ is convex and $\partial\Omega$ is at least of class C^1. We will use the *optics interpretation* of the billiard map. In this interpretation, $\partial\Omega$ is a perfect mirror, and Φ consists of light rays. A *caustic* is a closed, oriented curve $\gamma \subset \Omega$ with the following optics property: Each light ray tangent to γ remains, after reflection

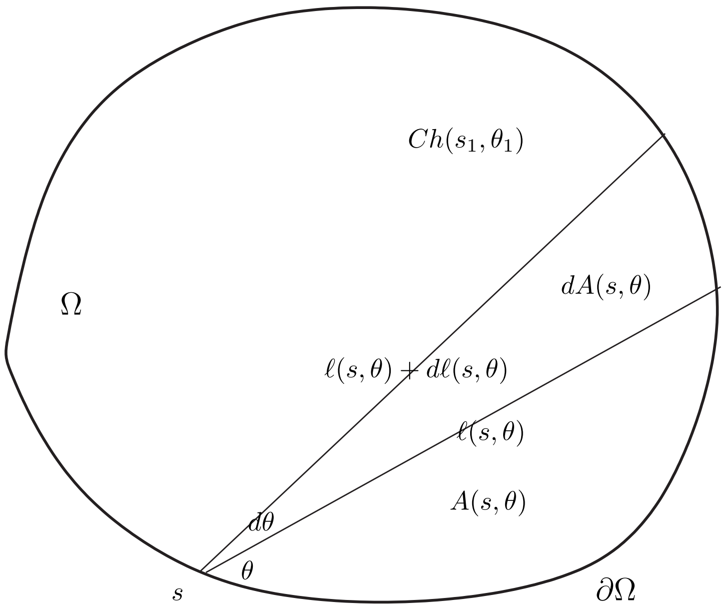

FIGURE 3. Sweeping Ω with the chord $Ch(s, \theta)$.

at $\partial\Omega$, tangent to γ. Let $\Gamma \subset \Phi$ be the set of light rays tangent to γ. The above property of γ is equivalent to $f(\Gamma) = \Gamma$, i.e., $\Gamma \subset \Phi$ is an *invariant curve* of the billiard map. Invariant curves $\Gamma \subset \Phi$ that are noncontractible topological circles are often called *invariant circles*. They are of of particular interest because then $\Phi \setminus \Gamma$ is a disjoint union of two invariant open sets. Since both of them have positive measure, the billiard on Ω is not ergodic. We will now discuss invariant circles $\Gamma \subset \Phi$, implicitly identifying them with the corresponding caustics $\gamma \subset \Omega$. Henceforth we will use the term "caustic" for both.

Let Ω be the disc of radius R. For all $r < R$ the concentric circles of radius r are caustics. Thus, the entire disc is foliated by caustics. Figure 4 shows the foliation of the round billiard table and the corresponding foliation of the billiard phase space. Let Ω be an ellipse with the foci F_1, F_2, and let $B \subset \Omega$ be the segment joining them. The confocal ellipses $\gamma \subset \Omega$ are caustics; they foliate the open set $\Omega \setminus B$. Since the disc is a special case of an ellipse, the above discussion implies the following statement.

Let the billiard table Ω be an ellipse. Then the caustics foliate an open set in Ω.

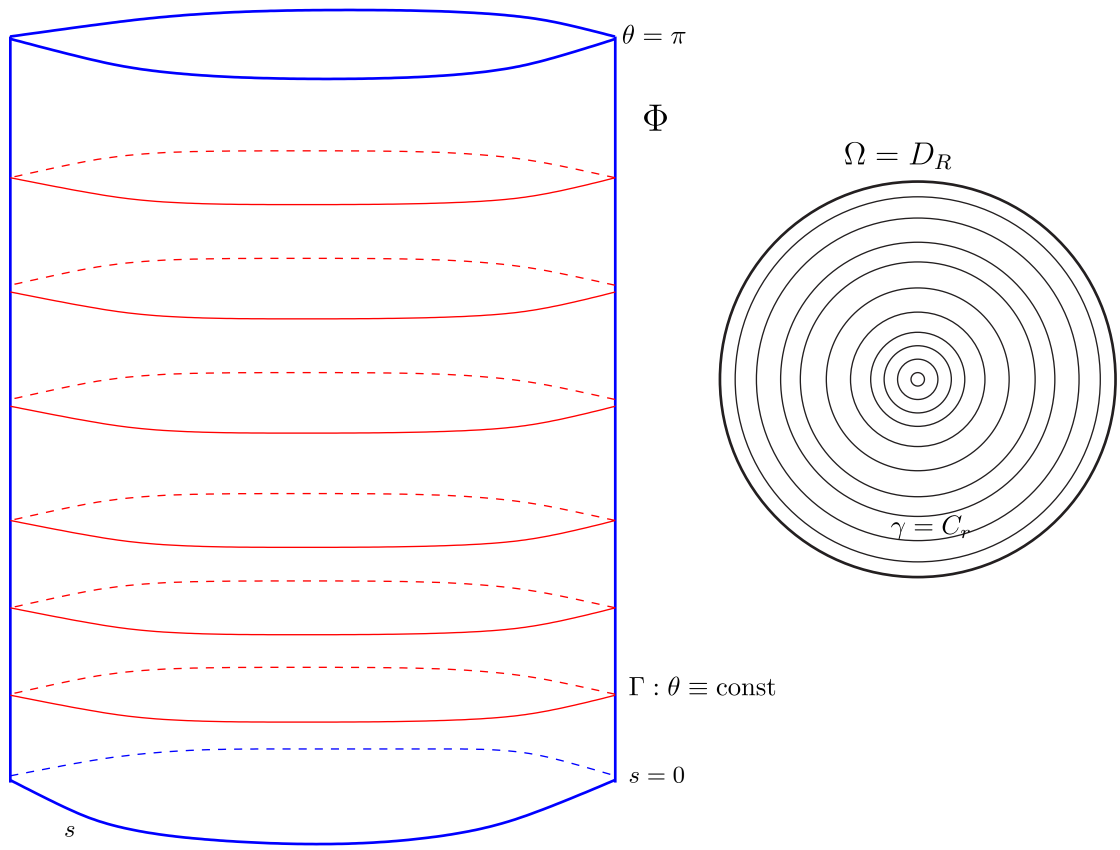

FIGURE 4. A circular billiard table foliated by concentric circles. The billiard map phase space is foliated by constant angle invariant circles.

The converse to this statement is an open question known as the *Birkhoff conjecture* [13]. The following theorem of M. Bialy [1] is a step towards the conjecture.

Theorem 1. *Let Ω be a billiard table. If caustics foliate the entire table, then Ω is a round disc.*

Bialy's proof of Theorem 1 is based on the variational, as opposed to the dynamical, approach. Wojtkowski's proof [23] is shorter, and uses a different approach. Let Ω be as in the assumption of Theorem 1. Exploring the billiard dynamics, Wojtkowski shows that Ω satisfies the inequality

$$|\Omega| \geq \frac{1}{4\pi}|\partial\Omega|^2. \tag{6}$$

The claim now follows, by the isoperimetric inequality theorem. Combining Wojtkowski's argument yielding equation (6) with the billiard derivation of the isoperimetric inequality in section 3, we obtain a proof of Theorem 1 entirely by billiard arguments.

The importance of caustics for the billiard research is not restricted to the Birkhoff conjecture. A caustic $\gamma \subset \Omega$ defines a positive function $\theta = h_\gamma(s)$ on $\partial\Omega$. We can think of it as the *height function* for the corresponding curve $\Gamma \subset \Phi$. Birkhoff proved that $h_\gamma(s)$ is a lipshitz function. Which lipshitz functions can occur this way? How many caustics (in an appropriate sense) does a billiard table typically have? What may prevent a billiard table from having any caustics? These and related questions have been widely studied [18, 5, 19, 16, 12, 15].

Let D (resp. D_r) denote any disc (resp. disc of radius r). The height functions corresponding to the caustics of D are especially simple: $h_\gamma(s) = \text{const}$. We will refer to these as *constant angle caustics.* Thus, every caustic of D is a constant angle caustic. Let Ω be a billiard table. If every constant function $h_\gamma(s) \equiv c$, $0 < c < \pi$, is the height function of a caustic in Ω, then, by Bialy's theorem, $\Omega = D$.

Suppose now that Ω has a constant angle caustic. Equivalently, let there exist $\theta \in (0, \pi)$ such that every chord in Ω making angle θ with $\partial\Omega$ at one endpoint makes angle θ with $\partial\Omega$ at the other endpoint. Is Ω circular then? Of course not: Any *domain of constant width* has this property with $\theta = \pi/2$: A straight line orthogonal to $\partial\Omega$ at one intersection point is orthogonal to $\partial\Omega$ at the other intersection point.

The popular example of a noncircular domain of constant width is the *Reuleaux triangle.* It is obtained via replacing each side of an equilateral triangle by the circular arc centered at the opposite corner of the triangle. See Figure 6. The Reuleaux triangle is not a C^1 curve: It has three corners. Many mathematicians think that all noncircular domains of constant width are singular. This is not so! Let $\rho(\cdot)$ be a C^k-smooth, strictly positive, 2π-periodic function on $\mathbb{R}$ such that

$$\int_0^{2\pi} \rho(\alpha) \cos \alpha d\alpha = \int_0^{2\pi} \rho(\alpha) \sin \alpha d\alpha = 0.$$

Then $\rho(\alpha)$ is the *radius of curvature* function of a C^{k+2}-smooth convex domain Ω. The domain Ω is a domain of constant width iff

$$\rho(\alpha) + \rho(\alpha + \pi) = \text{const}.$$

We have $\Omega = D_r$ iff $\rho(\cdot) \equiv r$. In particular, any nonconstant, real analytic function $\rho(\cdot)$ satisfying the above conditions yields a noncircular domain of constant width of class C^ω.

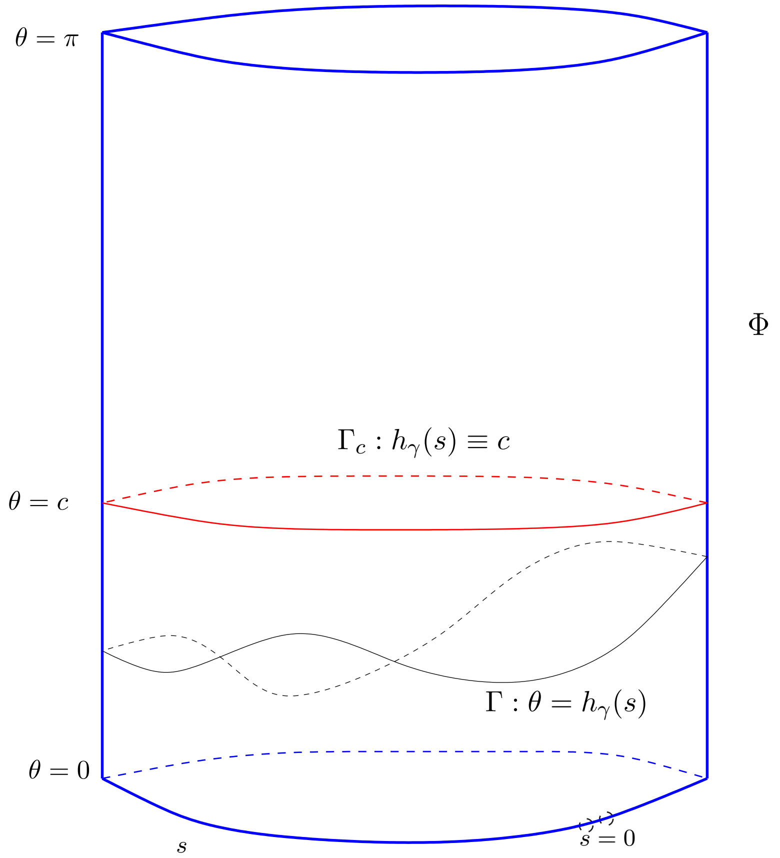

FIGURE 5. The billiard map phase space. A general caustic versus a constant angle caustic.

Excluding $\theta = \pi/2$, and using the reflection symmetry, we reformulate the above question as follows:

Which noncircular billiard tables Ω have a constant angle caustic Γ_θ, and for what $\theta \in (0, \pi/2)$?

Equivalently, for which noncircular Ω there exist $\theta \in (0, \pi/2)$ such that every straight line intersecting $\partial\Omega$ at the angle θ at one endpoint does the same at the other endpoint?

It turns out that these pairs Ω, θ exist, although rarely [11]. I published these results after I have learned from Bob Finn that the very same question comes up in the mathematical fluid mechanics [6, 7, 8]. Following up on the earlier work of Thomas Young [24], Finn has

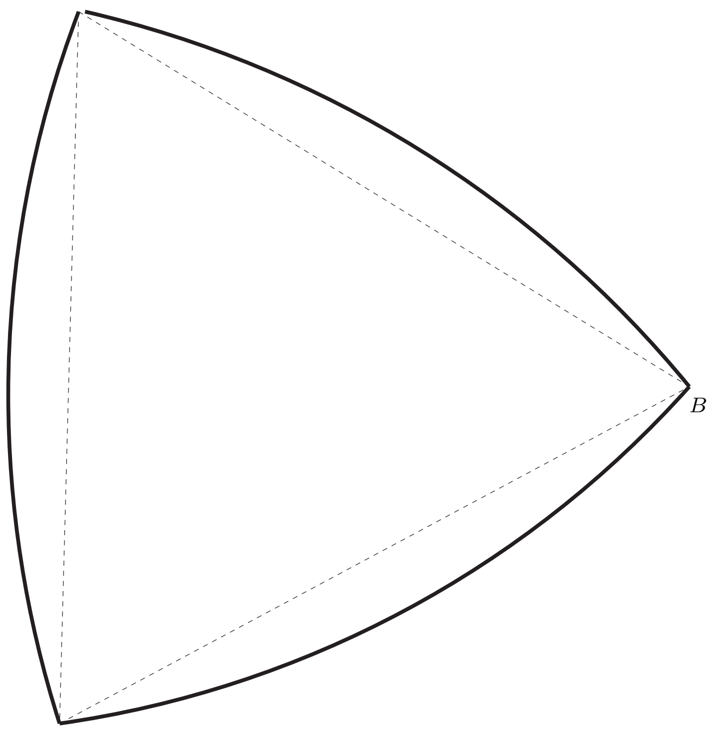

FIGURE 6. Reuleaux triangle: A domain of constant width with three corners.

formulated a theory of *floating in neutral equilibrium* where the *capillary forces* of the liquid are much stronger than the force of gravity. The Finn-Young theory of *floating in neutral equilibrium* is three-dimensional. However, in the special case when the solid is an infinite cylinder $\Omega \times \mathbb{R}$, the floating conditions become two-dimensional. In the Finn-Young floating theory the cylinder $\Omega \times \mathbb{R}$ floats in neutral equilibrium *in every orientation at the contact angle* $\pi - \theta$ iff the pair Ω, θ satisfies the above geometric condition. Figure 7 illustrates this concept.

Theorem 2. *Let* Ω *be a noncircular billiard table with a constant angle caustic* Γ_θ*,* $0 < \theta < \pi/2$*.*
1. Then there is an integer $n > 1$ *such that*

$$\tan n\theta = n \tan \theta. \tag{7}$$

2. For $n > 1$ *let* $A_n \subset (0, \pi/2)$ *be the set of angles satisfying equation* (7)*. Then*

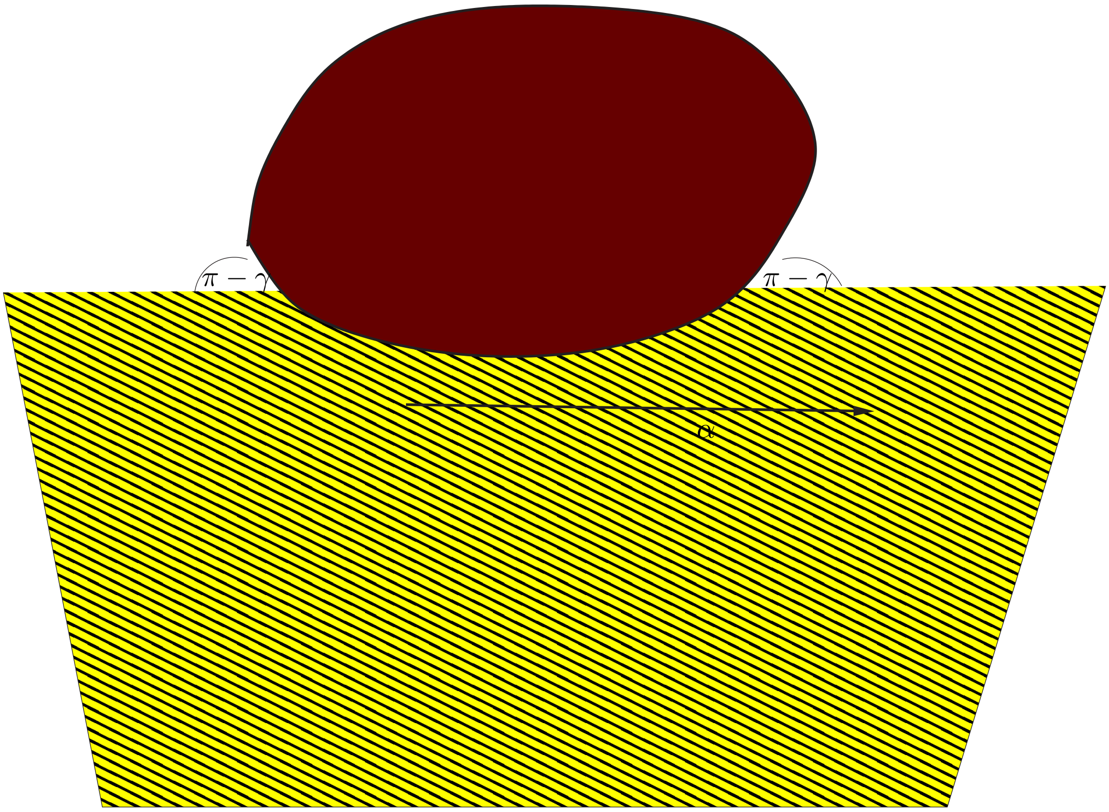

FIGURE 7. Finn-Young floating in neutral equilibrium at the orientation α with the contact angle $\pi - \gamma$.

i) For $n \geq 4$ the set A_n is not empty, $|A_n| < \infty$, and $\cup_n A_n = A \subset (0, \pi/2)$ is a countably infinite dense set;
ii) For any $\theta \in A_n$ there is a real analytic family $\{\Omega_{n,\tau} : 0 < \tau < 1\}$ of noncircular C^ω billiard tables;
iii) The limit set $\lim_{\tau \to 0} \Omega_{n,\tau}$ is the unit disc D_1;
iv) The limit set $\lim_{\tau \to 1} \Omega_{n,\tau}$ is a convex, piecewise real analytic billiard table with corners.

Proof. We parameterize $\partial\Omega$ by the direction $0 \leq \alpha < 2\pi$ of the tangent line. Let $\rho(\alpha)$ be the radius of curvature function, and let $z(\alpha) = (x(\alpha), y(\alpha))$ be the coordinates on $\partial\Omega$. The function $\rho(\cdot)$ determines $\partial\Omega$ via

$$x'(\alpha) = \rho(\alpha)\cos\alpha,\ y'(\alpha) = \rho(\alpha)\sin\alpha.$$

Using the assumption on intersection angles and expressing $(x(\alpha), y(\alpha))$ as integrals of $\rho(\cdot)$, we obtain

$$\frac{\int_{\alpha-\theta}^{\alpha+\theta} \rho(\xi) \sin \xi d\xi}{\int_{\alpha-\theta}^{\alpha+\theta} \rho(\xi) \cos \xi d\xi} = \tan \alpha.$$

This is equivalent to

$$\int_{\alpha-\theta}^{\alpha+\theta} \rho(\xi) \sin(\alpha - \xi) d\xi = 0.$$

Setting

$$k_\theta(\alpha) = (\sin \alpha)\, 1_{[-\theta,\theta]},$$

we rewrite the above identity as the convolution equation $\rho * k_\theta = 0$. Computing the fourier transform of the function $k_\theta(\cdot)$, we obtain equation (7). This proves claim 1. Claim 2i) is proved by an elementary study of equation (7). For $n \in \mathbb{N}$ and $0 < \tau < 1$ set $\rho_{n,\tau}(\alpha) = 1 + \tau \sin n\alpha$. Then for any θ satisfying equation (7), we have $\rho_{n,\tau} * k_\theta = 0$. Let $\partial\Omega_{n,\tau}$ be the curve with the radius of curvature $\rho_{n,\tau}(\cdot)$. The family of billiard tables $\{\Omega_{n,\tau} : 0 < \tau < 1\}$ satisfies the conclusions of claims 2ii), 2iii), and 2iv). I refer the reader to [14] for details. □

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