Hitting Probabilities of the Random Covering Sets

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Random covering problem on the circle
Covering model

- $\{\xi_n\}$ is a sequence of i.i.d. random variables uniformly distributed on the circle $\mathbb{T} := \mathbb{R}/\mathbb{Z}$ ($\xi_n : \Omega \to \mathbb{T}$, $\mathbb{P} \circ \xi_n^{-1} = \mathcal{L}$)
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- Random intervals: \( I_n(\omega) = \xi_n(\omega) + (0, l_n) \)
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- **Random covering set**

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E(\omega) := \{ t \in \mathbb{T} : t \in I_n(\omega) \text{ infinitely often} \} = \limsup_{n \to \infty} I_n(\omega)
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- Another writing as random series

\[
E(\omega) = \{ t \in \mathbb{T} : \sum_{n=1}^{\infty} \chi_{(0,l_n)}(t - \xi_n(\omega)) = +\infty \}
\]
Sizes of covering sets

- the roles of the two measures
  - \( P \): measures the randomness of the initial points of the random intervals
  - \( L \): measures the lengths of the random intervals

Kahane (1985) \( E \) is almost surely dense on \( T \) and is of second category.

Borel-Cantelli Lemma implies almost surely
\[
L(\{E(\omega)\}) = \begin{cases} 0 & \text{if } \sum_{n=1}^{\infty} l_n < \infty \\ 1 & \text{if } \sum_{n=1}^{\infty} l_n = \infty \end{cases}
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Hitting probability of covering set


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**Question:**
Given a sequence \( \{l_n\} \) with \( \sum_{n=1}^{\infty} l_n < +\infty \), under what conditions on measurable set \( G \), we have

\[ \mathbb{P}(E \cap G \neq \emptyset) > 0? \]
Hitting probability of covering set

- It can be shown that

\[ \limsup_{k \to \infty} \frac{\log_2 n_k}{k} = \alpha, \]

where

\[ n_k = \# \left\{ n \in \mathbb{N} : l_n \in [2^{-k+1}, 2^{-k+2}) \right\} \quad (k \geq 2). \]
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- **Condition (C)**: There exists an increasing sequence of positive integers \( \{k_i\} \) such that

\[ \lim_{i \to \infty} \frac{k_{i+1}}{k_i} = 1 \quad \text{and} \quad \lim_{i \to \infty} \frac{\log_2 n_{k_i}}{k_i} = \alpha < 1. \]
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Examples: \( l_n = \frac{a}{n^\gamma}, a > 0, \gamma > 1 \); \( l_n = \frac{1}{\beta n}, \beta > 1 \).
Hitting probability of covering set

**Theorem**

Let $E$ be the random covering set associated with the sequence $\{l_n\}$. If the condition (C) holds, then for every measurable set $G \subset \mathbb{T}$, we have

$$\mathbb{P}(E \cap G \neq \emptyset) = \begin{cases} 0 & \text{if } \text{dim}_P(G) < 1 - \alpha, \\ 1 & \text{if } \text{dim}_P(G) > 1 - \alpha. \end{cases}$$

**Remark**

The conclusion $\text{dim}_P(G) < 1 - \alpha$ implies $\mathbb{P}(E \cap G \neq \emptyset) = 0$ holds even without the condition (C).
Theorem

Let $E$ be the random covering set associated with the sequence $\{l_n\}$ which satisfies the condition (C). If $\dim_P(G') > 1 - \alpha$, then

$$\dim_P(E \cap G) = \dim_P(G) \quad a.s.$$  

and

$$\dim_H(G') - (1 - \alpha) \leq \dim_H(E \cap G) \leq \dim_P(G) - (1 - \alpha) \quad a.s.$$  

In particular, if $\dim_H(G) = \dim_P(G')$, then

$$\dim_H(E \cap G) = \dim_H(G') - (1 - \alpha) \quad a.s.$$
Construction of limsup random fractal subset
Limsup random fractal

- dyadic intervals

\[ \mathcal{D}_k = \left\{ \left[ \frac{i}{2^k}, \frac{i + 1}{2^k} \right] : i \in \mathbb{N} \right\} \]
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- random variables \((n \geq 1, J \in D_k)\)
  \[ Z_k(J) = \begin{cases} 
  1 & \text{if } J \text{ is picked,} \\
  0 & \text{otherwise.}
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- \(k\)-th level
  \[ A(k) = \bigcup_{J \in \mathcal{D}_k, Z_k(J) = 1} J^o \]
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- limsup random fractal (see Khoshnevisan, Peres, and Xiao, 2000)
  \[ A = \limsup_{k \to \infty} A(k) \]
Construction of subset

\[ \dim_{p}(G) > 1 - \alpha \implies \mathbb{P}(E \cap G \neq \emptyset) = 1 \]
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\[ \mathcal{I}_k = \{ n \in \mathbb{N} : l_n \in [2^{-k+1}, 2^{-k+2}) \} \]
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\[ Z_k(J) = \begin{cases} 
1 & \text{if } \exists \ n \in \mathcal{T}_k \text{ such that } J \subset I_n = (\xi_n, \xi_n + l_n), \\
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\[ E_* \subset E \]
Thanks for your attention!