Invariant measures for dissipative systems and generalized Banach limits
Constructions of invariant measures

Grzegorz Łukaszewicz

Joint work with José Real (University of Sevilla) and James Robinson (University of Warwick)

April 24, 2012
Overview

Overview of the material.

- History. Theorems of Krylov-Bogoliubov and Birkhoff.
- Context and motivation.
- Basic notions: dynamical system, global attractor.
- Construction of any Invariant Measure. Theorem 2.
- Proof of Theorem 1.
- References.
Theorem

(Krylov-Bogoliubov, 1937) In a **compact** phase space $X$ of a dynamical system $f(t, p)$ there exists an invariant probability measure.

Theorem

(Birkhoff, 1931) If in a phase space $X$ there is defined an invariant **transitive** measure $\mu$ with $\mu(X) = 1$ then for any absolutely summable function $\varphi$ and $\mu$-**almost all** $p \in X$

$$
\lim_{t \to \infty} \frac{1}{t} \int_0^t \varphi(f(s, p)) ds = \int_X \varphi(v) d\mu(v).
$$

- What if $X$ is **not compact** in Krylov-Bogoliubov?
- What if $\mu$ is **not transitive** in Birkhoff?
Our motivation comes from considerations of infinite-dimensional dynamical systems of mathematical physics, e.g. from turbulence studies, where the phase space is, say, a Hilbert space.

We look for the invariant measures describing statistical equilibria of the considered system.

The main tool in the construction is the notion of a generalized Banach limit used in the definition of time averages. It allows to avoid the "ergodic hypothesis", and get two formulas

\[
LIM_{t \to \infty} \frac{1}{t} \int_0^t \varphi(S(s)p) ds = \int_A \varphi(v) d\mu_p(v), \quad A - \text{global attractor}
\]

\[
LIM_{t \to \infty} \frac{1}{t} \int_0^t \int_H \varphi(S(s)p) d\mu^0(p) ds = \int_A \varphi(v) dm(v).
\]
Basic notion: dynamical system

Let us consider a **dissipative**, infinite-dimensional dynamical system:

\[
\frac{du}{dt} = F(u)
\]

\[u(0) = u_0 \in H \quad (H = \text{the phase space})\]

2D Navier-Stokes is a dissipative dynamical system.

\(H\) is a Banach or a Hilbert space (the phase space is infinite dimensional).

We assume that the solutions are unique and global in time.

Solution: \(u(t) = S(t)u_0, \ t \geq 0\), where \(\{S(t)\}_{t \geq 0}\) is a **semigroup**, \(S(t) : H \to H\).

In general we consider \(\{S(t)\}_{t \geq 0}\) acting in an arbitrary **metric space**.
For many dissipative dynamical systems there exists a subset $A$ (global attractor) in the phase space $H$ such that:

- $A$ is compact in $H$.
- $A$ is invariant: $S(t)A = A$ for $t \geq 0$.
- $A$ attracts bounded sets in $H$: $\text{dist}(S(t)B, A) \to 0$ as $t \to \infty$.

**Application** to the 2D NS turbulent flows (our claims):

States of statistical equilibria after a long time of evolution of a turbulent flow can be described by dynamics reduced to $A$, namely, by invariant measures ($= \text{stationary statistical solutions}$) of the dynamical system.
Banach Generalized Limit. Invariant Measure

Definition

A **Banach generalized limit** is any linear functional, denoted $LIM_{T \to \infty}$, defined on the space of all bounded real-valued functions on $[0, \infty)$ and satisfying

(i) $LIM_{T \to \infty} g(T) \geq 0$ for nonnegative functions $g$.
(ii) $LIM_{T \to \infty} g(T) = \lim_{T \to \infty} g(T)$ if the usual limit $\lim_{T \to \infty} g(T)$ exists.

Definition

A measure $\mu$ on $H$ is **invariant** for $\{S(t)\}_{t \geq 0}$ if and only if for all measurable sets $E$ and $t \geq 0$,

$$\mu(S(t)^{-1}(E)) = \mu(E)$$
Construction of Individual Invariant Measures

Theorem

Let $X$ be a metric space. Assume that there exists a global attractor $A$ for a semigroup $S(\cdot)$ in $X$. Let a Banach generalized limit $\lim_{t \to \infty}$ be fixed. Then for every $p \in X$ there exists an invariant probability measure $\mu_p$ on $X$ which is supported on $A$ and such that for all $\varphi \in C(X)$,

$$\lim_{t \to \infty} \frac{1}{t} \int_0^t \varphi(S(s)p) ds = \int_A \varphi(v) d\mu_p(v).$$

Basic facts:

- **Every** time averaged measure is invariant.
- **Every** invariant measure is supported on the global attractor.
Construction of Individual Invariant Measures

Example

Let $X$ be a metric space. Assume that there exists a trivial global attractor $\mathcal{A} = \{q\}$ for a semigroup $S(\cdot)$ in $X$. Then for every $p \in X$ there exists an invariant probability measure $\mu_p = \delta_q$ on $X$ which does not depend on $p \in X$, is supported on $\mathcal{A}$, and for all $\varphi \in C(X)$,

$$\lim_{t \to \infty} \frac{1}{t} \int_0^t \varphi(S(s)p)ds = \int_{\mathcal{A}} \varphi(v)d\delta_q(v) = \varphi(q).$$
Construction of any Invariant Measure

Theorem

Let $H$ be a Hilbert space. Assume that there exists a global attractor $A$ for a semigroup $S(\cdot)$ in $H$. Let a Banach generalized limit $\lim_{t \to \infty}$ be fixed. Then for any probability measure $\mu^0$ in $H$ there exists an invariant probability measure $m$ on $H$ which is supported on $A$ and such that for all bounded functions $\varphi$ from $C(H)$,

$$
\lim_{t \to \infty} \frac{1}{t} \int_0^t \int_H \varphi(S(s)p) d\mu^0(p) ds = \int_A \varphi(v) dm(v).
$$

Moreover, every invariant probability measure $m$ can be obtained as such limit.

- If $\mu^0$ is invariant then $m = \mu^0$.
- Here, for $H$ one can take any complete and separable metric space (Chekroun, Glatt-Holtz, 2011).
Proof.

(of the first theorem, where $X$ is a uniformly convex Banach space). Let $K$ be a closed convex hull of $A$, and let $t \to P(S(t)p)$ be the projection on $K$ of the trajectory through $p$. The function

$$[0, \infty) \ni t \to \varphi(P(S(t)u_0)) \in R$$

is continuous and bounded for $\varphi \in C(H)$. The trajectory through $p$ approaches the attractor, so

$$|\varphi(S(s)p) - \varphi(P(S(s)p))| \to 0 \text{ as } s \to \infty.$$  

Now, by a property of generalized Banach limits we conclude that

$$\lim_{t \to \infty} \frac{1}{t} \int_0^t \varphi(S(s)p) ds = \lim_{t \to \infty} \frac{1}{t} \int_0^t \varphi(P(S(s)p)) ds.$$  

The RHS defines a linear positive functional $L(\varphi)$ on $C(K)$, $K$ - compact. By the the Radon-Riesz representation theorem,

$$L(\varphi) = \int_K \varphi(v) d\mu_p(v).$$
Proof.
(continued)
We have thus
\[
\text{LIM}_{t \to \infty} \frac{1}{t} \int_0^t \varphi(S(s)p) ds = \int_K \varphi(v) d\mu_p(v).
\]
As a time averaged measure, \( \mu_p \) is invariant, and by invariance, \( \mu_p \) is supported on \( A \).
We extend the measure \( \mu_p \) by zero on outside of \( A \) and use the Tietze extension theorem to extend \( L(\varphi) \) to \( C(X) \), to get
\[
\text{LIM}_{t \to \infty} \frac{1}{t} \int_0^t \varphi(S(s)p) ds = \int_A \varphi(v) d\mu_p(v)
\]
for all \( \varphi \in C(X) \).


THANK YOU