Invariant measures for dissipative systems and generalized Banach limits

Constructions of invariant measures

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Overview of the material.

- History. Theorems of Krylov-Bogoliubov and Birkhoff.
- Context and motivation.
- Basic notions: dynamical system, global attractor.
- Construction of Individual Invariant Measures. Theorem 1.
- Construction of any Invariant Measure. Theorem 2.
- Proof of Theorem 1.
- References.

History

Theorem

*(**Krylov-Bogoliubov, 1937**) In a compact phase space X of a dynamical system $f(t,p)$ there exists an invariant probalility measure.*

Theorem

*(**Birkhoff, 1931**) If in a phase space X there is defined an invariant transitive measure μ with $\mu(X) = 1$ then for any absolutely summable function φ and μ-**almost all** $p \in X$*

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t \varphi(f(s,p))ds = \int_X \varphi(v)d\mu(v).$$

- What if X is **not compact** in Krylov-Bogoliubov?
- What if μ is **not transitive** in Birkhoff?

Context and Motivation

Our motivation comes from considerations of **infinite-dimensional** dynamical systems of mathematical physics, e.g. from **turbulence studies**, where the phase space is, say, a Hilbert space.

We look for the **invariant measures** describing statistical equilibria of the considered system.

The main tool in the construction is the notion of a **generalized Banach limit** used in the definition of time averages. It allows to avoid the "ergodic hypothesis", and get two formulas

$$LIM_{t\to\infty}\frac{1}{t}\int_0^t \varphi(S(s)p)ds = \int_{\mathcal{A}} \varphi(v)d\mu_p(v), \quad \mathcal{A} - \textbf{global attractor}$$

$$LIM_{t\to\infty}\frac{1}{t}\int_0^t \int_H \varphi(S(s)p)d\mu^0(p)ds = \int_{\mathcal{A}} \varphi(v)dm(v).$$

Basic notion: dynamical system

Let us consider a **dissipative**, infinite-dimensional dynamical system:

$$\begin{aligned} \frac{du}{dt} &= F(u) \\ u(0) &= u_0 \in H \quad (H = \textit{the phase space}) \end{aligned}$$

2D Navier-Stokes is a dissipative dynamical system.

H is a Banach or a Hilbert space (the phace space is infinite dimensional).

We assume that the solutions are unique and global in time.

Solution: $u(t) = S(t)u_0$, $t \geq 0$, where $\{S(t)\}_{t\geq 0}$ is a **semigroup**, $S(t) : H \to H$.

In general we consider $\{S(t)\}_{t\geq 0}$ acting in an arbitrary **metric space**.

Basic notion: global attractor

For many dissipative dynamical systems there exists a subset $\mathcal{A}$ (global attractor) in the phase space H such that:

- $\mathcal{A}$ is compact in H.
- $\mathcal{A}$ is invariant: $S(t)\mathcal{A} = \mathcal{A}$ for $t \geq 0$.
- $\mathcal{A}$ attracts bounded sets in H: $dist(S(t)B, \mathcal{A}) \to 0$ as $t \to \infty$.

Application to the 2D NS turbulent flows (our claims):

States of statistical equilibria after a long time of evolution of a turbulent flow can be described by dynamics reduced to $\mathcal{A}$, namely, by **invariant measures** (= stationary statistical solutions) of the dynamical system.

Banach Generalized Limit. Invariant Measure

Definition

A Banach generalized limit is any linear functional, denoted $\mathrm{LIM}_{T\to\infty}$, defined on the space of all bounded real-valued functions on $[0,\infty)$ and satisfying
(i) $\mathrm{LIM}_{T\to\infty} g(T) \geq 0$ for nonnegative functions g.
(ii) $\mathrm{LIM}_{T\to\infty} g(T) = \lim_{T\to\infty} g(T)$ if the usual limit $\lim_{T\to\infty} g(T)$ exists.

Definition

A measure μ on H is invariant for $\{S(t)\}_{t\geq 0}$ if and only if for all measurable sets E and $t \geq 0$,

$$\mu(S(t)^{-1}(E)) = \mu(E)$$

Construction of Individual Invariant Measures

Theorem

Let X be a metric space. Assume that there exists a global attractor $\mathcal{A}$ for a semigroup $S(\cdot)$ in X. Let a Banach generalized limit $LIM_{t\to\infty}$ be fixed. Then for **every** *$p \in X$ there* **exists** *an invariant probability measure μ_p on X which is supported on $\mathcal{A}$ and such that for all $\varphi \in C(X)$,*

$$LIM_{t\to\infty}\frac{1}{t}\int_0^t \varphi(S(s)p)ds = \int_{\mathcal{A}} \varphi(v)d\mu_p(v).$$

Basic facts:

- **Every** time averaged measure is invariant.
- **Every** invariant measure is supported on the global attractor.

Example

Let X be a metric space. Assume that there exists a **trivial** global attractor $\mathcal{A} = \{q\}$ for a semigroup $S(\cdot)$ in X. Then for **every** $p \in X$ there **exists** an invariant probability measure $\mu_p = \delta_q$ on X which does not depend on $p \in X$, is supported on $\mathcal{A}$, and for all $\varphi \in C(X)$,

$$\lim_{t \to \infty} \frac{1}{t} \int_0^t \varphi(S(s)p) ds = \int_{\mathcal{A}} \varphi(v) d\delta_q(v) = \varphi(q).$$

Construction of any Invariant Measure

Theorem

Let H be a Hilbert space. Assume that there exists a global attractor $\mathcal{A}$ for a semigroup $S(\cdot)$ in H. Let a Banach generalized limit $LIM_{t\to\infty}$ be fixed. Then for any probability measure μ^0 in H there exists an invariant probability measure m on H which is supported on $\mathcal{A}$ and such that for all bounded functions φ from $C(H)$,

$$LIM_{t\to\infty}\frac{1}{t}\int_0^t\int_H \varphi(S(s)p)d\mu^0(p)ds = \int_{\mathcal{A}}\varphi(v)dm(v).$$

Moreover, **every invariant** *probability measure m can be obtained as such limit.*

- If μ^0 is invariant then $m = \mu^0$.
- Here, for H one can take any complete and separable metric space (Chekroun, Glatt-Holtz, 2011).

Construction of Individual Invariant Measures.

Proof.

(of the first theorem, where X is a uniformly convex Banach space). Let K be a closed convex hull of $\mathcal{A}$, and let $t \to P(S(t)p)$ be the projection on K of the trajectory through p. The function

$$[0,\infty) \ni t \to \varphi(P(S(t)u_0)) \in R$$

is continuous and bounded for $\varphi \in C(H)$.
The trajectory through p approaches the attractor, so

$$|\varphi(S(s)p) - \varphi(P(S(s)p))| \to 0 \quad \text{as} \quad s \to \infty.$$

Now, by a property of generalized Banach limits we conclude that

$$LIM_{t\to\infty}\frac{1}{t}\int_0^t \varphi(S(s)p)ds = LIM_{t\to\infty}\frac{1}{t}\int_0^t \varphi(P(S(s)p))ds.$$

The RHS defines a linear positive functional $L(\varphi)$ on $C(K)$, K - **compact**. By the the Radon-Riesz representation theorem,

$$L(\varphi) = \int_K \varphi(v)d\mu_p(v).$$

□

Construction of Individual Measures

Proof.

(continued)
We have thus

$$LIM_{t\to\infty}\frac{1}{t}\int_0^t \varphi(S(s)p)ds = \int_K \varphi(v)d\mu_p(v).$$

As a time averaged measure, μ_p is invariant, and by invariance, μ_p is supported on $\mathcal{A}$.
We extend the measure μ_p by zero on ouside of $\mathcal{A}$ and use the Tietze extension theorem to extend $L(\varphi)$ to $C(X)$, to get

$$LIM_{t\to\infty}\frac{1}{t}\int_0^t \varphi(S(s)p)ds = \int_{\mathcal{A}} \varphi(v)d\mu_p(v)$$

for all $\varphi \in C(X)$.

□

References

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THANK YOU