Directed path spaces on skeleta of tori

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Directed spaces

Definition

A d-space is a pair $(X, \vec{P}(X))$, where

- X is a topological space,
- $\vec{P}(X)$ is a collection of paths on X called *d-paths*,

such that

- constant paths are d-paths,
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A trace space is

$$\vec{T}(X)_x^y := \vec{P}(X)_x^y / \text{increasing reparametrizations}$$

Examples of directed spaces:

- a full d-space $(X, P(X))$,
- a trivial d-space (X, X),
- a directed Euclidean space

$$\vec{\mathbb{R}}^n := (\mathbb{R}^n, \text{paths with non-decreasing coordinates})$$

- a directed cube $\vec{I}^n \subseteq \mathbb{R}^n$
- a directed torus $\vec{T}^n = \vec{\mathbb{R}}^n/\mathbb{Z}^n$

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- Processes: $A_1, \dots, A_n$
- Resources: $x_1, \dots, x_m$ having arities $c_1, \dots, c_m$
- Every process performs a sequence of operations Px_i (locking the resource x_i) and Vx_i (releasing the resource x_i)
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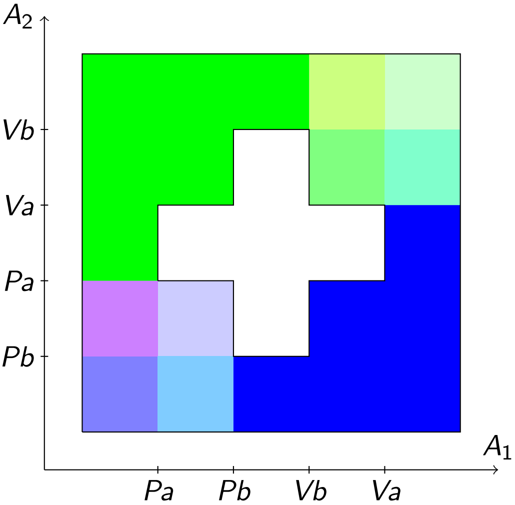
If processes contain no loops, the corresponding directed space X is a directed cube $\vec{I}^n$ with a collection of rectangular areas removed.

Example: Swiss cross

$A_1: \; Pa\, Pb\, Vb\, Va, \qquad A_2: \; Pb\, Pa\, Va\, Vb$



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The corresponding directed space is an $(n-1)$-skeleton of the directed n-torus. Hence we need to calculate

$$H_*(\vec{P}(\vec{T}^n_{(n-1)})^0_0).$$

Universal cover

Universal cover. After passing to universal cover we obtain a decomposition

$$\vec{P}(\vec{T}^n_{(n-1)})^0_0 \cong \coprod_{0 \leq \mathbf{k} \in \vec{\pi}_1(\vec{T}^n)} \vec{P}(\vec{\mathbb{R}}^n_{(n-1)})^{\mathbf{k}}_{\mathbf{0}}$$

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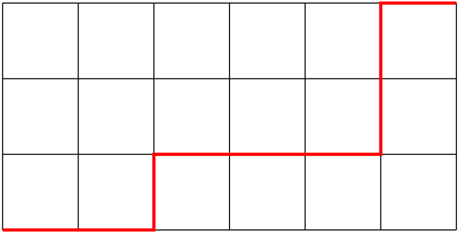
Case $n=2$**.** The space $\vec{P}(\vec{\mathbb{R}}^2_{(1)})^{(k,l)}_{(0,0)}$ is homotopy equivalent to a discrete space with $\binom{k+l}{k}$ components.



Euclidean cubical complexes

Notation: Bold letters denote sequences, e.g. $\mathbf{x} = (x_1, \dots, x_n)$.

$$\mathbf{x} \leq \mathbf{y} \Leftrightarrow x_i \leq y_i$$
$$\mathbf{x} < \mathbf{y} \Leftrightarrow \mathbf{x} \leq \mathbf{y} \wedge \mathbf{x} \neq \mathbf{y}$$
$$\mathbf{x} \ll \mathbf{y} \Leftrightarrow x_i < y_i$$

Definition

An elementary cube is a subset of $\mathbb{R}^n$ which has the form

$$[\mathbf{k}, \mathbf{k}+\mathbf{j}] := \{\mathbf{x} \in \mathbb{R}^n : \ \mathbf{k} \leq \mathbf{x} \leq \mathbf{k}+\mathbf{j}\}$$

where $\mathbf{k} \in \mathbb{Z}^n$ and $\mathbf{j} \in \{0,1\}^n$.

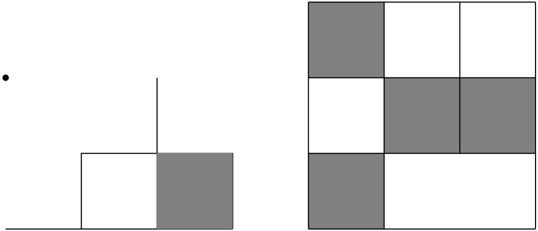
Euclidean cubical complexes (2)

Definition

A subset $K \subseteq \mathbb{R}^n$ is a Euclidean cubical complex if it is a sum of elementary cubes.

Equivalently, a Euclidean cubical complex is a subset of a cubical set $\mathbb{R}^n$.

Examples:



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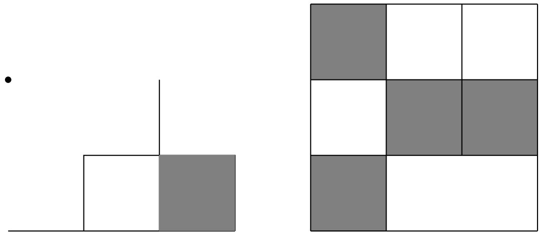

Main problem: $H^*(\vec{P}([\mathbf{0}, \mathbf{k}]_{(n-1)})_{\mathbf{0}}^{\mathbf{k}}) = ?$

The idea of the calculation

- Find a presentation of $\vec{P}([\mathbf{0},\mathbf{k}]_{(n-1)})_{\mathbf{0}}^{\mathbf{k}}$ as a homotopy colimit of spaces $\vec{P}([\mathbf{0},\mathbf{l}]_{(n-1)})_{\mathbf{0}}^{\mathbf{l}}$ for $\mathbf{l} < \mathbf{k}$.

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- Guess the formula for $H_*(\vec{P}([\mathbf{0},\mathbf{k}]_{(n-1)})_{\mathbf{0}}^{\mathbf{k}})$.
- Proceed inductively.

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$$\operatorname{hocolim}_{\mathcal{C}} F = \coprod_{c_0 \to \cdots \to c_n} \Delta^n \times F(c_0)/\sim$$

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- Homotopy equivalent diagrams have homotopy equivalent homotopy colimits (not true for colimits)
- In some cases $\mathrm{hocolim}_{\mathcal{C}} F \simeq \mathrm{colim}_{\mathcal{C}} F$
- Homology of homotopy colimit is calculated by the spectral sequence

$$E^2_{s,t} = H_s(\mathcal{C}; H_t(F(-))) \Rightarrow H_{s+t}(\mathrm{hocolim}_{\mathcal{C}} F).$$

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Extreme case 2:

- if $F(c)$ is contractible for all $c \in \mathcal{C}$, then $\operatorname{hocolim}_{\mathcal{C}} F \simeq N\mathcal{C}$
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Usually, the most effective are decompositions for which both the category and the values are non-trivial (but simplier than the original space X).

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- You can skip objects $J \subseteq I$ such that $\bigcap_{j\in J} U_j = \emptyset$.

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Intuitively: The set U_i contains paths $\alpha : [0, 1] \to K$ such that i-th coordinate of $\alpha(t)$ is less than k_i for $\alpha(t)$ close to $\mathbf{k}$ (process i goes last).

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Proposition

For every $\mathbf{j} \in \mathcal{J}_K$

$$F_{\mathbf{j}}\vec{P}(K)_{\mathbf{0}}^{\mathbf{k}} \simeq \vec{P}(K)_{\mathbf{0}}^{\mathbf{k}-\mathbf{j}} \simeq \vec{P}([\mathbf{0}, \mathbf{k}-\mathbf{j}] \cap K)_{\mathbf{0}}^{\mathbf{k}-\mathbf{j}}.$$

An application - the path space of $\partial\square^n$

For $K = [\mathbf{0}, \mathbf{1}]_{(n-1)}$:

- $\Delta_K = \partial\Delta^{n-1}$
- $\mathcal{J}_K = \{\mathbf{0} < \mathbf{j} < \mathbf{1} \in \{0,1\}^n\}$
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Then

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Let L be a simplicial complex with a set of vertices $\{1,\dots,n\}$. Define a Euclidean cubical complex $K \subseteq [\mathbf{0},\mathbf{1}]$ by

$$[\mathbf{k},\mathbf{l}] \subseteq K \Leftrightarrow \mathbf{l} < \mathbf{1} \vee \{i :\ k_i = 0\} \in L$$

Then $\mathcal{J}_K$ is the category of simplices of L (reversed). Hence

$$\vec{P}(K)_{\mathbf{0}}^{\mathbf{k}} \simeq \operatorname{hocolim}_{\mathbf{j}\in\mathcal{J}_K} F_{\mathbf{j}}\vec{P}(K)_{\mathbf{0}}^{\mathbf{k}} \simeq N\mathcal{J}_K \simeq |L|.$$

Homology of $\vec{P}(K)_{\mathbf{0}}^{\mathbf{k}}$

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A cube sequence in K is a sequence

$$[\mathbf{a}^*] = [\mathbf{0} \ll \mathbf{a}^1 \ll \dots, \ll \mathbf{a}^r \leq \mathbf{k}]$$

such that $[\mathbf{a}^s - \mathbf{1}, \mathbf{a}^s] \not\subseteq K$ for $s = 1, \dots, r$.

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Let $CS_r(K)$ be a set of cube sequences in K having length r. Finally, define

$$A_m(K) = \begin{cases} \mathbb{Z}[CS_{m/(n-2)}(K)] & \text{for } m \equiv 0 \ (n-2) \\ 0 & \text{otherwise} \end{cases}$$

The main theorem

For a cube sequence $[\mathbf{a}^*]$ define $\Phi_K([\mathbf{a}^*]) \in H_{r(n-2)}(\vec{P}(K)_{\mathbf{0}}^{\mathbf{k}})$ as the image of the generator in $H_{r(n-2)}((S^{n-2})^r)$ under the composition

$$(S^{n-2})^r \simeq (\partial\Delta^{n-1})^r \simeq \prod_{s=1}^{r} \vec{P}(K)_{\mathbf{a}^{s-1}}^{\mathbf{a}^s} \xrightarrow{concat} \vec{P}(K)_{\mathbf{0}}^{\mathbf{k}}.$$

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Proof: Induction on **k** using the homotopy decomposition.

Proof - case $[\mathbf{k}-\mathbf{1},\mathbf{k}] \subseteq K$

Assume that $[\mathbf{k}-\mathbf{1},\mathbf{k}] \subseteq K$. Denote $K_{\mathbf{j}} := K \cap [\mathbf{0},\mathbf{k}-\mathbf{j}]$.

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- $\Delta_K = \Delta^{n-1}$
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- there is a spectral sequence

$$E^2_{s,t} = H_s(\mathcal{J}_K; H_t(\vec{P}(K_{\mathbf{j}})_{\mathbf{0}}^{\mathbf{k}-\mathbf{j}})) = H_s(\mathcal{J}_K; A_t(K_{\mathbf{j}})) \Rightarrow H_{s+t}(\vec{P}(K)_{\mathbf{0}}^{\mathbf{k}}).$$

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Hence $E^2_{s,t} = 0$ for $s \neq 0$ and then

$$H_*(\vec{P}(K)_{\mathbf{0}}^{\mathbf{k}}) = E^2_{0,*} = \operatorname{colim}_{\mathcal{J}_K} A_*(K_{\mathbf{j}}) = A_*(K).$$

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Let $L = K \cup [\mathbf{k}-\mathbf{1}, \mathbf{k}]$. We will compare the decomposition diagrams of $\vec{P}(K)_{\mathbf{0}}^{\mathbf{k}}$ and $\vec{P}(L)_{\mathbf{0}}^{\mathbf{k}}$.

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- There is the diagram of cofibrations

$$\begin{array}{ccccc}
\vec{P}(K)_{\mathbf{0}}^{\mathbf{k}} & \xrightarrow{\subseteq} & \vec{P}(L)_{\mathbf{0}}^{\mathbf{k}} & \longrightarrow & \vec{P}(L)_{\mathbf{0}}^{\mathbf{k}}/\vec{P}(K)_{\mathbf{0}}^{\mathbf{k}} \\
\downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq \\
\operatorname{hocolim}_{\mathcal{J}_K} \vec{P}(K_{\mathbf{j}})_{\mathbf{0}}^{\mathbf{k-j}} & \xrightarrow{\subseteq} & \operatorname{hocolim}_{\mathcal{J}_L} \vec{P}(K_{\mathbf{j}})_{\mathbf{0}}^{\mathbf{k-j}} & \longrightarrow & \Sigma^{n-1}\vec{P}(K_1)_{\mathbf{0}}^{\mathbf{k-1}}
\end{array}$$

Finally, we get a transformation of long exact sequences:

$$\begin{array}{ccccc}
A_{*-(n-2)}(K_{\mathbf{1}}) & \longrightarrow & A_*(K) & \longrightarrow & A_*(L) \\
\Sigma^{n-1}\Phi_{K_{\mathbf{1}}} \downarrow \simeq & & \Phi_K \downarrow & & \Phi_L \downarrow \simeq \\
H_{*+1}(\Sigma^{n-1}\vec{P}(K_1)_{\mathbf{0}}^{\mathbf{k-1}}) & \longrightarrow & H_*(\vec{P}(K)) & \longrightarrow & H_*(\vec{P}(L))
\end{array}$$

By Five Lemma the middle homomorphism Φ_K is also an isomorphism. The induction step is complete.

Cohomology ring

By Universal Coefficients Formula

$$H^*(\vec{P}(K)_{\mathbf{0}}^{\mathbf{k}}) = \mathrm{Hom}(H_*(\vec{P}(K)_{\mathbf{0}}^{\mathbf{k}}), \mathbb{Z}) = \mathrm{Hom}(A_*(K), \mathbb{Z}) =: A^*(K)$$

Let $\overline{[\mathbf{a}^*]} \in A^*(K)$ be a generator dual to $[\mathbf{a}^*]$.

Proposition

The algebra $A^*(K) = H^*(\vec{P}(K)_{\mathbf{0}}^{\mathbf{k}})$ *is generated by cube sequences of length 1.*

$$\overline{[\mathbf{a}^*]} \smile \overline{[\mathbf{b}^*]} = \begin{cases} (-1)^{s(\mathbf{a},\mathbf{b})}[\mathbf{a}^* \cup \mathbf{b}^*] & \textit{if } \mathbf{a}^* \cup \mathbf{b}^* \textit{ is a cube sequence} \\ 0 & \textit{otherwise} \end{cases}$$

Thank you for your attention