Topological Dynamical Embedding and Jaworski-type Theorems

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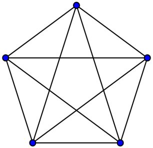
Ergodic Methods in Dynamics
On the occasion of the 60th birthday of Professor Feliks Przytycki
Będlewo, April 23-27, 2012

Embedding a topological space in a Euclidean cube

Consider K_5, the complete graph on 5 vertices.



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Theorem (Menger, 1926, Nöbeling, 1930)

If X is a metric compact space with $dim(X) = n$ then there exists an embedding $X \hookrightarrow [0,1]^{2n+1}$.

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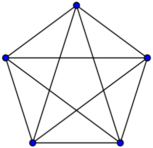

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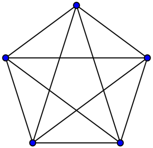

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Realizing a measure-preserving system as a symbolic shift

Let $A \in SL_n(\mathbb{Z})$ be considered as an invertible transformation of the n-torus $\mathbb{T}^n$. Assume $(\mathbb{T}^n, \mathcal{B}, \mu_{haar}, A)$ is ergodic. E.g.:

$$A = \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}$$

$$h(\mathbb{T}^n, \mathcal{B}, \mu_{haar}, A) = \log \frac{5 + \sqrt{21}}{2} \approx \log 4.79$$

Krieger's Generator Theorem: $\exists$ **generating partition** $\mathcal{P}$ $|\mathcal{P}| = 5$.

$\Leftrightarrow$

$$(\mathbb{T}^2, \mathcal{B}, \mu_{haar}, A) \hookrightarrow (\{1,2,3,4,5\}^{\mathbb{Z}}, shift)$$

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Generating Functions and full topological shifts

A generating partition with m elements is equivalent to a **measurable** function $P : X \to \{1, 2, \ldots, m\}$ so that the orbit-map $I_P(x) = (P(T^k(x)))_{k\in\mathbb{Z}}$ a.s. separates points.
Let (X, T) (or $(\mathbb{Z}, X)$) be a topological dynamical system (t.d.s.) given by the homeomorphism $T : X \to X$.

Definition

We call a continuous $f : X \to [0,1]^d$ a **generating function** if $I_f(x) \triangleq (f(T^k(x)))_{k\in\mathbb{Z}}$ is *separating points*, i.e.:

$$I_f : (X, T) \hookrightarrow (([0,1]^d)^{\mathbb{Z}}, shift)$$

$(([0,1]^d)^{\mathbb{Z}}, shift)$ is referred to as the full topological shift on the alphabet $[0,1]^d$ or simply as the d-cubical shift.

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Canonical Embedding Spaces and Embedding Dimension

- Let X be a compact metric space.
- There is an embedding:

$$X \hookrightarrow [0,1]^{\mathbb{N}}$$

- Let (X,T) be a t.d.s. There is an equivariant embedding:

$$(X,T) \hookrightarrow (([0,1]^{\mathbb{N}})^{\mathbb{Z}}, shift)$$

- Under which conditions is there an embedding into the d-cubical shift:

$$(X,T) \hookrightarrow (([0,1]^{d})^{\mathbb{Z}}, shift)\ (d \in \mathbb{N})?$$

- Define the embedding-dimension:

$$edim(X,T) = \min\{d \in \mathbb{N} \cup \{\infty\} \,|\, \exists \theta : (X,T) \hookrightarrow (([0,1]^{d})^{\mathbb{Z}}, shift)\}$$

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A Question

Let X be finite dimensional and (X,T) aperiodic with finite topological entropy. E.g.:

$$X=\mathbb{T}^n,\ \ T:\mathbb{T}^n\to\mathbb{T}^n\ (x_1,\ldots,x_n)\mapsto(x_1+\beta_1,\ldots,x_n+\beta_n)$$

Question: Does (X,T) have a generating function $f:X\to[0,1]$?

More generally, let $T_i:X\to X$, $i=1,2,\ldots$ be aperiodic homeomorphisms of X with finite topological entropy. Consider the product $\prod X\triangleq X\times X\times\ldots$ with the (compact) Tychonoff topology and the "diagonal" homeomorphism $\prod_i T_i\triangleq T_1\times T_2\times\ldots$

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Mean Dimension

- $ord(\alpha) = \max_{x \in X} \sum_{U \in \alpha} 1_U(x) - 1$
- $D(\alpha) = \min_{\beta \succ \alpha} ord(\beta)$
- $dim(X) = \sup_\alpha D(\alpha)$
- (Gromov) $\boxed{mdim(X, T) = \sup_\alpha \lim_{n \to \infty} \frac{1}{2n+1} D(\bigvee_{k=-n}^{k=n} T^k \alpha)}$
- Compare: $h_{top}(X, T) = \sup_\alpha \lim_{n \to \infty} \frac{\log N(\bigvee_{k=-n}^{k=n} T^k \alpha)}{2n+1}$, $N(\alpha) = \min_{\beta \subset \alpha} |\beta|$

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Mean Dimension - Properties

- $h_{top}(X,T) < \infty \Rightarrow mdim(X,T) = 0$
- $dim(X) < \infty \Rightarrow mdim(X,T) = 0$
- $mdim(([0,1]^d)^{\mathbb{Z}}, shift) = d$
- If $Y \subset X$ is closed and T-invariant then $mdim(Y,T) \leq mdim(X,T)$
- $edim(X,T) \geq mdim(X,T)$.

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Let $P_n = \{x \in X | \exists 1 \leq m \leq n \; T^m x = x\}$, be the set of **periodic points of period** $\leq n$ of X. Define:

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Conjecture (Lindenstrauss)

Let $d \in \mathbb{N}$ be such that:

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Theorem (Jaworski, 1974)

If $\dim(X) = n$ *and* $P_{(4n+3)^2} = \varnothing$, *then* (X, T) *can be equivariantly embedded in* $(([0,1])^{\mathbb{Z}}, shift)$.

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Theorem (G 2011)

There exist constants $C(k) = \frac{3(\frac{k}{2})!200^k}{\pi^{\frac{k}{2}}}$, $k = 1, 2, \ldots$ so that if $(\mathbb{Z}^k, X)$ is an extension of an aperiodic zero-dimensional t.d.s and $mdim(\mathbb{Z}^k, X) < \frac{d}{C(k)}$, then $(\mathbb{Z}^k, X)$ can be equivariantly embedded in $(([0,1]^d)^{\mathbb{Z}^k}, \mathbb{Z}^k - shift)$.

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