On almost specification and average shadowing properties

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Introduction

Theories of shadowing and specification, originating with the works of Anosov and Bowen have been developing parallel with the theory of hyperbolic systems. In some crude sense, one may say that these notions are similar. The common goal is to find a true trajectory near an approximate one, but they differ in understanding what constitutes the approximate trajectory. In shadowing one traces a pseudo-orbit, while in specification arbitrarily assembled finite pieces of orbits are supposed to be followed by a true orbit.

Introduction

A template definition for any generalization of shadowing (or specification) might be: every approximate orbit can be traced by a true one. Moreover, given a quantitative methods of measuring how well an approximate orbit resembles a true trajectory, and how close it is traced by an orbit of some point, we restate our template definition as follows: for every ε there is a δ such that every δ-approximate orbit can be traced with the error not greater than ε. As we will see that template was a base for the subsequent generalizations of both notions: almost specification property, average shadowing property and asymptotic average shadowing property.

Introduction

In 1980s Blank introduced the notion of average pseudo-orbits and he proved that for a certain kind of perturbed hyperbolic systems have the average shadowing property (see [Blank1, Blank2]). Average pseudo-orbits arise naturally in the realizations of independent Gaussian random perturbations with zero mean and in the investigations of the most probable orbits of the dynamical system with general Markov perturbations, etc. (see, [Blank3, p. 20]). It is proved in [Blank1, Theorem 4] that if Λ is a basic set of a diffeomorphism f satisfying Axiom A, then $f|_\Lambda$ has the average shadowing property.

Introduction

The notion gained considerable attention of researchers see [Blank1, Blank2, Blank3, Blank4, Niu, Sakai2, Sakai3, Sakai1, Zhang]. In [Sakai2] Sakai analyzed the dynamics of diffeomorphisms satisfying the average-shadowing property on a two-dimensional closed manifold. Then the same author compared various shadowing properties of positively expansive maps in [Sakai3, Sakai1]. Note that the results of [Sakai1] were generalized and completed in [KwOp]. In [Niu] Niu proved that if f has the average-shadowing property and the minimal points of f are dense in X, then f is weakly mixing and totally strongly ergodic.

Introduction

The next property we are going to consider is *asymptotic average shadowing* introduced by Gu in [Gu1]. Gu followed the same scheme as Blank, but with limit shadowing instead of shadowing as the starting point for generalization. The asymptotic average shadowing property was examined, inter alia, in [KuOp1, KuOp2]. It was proved that there is a large class of systems with asymptotic average shadowing property, including all mixing maps of the unit interval and their Denjoy extensions.

Introduction

More recently, Climenhaga and Thompson ([CT, Thompson]), inspired by the work of Pfister and Sullivan, examined some properties of systems with the almost specification property, which it turns generalizes the notion of specification. As all beta shifts have the almost specification property their results apply to those important symbolic systems.

We believe that techniques and notions described above deserve deeper study and the results scattered through the literature should be put into a unified framework. Therefore our main goal is to explore the general properties of systems possessing generalized shadowing and/or specification.

Specification, shadowing and their generalizations

- a dynamical system $f \colon X \mapsto X$.
- an orbit $x, f(x), f^2(x), \ldots$

Template definition

Every [approximate] orbit can be traced by a [true one].

Given a quantitative methods of measuring how well an approximate orbit resembles a true trajectory, and how close it is traced by an orbit of some point, we may state:

Template definition

For every $\varepsilon > 0$ there is a $\delta > 0$ such that every δ-approximate orbit can be traced with the error not greater than ε (ε-tracing).

Specification

Definition

We say that f has the *periodic specification property* if, for any $\varepsilon > 0$, there is an integer $N_\varepsilon > 0$ such that for any integer $s \geq 2$, any set $\{y_1, \dots, y_s\}$ of s points of X, and any sequence $0 = j_1 \leq k_1 < j_2 \leq k_2 < \dots < j_s \leq k_s$ of $2s$ integers with $j_{l+1} - k_l \geq N_\varepsilon$ for $l = 1, \dots, s-1$, there is a point $x \in X$ such that, for each $1 \leq m \leq s$ and any i with $j_m \leq i \leq k_m$, the following conditions hold:

$$d(f^i(x), f^i(y_m)) < \varepsilon, \tag{1}$$

$$f^n(x) = x, \quad \text{where} \quad n = N_\varepsilon + k_s. \tag{2}$$

Shadowing

Definition

A sequence of points $\{x_n\}_{n=0}^{\infty}$ is called a *δ-pseudo-orbit* if $d(f(x_n), x_{n+1}) < \delta$ for $n = 0, 1, 2, \ldots$.

Definition

We say that a δ-pseudo-orbit $\{x_n\}_{n=0}^{\infty}$ is *ε-traced* by a point $y \in X$ when $d(f^n(y), x_n) < \varepsilon$ for $n = 0, 1, \ldots$.

Definition

We say that f has the *pseudo-orbit tracing property* (*shadowing* for short) if for every $\varepsilon > 0$ there exists $\delta > 0$ such that every δ-pseudo-orbit for f is ε-traced by some point in X.

Average shadowing property

Definition

A sequence $\{x_n\}_{n=0}^{\infty}$ is a δ-average-pseudo-orbit of f, if there is an integer $N > 0$ such that:

$$\frac{1}{n}\sum_{i=0}^{n-1} d(f(x_{i+k}), x_{i+k+1}) < \delta \quad \text{for all } n \geq N,\ k \geq 0.$$

Definition

The sequence $\{x_n\}_{n=0}^{\infty}$ is ε*-shadowed on average by a point* $y \in X$ if

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} d(f^i(y), x_i) < \varepsilon.$$

Average shadowing property (cont.)

Definition

We say that f has the *average shadowing property* if for every $\varepsilon > 0$ there is $\delta > 0$ such that every δ-average-pseudo-orbit of f is ε*-shadowed on average* by some point in X.

Asymptotic average shadowing property

Definition

The sequence $\{x_n\}_{n=0}^{\infty} \subset X$ is an *asymptotic average pseudo-orbit of f* if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} d(f(x_i), x_{i+1}) = 0.$$

Definition

We say that the sequence $\{x_n\}_{n=0}^{\infty} \subset X$ is *asymptotically shadowed in average* by the point $y \in X$ if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} d(f^i(y), x_i) = 0.$$

Asymptotic average shadowing property (cont.)

Definition

The map f has the *asymptotic average shadowing property* provided that every asymptotic average pseudo-orbit of f is asymptotically shadowed in average by some point in X.

Almost specification property

Definition

Let $\varepsilon_0 > 0$. A function $g\colon \mathbb{N} \times (0, \varepsilon_0) \mapsto \mathbb{N}$ is called a *mistake function* if for all $\varepsilon \in (0, \varepsilon_0)$ and all $n \in N$, we have $g(n, \varepsilon) \leq g(n+1, \varepsilon)$ and

$$\lim_{n\to\infty} \frac{g(n, \varepsilon)}{n} = 0.$$

Given a mistake function g, if $\varepsilon > \varepsilon_0$, then we define $g(n, \varepsilon) := g(n, \varepsilon_0)$.

Almost specification property

Definition

For a finite set of indices $\Lambda \subset \{0, 1, \ldots, n-1\}$, we define the *Bowen distance between* $x, y \in X$ *along* Λ by

$$d_\Lambda(x, y) = \max\left\{d(f^j(x), f^j(y)) : j \in \Lambda\right\}$$

and the *Bowen ball (of radius* ε *centered at* $x \in X$*) along* Λ by

$$B_\Lambda(x, \varepsilon) = \{y \in X : d_\Lambda(x, y) < \varepsilon\}.$$

Almost specification property

Definition

Let g be a mistake function and $\varepsilon > 0$. For n sufficiently large so that $g(n, \varepsilon) < n$, we define the set of $(g; n, \varepsilon)$ *almost full* subsets of $\{0, \ldots, n-1\}$ to be the family $I(g, n, \varepsilon)$ consisting of subsets of $\{0, 1, \ldots, n-1\}$ with at least $n - g(n, \varepsilon)$ elements, that is,

$$I(g, n, \varepsilon) := \{\Lambda \subset \{0, 1, \ldots, n-1\} : |\Lambda| \geq n - g(n, \varepsilon)\}.$$

Definition

For $x \in X$ a $(g; n, \varepsilon)$-*Bowen ball of radius* ε*, center* x*, and length* n is given by

$$\begin{aligned} B_n(g; x, \varepsilon) &:= \{y \in X : y \in B_\Lambda(x, \varepsilon) \text{ for some } \Lambda \in I(g; n, \varepsilon)\} \\ &= \bigcup_{\Lambda \in I(g; n, \varepsilon)} B_\Lambda(x, \varepsilon). \end{aligned}$$

Almost specification property

Definition

A continuous map $f \colon X \mapsto X$ satisfies the *almost specification property* if there exists a mistake function g such that for any $k \geq 1$ and any $\varepsilon_1, \ldots, \varepsilon_k > 0$, there exist integers $N(g, \varepsilon_1), \ldots, N(g, \varepsilon_k)$ such that for any points $x_1, \ldots, x_k$ in X and integers $n_1 \geq N(g, \varepsilon_i), \ldots, n_k \geq N(g, \varepsilon_k)$, setting $n_0 = 0$, and

$$l_j = \sum_{t=0}^{j-1} n_s, \text{ for } j = 1, \ldots, k$$

we can find a point $z \in X$ such that for every $j = 1, \ldots, k$ we have

$$f^{l_j}(z) \in B_{n_j}(g; x, \varepsilon_j).$$

In other words, the appropriate part of the orbit of z ε_j-traces with at most $g(\varepsilon_j, n_j)$ mistakes the orbit of x_j.

In general

Theorem

Let (X, f) is a surjective compact dynamical system.

- *f has the specification property*

$\Longrightarrow$ *f has the almost specification property*

$\overset{\text{(KKO)}}{\Longrightarrow}$ *f has the asymptotic average shadowing property*

$\overset{\text{(KKO)}}{\Longrightarrow}$ *f has the average shadowing property.*

Under shadowing

Theorem ([KKO])

If (X, f) is a compact dyn. sys. with the shadowing property, then the following conditions are equivalent:

1. *f is totally transitive,*
2. *f is topologically weakly mixing,*
3. *f is topologically mixing,*
4. *f is surjective and has the specification property,*
5. *f is surjective and has the almost specification property,*
6. *f is surjective and has the average shadowing property.*
7. *f is surj. and has the asymptotic aver. shadowing property,*

Moreover, if f is in addition c-expansive, then any of the above conditions is equivalent to the periodic specification property of f.

Weak mixing

Theorem ([KKO])

Assume that the compact dynamical system (X, f) *has an invariant measure with full support. If* f *has at least one of the following properties*

1. *the almost specification property,*
2. *the asymptotic average shadowing property,*
3. *the average shadowing property,*

then f *is topologically weakly mixing.*

On measure center

Definition

The measure center of a compact dynamical system is the closure of a set-theoretic union of topological supports of all invariant measures.

Theorem ([KKO])

If a compact dynamical system (X, f) *restricted to some subsystem containing its measure center has the almost specification property ((asymptotic) average shadowing), then* f *has the almost specification ((asymptotic) average shadowing) property.*

Some corollaries

Corollary ([KKO])

If a compact dynamical system (X, f) is uniquely ergodic and proximal, then it has the almost specification property.

Corollary ([KKO])

If a compact dynamical system (X, f) is distal and has the almost specification property (or average shadowing), then it is trivial.

Some examples

Remark

It is well known that for a compact dynamical system (X, f) we have

$$\text{mixing} \implies \text{weak mixing} \implies \text{total trans.} \implies \text{transitivity}$$

Corollary ([KKO])

For every implication above there is a compact dynamical system (X, f) which has the almost specification property (or average shadowing) and is a counterexample for the reverse implication.

Some (non-compact) counterexamples

Theorem ([KKO])

In general (in non-compact case) neither average shadowing property implies asymptotic average shadowing property, nor asymptotic average shadowing property implies average shadowing property.

Questions

1. Does average shadowing property imply almost specification property?
2. Does average shadowing property imply asymptotic average shadowing property?
3. Does asymptotic average shadowing property imply almost specification property?
4. Does any of the above properties imply topological mixing provided there is a full invariant measure?

Conjectures

1. Almost specification property and a full invariant measure imply (uniform) positive entropy.
2. Average shadowing property implies average shadowing property on the measure center, and the same holds for the almost specification property and the asymptotic average shadowing property.

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