Hitting Probabilities of the Random Covering Sets

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Będlewo, 26th April, 2012

Random covering problem on the circle

Covering model

- $\{\xi_n\}$ is a sequence of i.i.d. random variables uniformly distributed on the circle $\mathbb{T} := \mathbb{R}/\mathbb{Z}$ ($\xi_n : \Omega \to \mathbb{T}$, $\mathbb{P} \circ \xi_n^{-1} = \mathcal{L}$)

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- Another writing as random series

$$E(\omega) = \{t \in \mathbb{T} : \sum_{n=1}^{\infty} \chi_{(0,l_n)}(t - \xi_n(\omega)) = +\infty\}$$

Sizes of covering sets

- the roles of the two measures
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- Borel-Cantelli Lemma implies almost surely

$$\mathcal{L}(E(\omega)) = \begin{cases} 0 & \text{if } \sum_{n=1}^{\infty} l_n < \infty \\ 1 & \text{if } \sum_{n=1}^{\infty} l_n = \infty. \end{cases}$$

Hitting probability of covering set

- Fan and Wu (2004), Durand (2010)

$$\dim_{\mathrm{H}}(E) = \alpha := \inf \left\{ s > 0 : \sum_{n=1}^{\infty} l_n^s < \infty \right\}$$

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- Question :
  Given a sequence $\{l_n\}$ with $\sum_{n=1}^{\infty} l_n < +\infty$, under what conditions on measurable set G, we have

$$\mathbb{P}\big(E \cap G \neq \emptyset\big) > 0?$$

Hitting probability of covering set

- It can be shown that

$$\limsup_{k\to\infty} \frac{\log_2 n_k}{k} = \alpha,$$

where

$$n_k = \#\Big\{ n \in \mathbb{N} : l_n \in [2^{-k+1}, 2^{-k+2}) \Big\} \qquad (k \geq 2).$$

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- Condition (C) : There exists an increasing sequence of positive integers $\{k_i\}$ such that

$$\lim_{i\to\infty} \frac{k_{i+1}}{k_i} = 1 \quad \text{and} \quad \lim_{i\to\infty} \frac{\log_2 n_{k_i}}{k_i} = \alpha < 1.$$

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Examples : $l_n = \frac{a}{n^\gamma}, a > 0, \gamma > 1\,;\, l_n = \frac{1}{\beta^n}, \beta > 1.$

Hitting probability of covering set

Theorem

Let E be the random covering set associated with the sequence $\{l_n\}$. If the condition (C) holds, then for every measurable set $G \subset \mathbb{T}$, we have

$$\mathbb{P}(E \cap G \neq \emptyset) = \begin{cases} 0 & \textit{if } \dim_{\mathrm{P}}(G) < 1 - \alpha, \\ 1 & \textit{if } \dim_{\mathrm{P}}(G) > 1 - \alpha. \end{cases}$$

Remark

The conclusion $\dim_{\mathrm{P}}(G) < 1 - \alpha$ implies $\mathbb{P}(E \cap G \neq \emptyset) = 0$ holds even without the condition (C).

Hitting probability of covering set

Theorem

Let E be the random covering set associated with the sequence $\{l_n\}$ which satisfies the condition (C). If $\dim_{\mathrm{P}}(G) > 1-\alpha$, then

$$\dim_{\mathrm{P}}(E \cap G) = \dim_{\mathrm{P}}(G) \qquad a.s.$$

and

$$\dim_{\mathrm{H}}(G) - (1-\alpha) \leq \dim_{\mathrm{H}}(E \cap G) \leq \dim_{\mathrm{P}}(G) - (1-\alpha) \quad a.s.$$

In particular, if $\dim_{\mathrm{H}}(G) = \dim_{\mathrm{P}}(G)$, then

$$\dim_{\mathrm{H}}(E \cap G) = \dim_{\mathrm{H}}(G) - (1-\alpha) \quad a.s.$$

Construction of limsup random fractal subset

Limsup random fractal

- dyadic intervals

$$\mathcal{D}_k = \left\{ [\frac{i}{2^k}, \frac{i+1}{2^k}] : i \in \mathbb{N} \right\}$$

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- limsup random fractal (see Khoshnevisan, Peres, and Xiao, 2000)

$$A = \limsup_{k \to \infty} A(k)$$

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- $E_* \subset E$

Thanks for your attention !