Existence of Absorbing domains: Proof of Theorem A

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Previous results and definitions

- $\text{dist}(\cdot,\cdot)$ denotes the Euclidean distance.
- $\mathbb{D}(z,r)$ is the Euclidean disc of radius r centred at $z \in \mathbb{C}$.
- A domain $U \subset \mathbb{C}$ is hyperbolic if its boundary in $\overline{\mathbb{C}}$ contain at least three points. By the Uniformization Theorem, in this case there exists a universal holomorphic covering from $\mathbb{D}$ (or $\mathbb{H}$) onto U.
- $\varrho_U(\cdot)$ and $\varrho_U(\cdot,\cdot)$ denote respectively the density of the hyperbolic metric and the hyperbolic distance in U.

Previous results and definitions

- In particular, we consider the open unit disc $\mathbb{D}$ equipped with the hyperbolic metric of density

$$\varrho_{\mathbb{D}}(z) = \frac{2}{1 - |z|^2}$$

and the right half-plane

$$\mathbb{H} = \{z \in \mathbb{C} : \Re(z) > 0\}$$

with the hyperbolic metric of density

$$\varrho_{\mathbb{H}}(z) = \frac{1}{\Re(z)}.$$

- Finally $\mathcal{D}_U(z, r)$ denotes the disc of radius r, centred at $z \in U$, with respect to the hyperbolic metric in U.

Previous results and definitions

Denjoy–Wolf's Theorem: Let $g : \mathbb{D} \mapsto \mathbb{D}$ holomorphic. Assume $g \not\equiv c$ and g not being an automorphism of $\mathbb{D}$ (Möbius transformation preserving $\mathbb{S}$).
Then there exists $z_0 \in \overline{\mathbb{D}}$ such that g^n tends to z_0 (uniformly in compact subsets of $\mathbb{D}$) as n tends to ∞. If there are no fixed points of g in $\mathbb{D}$ then $z_0 \in \partial\mathbb{D}$.

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Definition (Absorbing domain) Let U be a hyperbolic domain in $\mathbb{C}$ and let $F : U \to U$ be a holomorphic map. An invariant domain $W \subset U$ is absorbing in U for F if for every compact set $K \subset U$ there exists $n > 0$, such that $F^n(K) \subset W$.

Cowen's Theorem: Let $g : \mathbb{H} \to \mathbb{H}$ be a holomorphic map such that $g^n \to \infty$ as $n \to \infty$. Then there exists a simply connected domain $V \subset \mathbb{H}$, a domain Ω equal to $\mathbb{H}$ or $\mathbb{C}$, a holomorphic map $\varphi : \mathbb{H} \to \Omega$, and a Möbius transformation T mapping Ω onto itself, such that:

(a) $g(V) \subset V$,

(b) V is absorbing in $\mathbb{H}$ for g,

(c) $\varphi(V)$ is absorbing in Ω for T,

(d) $\varphi \circ g = T \circ \varphi$ on $\mathbb{H}$,

(e) φ is univalent on V.

Previous results

Moreover, φ, T depend only on g. In fact (up to a conjugation of T by a Möbius transformation preserving Ω), one of the following cases holds:

- $\Omega = \mathbb{C}$, $T(\omega) = \omega + 1$,
- $\Omega = \mathbb{H}$, $T(\omega) = a\omega$ for some $a > 1$,
- $\Omega = \mathbb{H}$, $T(\omega) = \omega \pm i$.

Previous results

Hence, by Cowen's Theorem, g defined in $\mathbb{H}$ is semi-conjugated to a Möbius transformation T on Ω by the map φ, which is univalent on V. In other words, we have the following (blue part) commutative diagram.

$$
\begin{array}{ccccc}
\varphi(V) & \subset & \Omega & \xrightarrow{T} & \Omega \\
\downarrow \varphi^{-1} & & \uparrow \varphi & & \uparrow \varphi \\
V & \subset & \mathbb{H} & \xrightarrow{g} & \mathbb{H} \\
 & & \downarrow \pi & & \downarrow \pi \\
 & & U & \xrightarrow{F} & U
\end{array}
$$

with $\Omega = \{\mathbb{C}, \mathbb{H}\}$.

Previous results

König's Theorem: Let U be a hyperbolic domain in $\mathbb{C}$ and let $F : U \to U$ be a holomorphic map, such that $F^n \to \infty$ as $n \to \infty$. Suppose that for every closed curve $\gamma \subset U$ there exists $n > 0$ such that $F^n(\gamma)$ is contractible in U (this is guarantied by F having a finite number of poles). Then there exists a simply connected domain $W \subset U$, a domain Ω and a transformation T as in Cowen's Theorem, and a holomorphic map $\psi : U \to \Omega$, such that:

(a) $F(W) \subset W$,

(b) W is absorbing in U for F,

(c) $\psi(W)$ is absorbing in Ω for T,

(d) $\psi \circ F = T \circ \psi$ on U,

(e) ψ is univalent on W.

In fact, if we take V and φ from Cowen's Theorem for g being a lift of F by a universal covering $\pi : \mathbb{H} \to U$, then π is univalent in V and one can take $W = \pi(V)$ and $\psi = \varphi \circ \pi^{-1}$, which is well defined in U.

Previous results

$$
\begin{array}{ccccc}
\varphi(V) & \subset & \Omega & \xrightarrow{T} & \Omega \\
\downarrow{\scriptstyle \varphi^{-1}} & & \uparrow{\scriptstyle \varphi} & & \uparrow{\scriptstyle \varphi} \\
V & \subset & \mathbb{H} & \xrightarrow{g} & \mathbb{H} \\
\downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\
W := \pi(V) & \subset & U & \xrightarrow{F} & U
\end{array}
$$

with $\psi = \varphi \circ \pi^{-1}$. Notice that we assume here that F has a finite number of poles.

Theorem A

Theorem A (Existence of Absorbing domains) Let U be a hyperbolic domain in $\mathbb{C}$ and let $F : U \to U$ be a holomorphic map such that $F^n(z) \to \infty$ as $n \to \infty$ for $z \in U$. Then there exists a domain $W \subset U$, such that:

(a) $\overline{W} \subset U$,

(b) $F(\overline{W}) = \overline{F(W)} \subset W$,

(c) $\bigcap_{n=1}^{\infty} F^n(\overline{W}) = \emptyset$,

(d) W is absorbing in U for F.

Moreover, for every point $z \in U$ and every sequence of positive numbers r_n, $n \geq 0$ with $\lim_{n\to\infty} r_n = \infty$, the domain W can be chosen such that

$$W \subset \bigcup_{n=0}^{\infty} \mathcal{D}_U(F^n(z), r_n).$$

The main tool

Main Proposition: (Assume all previous notation and results.)
There exists a simply connected domain $A \subset \Omega$ with the following properties:

(a) $\overline{A} \subset \varphi(V)$ (V Cowen's absorbing set in $\mathbb{H}$),

(b) $T(\overline{A}) \subset A$,

(c) A is absorbing in Ω for T

(d) for every $\omega \in \Omega$ and every sequence of positive numbers b_n with $\lim_{n\to\infty} b_n = \infty$ there exists $m \in \mathbb{N}$ such that the domain A can be chosen with

$$A \subset \bigcup_{n=m}^{\infty} \mathcal{D}_{\varphi(V)}(T^n(\omega), b_n).$$

Remark: In the future $W := \pi\left(\varphi^{-1}(A)\right)$

Proof of the Main Proposition

We will proof this result for $\Omega = \mathbb{H}$ (and so $T(w) = aw, \; a > 1$ or $T(w) = \omega \pm i$). The other case is similar but different...

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The proof splits in many STEPS. See Figure 1 (by hand!)

STEP 1:

- Notice that T acting on $\Omega \equiv \mathbb{H}$ is an isometry with respect to the hyperbolic metric in $\mathbb{H}$ (It sends hyperbolic discs to hyperbolic discs of the same radius).
- Take $V \in \mathbb{H}$ as in Cowen's Theorem. Since $\varphi(V)$ is absorbing in $\mathbb{H}$ for T then $T(\varphi(V)) \subset \varphi(V)$.

Proof of the Main Proposition

STEP 2: Take $w \in \Omega \equiv \mathbb{H}$ and take $\{b_n\}_{n\geq 0}$ such that $b_n > 0$ for all $n \geq 0$ and $\lim_{n\to\infty} b_n = \infty$.

Claim: There exists $m > 0$ and $\{d_n\}_{n\geq 0}$ such that $d_n > 0$ for all $n \geq 0$ and $\lim_{n\to\infty} d_n = \infty$ such that

$$\mathcal{D}_{\mathbb{H}}(T^n(w), d_n) \subset \varphi(V), \forall n \geq m.$$

Assume it is not true: Then there exists $d > 0$ such that $\mathcal{D}_{\mathbb{H}}(T^n(w), d) \not\subset \varphi(V)$ for infinitely many n's. Take $K = \overline{\mathcal{D}_{\mathbb{H}}(w, d)} \subset \mathbb{H}$. Since T is an isometry we have

$$T^n(\mathcal{D}_{\mathbb{H}}(w, d)) = \mathcal{D}_{\mathbb{H}}(T^n(w), d) \not\subset \varphi(V)$$

for infinitely many n's. A contradiction with STEP 1 since $\varphi(V)$ is absorbing in $\mathbb{H}$ for T.

Proof of the Main Proposition

STEP 3: Given the sequences $\{b_n\}_{n\geq 0}$ (arbitrary) and $\{d_n\}_{n\geq 0}$ as in STEP 2 define a new sequence c_n as follows:

$$c_n = \frac{n}{n+1}\min\left\{\inf_{k\geq n} \ln \frac{1+B_kD_k}{1-B_kD_k}\ ,\ \frac{\rho_{\mathbb{H}}(T^n(w),w)}{2}\right\}$$

where

$$B_n = \frac{\exp(b_n)-1}{\exp(b_n)+1} < 1 \quad \text{and} \quad D_n = \frac{\exp(d_n)-1}{\exp(d_n)+1} < 1$$

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- $c_n \geq 0$ for all $n \geq 0$.
- $c_{n+1} > c_n$ and $\{c_n\} \to \infty$ as $n \to \infty$.
- $c_n < d_n$ for all $n \geq 0$ and $C_n < D_nB_n$ where

$$C_n = \frac{\exp(c_n)-1}{\exp(c_n)+1} \quad \left(c_n < \ln\frac{1+B_nD_n}{1-B_nD_n}\right)$$

Proof of the Main Proposition

STEP 4:

Claim: The set

$$A := \bigcup_{n=m}^{\infty} \mathcal{D}_{\mathbb{H}}(T^n(\omega), c_n) \quad \left(\subset \bigcup_{n=m}^{\infty} \mathcal{D}_{\varphi(V)}(T^n(\omega), b_n) \right)$$

is a simply connected domain if m is large enough. The green part is what we will see!!

Choosing m larger if necessary we can assume

$$c_n > \rho_{\mathbb{H}}(w, T(w)) = \rho_{\mathbb{H}}(T^n(w), T^{n+1}(w)) \; \forall n > m,$$

where the equality follows becuase T is an isometry in $\mathbb{H}$. We have:

- A is the union of hyperbolic discs in $\mathbb{H}$, and hyperbolic discs in $\mathbb{H}$ are Euclidian (and so convex sets).
- This union of convex sets contains the line of the trajectory of $T^m(w)$.

So A is a simply connected domain.

Proof of the Main Proposition

STEP 5: Claim:

$$\overline{A} := \overline{\bigcup_{n=m}^{\infty} \mathcal{D}_{\mathbb{H}}(T^n(\omega), c_n)} \subset \bigcup_{n=m}^{\infty} \overline{\mathcal{D}_{\mathbb{H}}(T^n(\omega), c_n)}.$$

Remark: we want to see that $T(\overline{A}) \subset A$...

STEP 5: Claim:

$$\overline{A} := \overline{\bigcup_{n=m}^{\infty} \mathcal{D}_{\mathbb{H}}(T^n(\omega), c_n)} \subset \bigcup_{n=m}^{\infty} \overline{\mathcal{D}_{\mathbb{H}}(T^n(\omega), c_n)}.$$

Remark: we want to see that $T(\overline{A}) \subset A$...

To see the claim let $v \in \overline{A}$ and let $v_k \to v$ with $v_k \in A$. Thus there exist $n_k > m$ such that

$$v_k \in \mathcal{D}_{\mathbb{H}}(T^{n_k}(\omega), c_{n_k}) \quad \left(c_{n_k} < \frac{\rho_{\mathbb{H}}(T^{n_k}(w), w)}{2} \right)$$

Thus (see Figure 2)

Proof of the Main Proposition

$$\frac{\rho_{\mathbb{H}}(T^{n_k}(w), w)}{2} > c_{n_k} > \rho_{\mathbb{H}}(T^{n_k}(w), v_k) \geq \rho_{\mathbb{H}}(T^{n_k}(w), w) - \rho_{\mathbb{H}}(v_k, w)$$

and consequently

$$\rho_{\mathbb{H}}(v_k, w) > \frac{1}{2}\rho_{\mathbb{H}}(T^{n_k}(w), w).$$

But $\rho_{\mathbb{H}}(v_k, w)$ is bounded (since $v_k \to v$) and so $n_k \equiv \hat{n}$ for $k \geq k_0$. We conclude

$$v \in \overline{\mathcal{D}_{\mathbb{H}}(T^{\hat{n}}(\omega), c_{\hat{n}})}$$

STEP 6: The statements of the Proposition...

Statement (a): $\overline{A} \subset \varphi(V)$.

- $c_n < d_n$ (this follows from $C_n < B_n D_n$, $B_n < 1$ and $f(x) = \frac{e^x - 1}{e^x + 1}$ increasing).
- So,

$$\mathcal{D}_{\mathbb{H}}(T^n(\omega), c_n) \subset \mathcal{D}_{\mathbb{H}}(T^n(\omega), d_n) \subset \varphi(V) \quad \forall n \geq m$$

and consequently

$$\overline{A} \subset \bigcup_{n=m}^{\infty} \overline{\mathcal{D}_{\mathbb{H}}(T^n(\omega), c_n)} \subset \varphi(V)$$

Proof of the Main Proposition

STEP 6: Statement (b): $T(\overline{A}) \subset A$.

- First notice that

$$\overline{A} \subset \bigcup_{n=m}^{\infty} \overline{\mathcal{D}_{\mathbb{H}}(T^n(\omega), c_n)}$$

- Second observe that

$$T(\mathcal{D}_{\mathbb{H}}(T^n(\omega), c_n) = \mathcal{D}_{\mathbb{H}}(T^{n+1}(\omega), c_n) \subset \mathcal{D}_{\mathbb{H}}(T^{n+1}(\omega), c_{n+1})$$

where the equality follows since T is an isometry with respect to the hyperbolic metric in $\mathbb{H}$ and the inclusion follows since c_n is increasing.

Proof of the Main Proposition

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  where the equality follows since T is an isometry with respect to the hyperbolic metric in $\mathbb{H}$ and the inclusion follows since c_n is increasing.

- Finally

$$T(\overline{A}) \subset \bigcup_{n=m}^{\infty} \overline{T(\mathcal{D}_{\mathbb{H}}(T^n(\omega), c_n))} \subset \bigcup_{n=m+1}^{\infty} \mathcal{D}_{\mathbb{H}}(T^n(\omega), c_n) \subset A.$$

STEP 6: Statement (c): A is absorbing in $\mathbb{H}$ for T.

- First notice that for all compact set K in $\mathbb{H}$ there exists $r > 0$ such that $K \subset \mathcal{D}_{\mathbb{H}}(w, r)$.
- Second observe that

$$T^n(K) \subset T^n\left(\mathcal{D}_{\mathbb{H}}(w, r)\right) = \mathcal{D}_{\mathbb{H}}(T^n(w), r) \subset \mathcal{D}_{\mathbb{H}}(T^n(w), c_n) \subset A$$

for some large enough n.

Proof of the Main Proposition

STEP 6: Statement (d): $A \subset \bigcup_{n=m}^{\infty} \mathcal{D}_{\varphi(V)}(T^n(\omega), b_n)$

The proof of this needs a little bit of work. We need to show that

$$\mathcal{D}_{\mathbb{H}}(T^n(w), c_n) \subset \mathcal{D}_{\varphi(V)}(T^n(w), b_n) \qquad (1)$$

for every $n \geq m$.

- From Schwartz-Pick Lemma, since $\mathcal{D}_{\mathbb{H}}(T^n(w), d_n) \subset \varphi(V)$ for all $n \geq m$, we have

$$\mathcal{D}_{\mathcal{D}_{\mathbb{H}}(T^n(w), d_n)}(T^n(w), b_n) \subset \mathcal{D}_{\varphi(V)}(T^n(w), b_n)$$

- Consequently to prove (1) it is enough to show

$$\mathcal{D}_{\mathbb{H}}(T^n(w), c_n) \subset \mathcal{D}_{\mathcal{D}_{\mathbb{H}}(T^n(w), d_n)}(T^n(w), b_n)$$

Proof of the Main Proposition

$$\mathcal{D}_{\mathbb{H}}(T^n(w), c_n) \subset \mathcal{D}_{\mathcal{D}_{\mathbb{H}}(T^n(w), d_n)}(T^n(w), b_n)$$

Let $h_1 : \mathbb{C} \to \mathbb{C}$ a Möbius transformation (isometry) such that $h_1(\mathbb{H}) = \mathbb{D}$ (onto) and $h_1(T^n(w)) = 0$. Then

- $$h_1\left(\mathcal{D}_{\mathbb{H}}(T^n(w), c_n)\right) = \mathcal{D}_{\mathbb{D}}(0, c_n) = \mathbb{D}(0, C_n)$$

- $$h_1\left(\mathcal{D}_{\mathcal{D}_{\mathbb{H}}(T^n(w), d_n)}(T^n(w), b_n)\right) = \mathcal{D}_{\mathcal{D}_{\mathbb{D}}(0, d_n)}(0, b_n) = \mathcal{D}_{\mathbb{D}(0, D_n)}(0, b_n)$$

- We must show $\mathbb{D}(0, C_n) \subset \mathcal{D}_{\mathbb{D}(0, D_n)}(0, b_n)$

Proof of the Main Proposition

Before this we show that $\mathcal{D}_{\mathbb{D}}(0, d_n) = \mathbb{D}(0, D_n)$ (or equivalently $\mathcal{D}_{\mathbb{D}}(0, c_n) = \mathbb{D}(0, C_n)$). By definition

$$\mathcal{D}_{\mathbb{D}}(0, d_n) = \{z \in \mathbb{C} \mid \rho_{\mathbb{D}}(0, z) \leq d_n\}$$

$$\mathbb{D}(0, D_n) = \{z \in \mathbb{C} \mid d(0, z) \leq D_n\}$$

where $\rho_{\mathbb{D}}(0, z) = \ln \frac{1+|z|}{1-|z|}$. Finally

$$|z| = D_n = \frac{\exp(d_n) + 1}{\exp(d_n) - 1} \leftrightarrow d_n = \ln \frac{1 + |z|}{1 - |z|} = \rho_{\mathbb{D}}(0, z)$$

Remember we must show $\mathbb{D}(0, C_n) \subset \mathcal{D}_{\mathbb{D}(0, D_n)}(0, b_n)$

Proof of the Main Proposition

Remember (again!) we must show $\mathbb{D}(0, C_n) \subset \mathcal{D}_{\mathbb{D}(0,D_n)}(0, b_n)$
Let $h_2 : \mathbb{C} \to \mathbb{C}$ the Möbius transformation given by $h_2(v) = \frac{v}{D_n}$.
Then

- $$h_2\left(\mathbb{D}(0, C_n)\right) = \mathbb{D}(0, \frac{C_n}{D_n})$$
- $$h_2\left(\mathcal{D}_{\mathbb{D}(0,D_n)}(0, b_n)\right) = \mathcal{D}_{\mathbb{D}}(0, b_n) = \mathbb{D}(0, B_n)$$
- Since $C_n < B_n D_n$ we conclude that $\mathbb{D}(0, \frac{C_n}{D_n}) \subset \mathbb{D}(0, B_n)$, as desired.

Theorem A (Existence of Absorbing domains) Let U be a hyperbolic domain in $\mathbb{C}$ and let $F : U \to U$ be a holomorphic map such that $F^n(z) \to \infty$ as $n \to \infty$ for $z \in U$. Then there exists a domain $W \subset U$, such that:

(a) $\overline{W} \subset U$,

(b) $F(\overline{W}) = \overline{F(W)} \subset W$,

(c) $\bigcap_{n=1}^{\infty} F^n(\overline{W}) = \emptyset$,

(d) W is absorbing in U for F.

Moreover, for every point $z \in U$ and every sequence of positive numbers r_n, $n \geq 0$ with $\lim_{n\to\infty} r_n = \infty$, the domain W can be chosen such that

$$W \subset \bigcup_{n=0}^{\infty} \mathcal{D}_U(F^n(z), r_n).$$

Proof of Theorem A

As before, the proof this theorem splits in many STEPS.

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STEP 1: Choosing W

- Let $z \in U$ and let $\{r_n\}_{n\geq 0}$ a sequence of positive numbers such that $\lim_{n\to\infty} r_n = \infty$. Fix $v_0 \in \varphi(V)$ and let $z_0 = \pi\varphi^{-1}(v_0)$.
- It can be proven (hyperbolic metric) that there exist $c > 0$ and $r > 1$ such that

$$\rho_U(w) > \frac{c}{|w| \ log|w|} \quad \text{if} \quad w \in U, \ |w| > r.$$

- Since $F^n(z_0) \to \infty$ as $n \to \infty$, we may assume that v_0 is such that for z_0 we have $|F^n(z_0) \geq r|$ for all $n \geq 0$.
- Define

$$a_n = \min\left\{ \frac{r_n}{2} \ , \ \frac{c}{4} \inf_{k\geq n} \ln \ln |F^k(z_0)| \right\}$$

Clearly $\{a_n\} \to \infty$ as $n \to \infty$.

Proof of Theorem A

STEP 1: Choosing W

- Take $n_0 \in \mathbb{N}$ such that $r_n > 2\rho_U(z, z_0)$ for all $n \geq n_0$.
- Let $A \in \Omega$ as in Main Proposition with $w = T^{n_0}(v_0)$ and $b_n = a_{n+n_0}$. That is

$$A = \bigcup_{n=m}^{\infty} \mathcal{D}_{\mathbb{H}}(T^n(T^{n_0}(v_0)), c_n).$$

- Finally $W = \pi\varphi^{-1}(A)$.

Proof of Theorem A

- We have the following diagram

$$
\begin{array}{ccccccc}
A & \subset & \varphi(V) & \subset & \Omega & \xrightarrow{T} & \Omega \\
\downarrow \varphi^{-1} & & \downarrow \varphi^{-1} & & \uparrow \varphi & & \uparrow \varphi \\
\varphi^{1}(A) & \subset & V & \subset & \mathbb{H} & \xrightarrow{g} & \mathbb{H} \\
\downarrow \pi & & \downarrow \pi & & \downarrow \pi & & \downarrow \pi \\
W & \subset & \pi(V) & \subset & U & \xrightarrow{F} & U
\end{array}
$$

- We also have

$$A \subset \bigcup_{n=m}^{\infty} \mathcal{D}_{\varphi(V)}(T^{n}(T^{n_0}(v_0)), b_n) = \bigcup_{n=m+n_0}^{\infty} \mathcal{D}_{\varphi(V)}(T^{n}(v_0), a_n) \subset \bigcup_{n=n_0}^{\infty} \mathcal{D}_{\varphi(V)}(T^{n}(v_0), a_n).$$

Proof of Theorem A

STEP 2: Proving $\overline{F^j(W)} \subset F^j(\overline{W})$ for all $j \geq 0$ Again, this would take a little longer...

Notice that we want to see Statement (b)

$$F(\overline{W}) = \overline{F(W)} \subset W$$

$$\bigcap_{n=1}^{\infty} F^n(\overline{W}) = \emptyset,$$

The inclusion $F(\overline{W}) \subset \overline{F(W)}$ follows from continuity of F in W. While the converse does not (points which are not on the boundary of any of the discs...if you wish)

Proof of Theorem A

- Take $j \geq 1$ ($j = 0$ is trivial) and $u \in \overline{F^j(W)}$. We want to conclude that $u \in F^j(\overline{W})$.
- Let $u_k \in F^j(W)$ such that $u_k \to u$ as $k \to \infty$. By definition $u_k = F^j(w_k)$ where $w_k \in W$.
- Since $W = \pi\varphi^{-1}(A)$ there must exist $v_k \in A$ such that $w_k = \pi\varphi^{-1}(v_k)$.
- Since $A \subset \bigcup_{n=n_0}^{\infty} \mathcal{D}_{\varphi(V)}(T^n(v_0), a_n)$, for each k, there exists $n_k \geq n_0$ such that

$$v_k \in \mathcal{D}_{\varphi(V)}(T^{n_k}(v_0), a_{n_k})$$

$$T^j(v_k) \in \mathcal{D}_{\varphi(V)}(T^{n_k+j}(v_0), a_{n_k})$$

- We want to project downstairs: $\pi\varphi^{-1} : \varphi(V) \to U$ (holomorphic) and apply Schwartz-Pick Lemma (hyperbolic distance is lower if the set is bigger).

Proof of Theorem A

- $$a_{n_k} \geq \rho_{\varphi(V)}(T^{n_k}(v_0), v_k) \geq \rho_U(\pi\varphi^{-1}(T^{n_k}(v_0)), \pi\varphi^{-1}(v_k)) = \rho_U(F^{n_k}(z_0), w_k)$$
- Then
$$w_k \in \mathcal{D}_U(F^{n_k}(z_0), a_{n_k})$$
$$u_k \in \mathcal{D}_U(F^{n_k+j}(v_0), a_{n_k})$$
- As before the key point here is to prove that because $F^{n_k}(z_0)$ tends to infinity as n tends to infinity but the sequence u_k is bounded, we must have $n_k \equiv \hat{n}$ for all $k \geq k_0$.
- We argue by contradiction: Assume n_k tends to infinity.

Proof of Theorem A

- Consider $\gamma_k : [0,1] \to U$ with $\gamma_k(0) = F^{n_k+j}(z_0)$ and $\gamma_k(1) = u_k$. We may assume that

$$\int_{\gamma_k} \rho(\eta)|d\eta| < 2\rho_U\left(F^{n_k+j}(z_0), u_{n_k}\right).$$

- By construction we know that $a_n \leq \frac{c}{4} \inf_{k \geq n} \ln \ln |F^k(z_0)|$.
- Let

$$t_k = \sup\{t \in [0,1] : |\gamma_k(t')| \geq r, \text{ for all } 0 < t' < t\ .$$

- The following is a chain of inequalities...

Proof of Theorem A

$$\tfrac{c}{4}\log\log|F^{n_k+j}(z_0)| \geq a_{n_k} > \varrho_U(F^{n_k+j}(z_0), u_k) >$$

$$\tfrac{1}{2}\int_{\gamma_k}\varrho_U(\xi)|d\xi| \geq \frac{1}{2}\int_{\gamma_k([0,t_k])}\varrho_U(\xi)|d\xi| \geq$$

$$\geq \tfrac{c}{2}\int_{\gamma_k([0,t_k])}\frac{|d\xi|}{|\xi|\ln|\xi|} = \frac{c}{2}\int_0^{t_k}\frac{|z'(t)|dt}{|z(t)|\ln|z(t)|}$$

where the later equality follows from the change of variables
$\xi = z(t),\ d\xi = z'(t)dt,\ |d\xi| = |z'(t)|dt.$

Proof of Theorem A

$$\frac{c}{2}\int_0^{t_k} \frac{|z'(t)|dt}{|z(t)|\ln|z(t)|} \geq \frac{c}{2}\int_0^{t_k} \frac{|z'(t)|\cos\alpha(t)dt}{|z(t)\ln|z(t)|} = \frac{c}{2}\int_{|\gamma_k(0)|}^{|\gamma_k(t_k)|} \frac{ds}{s\ln s}$$

where the later equality follows from the change of variables $|z(t)| = s,\ \left(\frac{d}{dt}|z(t)|\right) dt = ds,\ |z'(t)|\cos\alpha(t)dt = ds.$

And finally

$$\frac{c}{2}\int_{|\gamma_k(0)|}^{|\gamma_k(t_k)|} \frac{ds}{s\ln s} \geq \frac{c}{2}\int_{|\gamma_k(t_k)|}^{|\gamma_k(0)|} \frac{ds}{s\ln s} = \frac{c}{2}\left(\ln\ln|F^{n_k+j}(z_0)| - \ln\ln|\gamma(t_k)|\right)$$

And consequently:

$$\frac{c}{4}\log\log|F^{n_k+j}(z_0)| \geq \frac{c}{2}\left(\ln\ln|F^{n_k+j}(z_0)| - \ln\ln|\gamma(t_k)|\right)$$

Proof of Theorem A

By reordering the previous inequality we get

$$|\gamma_k(t_k)| > \exp\left(\sqrt{\log|F^{n_k+j}(z_0)|}\right).$$

If n_k tends to infinity as k tends to infinity then $|\gamma_k(t_k)|$ also tends to infinity and consequently $|\gamma_k(t_k)| > r$ which implies $t_k = 1$ and

$$|u_k| > \exp\left(\sqrt{\log|F^{n_k+j}(z_0)|}\right).$$

Since $|u_k|$ is bounded for all k we conclude that (taking a subsequence) $n_k \equiv \hat{n} \equiv n$ for every k. So,

$$v_k \in \mathcal{D}_{\varphi(V)}(T^n(v_0), a_n),\; w_k \in \mathcal{D}_U(F^n(z_0), a_n),\; u_k \in \mathcal{D}_U(F^{n+j}(z_0), a_n).$$

From the Main Proposition we have $v_k \to v \in \overline{A}$.

Proof of Theorem A

- $v_k \to v \in \overline{A}$. Moreover $T^j(v) \in A,\ j \geq 1$.
- Since $\overline{A} \subset \varphi(V)$ the continuity of π gives $w_k \to w = \pi\varphi^{-1}(v) \in \overline{W}$.
- Now we take $F^j(w) = u$. So $u \in F(\overline{W})$, and

$$\overline{F^j(W)} \subset F^j(\overline{W})$$

as desired.

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as desired.

Since points $u \in \overline{F^j(W)}$ are such that $u \in \overline{\mathcal{D}_U(F^{n+j}(z_0), a_n)}$ for some $n \geq n_0$, we have

$$\overline{F^j(W)} \subset \bigcup_{n=n_0}^{\infty} \overline{\mathcal{D}_U(F^{n+j}(z_0), a_n)} \subset U.$$

$$\overline{F^j(W)} \subset \mathbb{C} \setminus \mathbb{D}\left(0, e^{\sqrt{\log \inf_{k \geq j} |F^k(z_0)|}}\right).$$

$$\overline{F^j(W)} \subset W \qquad \text{for } j \geq 1 \quad (u = \pi(\varphi^{-1}(T^j(v))) \in W).$$

Proof of Theorem A

Now we prove the statements of the Theorem, one by one.

Statement (a): $\overline{W} \subset U$.

This follows from

$$\overline{F^j(W)} \subset \bigcup_{n=n_0}^{\infty} \overline{\mathcal{D}_U(F^{n+j}(z_0), a_n)} \subset U, \; j = 0.$$

Statement (b): $F(\overline{W}) = \overline{F(W)} \subset W$.

This follows from

$$F(\overline{W}) \subset \overline{F(W)} \quad \text{(continuity)}$$
$$\overline{F(W)} \subset F(\overline{W}) \subset W \quad \text{(previous arguments)}.$$

Proof of Theorem A

Statement (c): $\bigcap_{n=1}^{\infty} F^n(\overline{W}) = \emptyset$.

This follows from

$$\overline{F^j(W)} \subset \mathbb{C} \setminus \mathbb{D}\left(0, e^{\sqrt{\log \inf_{k\geq j} |F^k(z_0)|}}\right)$$

by taking $j \to \infty$.

Statement (d): W is absorbing.

- Take $K \in U$, and $u \in K$.
- Let $w \in \mathbb{H}$ be such that $\pi(w) = u$, and let N be an open neighborhood of $w \in \mathbb{H}$.
- Then (continuity of π) $\pi(N(w))$ is an open neighborhood of u in U.
- By compactness of K we can choose $u_1, \ldots u_k \in K$ such that

$$K \subset \bigcup_{j=1}^{k} \pi(N(w_j)).$$

Proof of Theorem A

- Going upstairs we have that

$$L = \bigcup_{j=1}^{k} \varphi(\overline{N(w_j)})$$

is a compact set in Ω.

- So, because the Main Proposition, there is $n > 0$ such that $T^n(L) \in A$.
- Finally

$$\bigcup_{j=1}^{k} g^n(N(w_j)) \subset \varphi^{-1}\left(\bigcup_{j=1}^{k} T^n(\varphi(N(w_j)))\right) = \varphi^{-1}\left(\bigcup_{j=1}^{k} \varphi(g^n(N(w_j)))\right) \subset \varphi^{-1}(A),$$

$$F^n(K) \subset \bigcup_{j=1}^{k} F^n(\pi(N(w_j))) = \bigcup_{j=1}^{k} \pi(g^n(N(w_j))) = \pi\left(\bigcup_{j=1}^{k} g^n(N(w_j))\right) \subset \pi\varphi^{-1}(A) = W,$$