Discrete Lusternik–Schnirelmann category

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Introduction

The **Lusternik-Schnirelmann category** of a topological space $X$, denoted $\text{cat}(X)$, is the least integer $m$ such that there exists open subsets $\{U_0, U_1, \ldots, U_m\}$ covering $X$ with each $U_i$ contractible in $X$.

Example

$\text{cat}(S^n) = 1$

In general, $\text{cat}(\Sigma X) \leq 1$.

**Theorem** (Lusternik, Schnirelmann 1934) If $M$ is a smooth manifold satisfying additional properties and $f: M \to \mathbb{R}$ is smooth with $m$ critical points, then $\text{cat}(M) + 1 \leq m$. 

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Collapsibility

Recall that if a simplicial complex $K$ contains a pair of simplicies $(\sigma, \tau)$ such that $\sigma$ is a face of $\tau$ and $\sigma$ has no other cofaces, then $K$ is said to **collapse** onto $K - \{\sigma, \tau\}$. The complex $K$ is said to be **collapsible** if there exists a sequence of collapses $K \rightarrow K_1 \rightarrow \ldots \rightarrow \{v\}$. 
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\[ \begin{tikzpicture}
    \coordinate (A) at (0,0);
    \coordinate (B) at (1,0);
    \coordinate (C) at (0.5,0.866);
    \draw (A) -- (B) -- (C) -- cycle;
\end{tikzpicture} \]
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![Diagram of a simplicial complex](image)
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Bing's house with two rooms and the dunce cap are examples of simplicial complexes which are not collapsible, but whose geometric realization is contractible.
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Collapsibility Caveats

• W. Lickorish and J. Martin have shown that collapsibility is not invariant under barycentric subdivision.

• B. Benedetti and F. Lutz have shown the existence of a (collapsible) 3-ball with 8 vertices, but that has a collapse onto the dunce cap.

• Allowing elementary expansions (i.e. working up to simple homotopy type) would eliminate Bing's house and dunce cap as interesting examples (and “too close” to homotopy type).

• If $L \subseteq K$ and $L \downarrow L'$ is a collapse, we would like $\text{dcat}_K(L) = \text{dcat}_K(L')$. 
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**Discrete LS pre-category**

**Definition**

Let $L$ be a (closed) subcomplex of $K$. We say that $L$ has **discrete pre-category** $m$, denoted $\widetilde{\text{dcat}}_K(L) = m$, if $m$ is the least integer such that there exists $m + 1$ closed subcomplexes $\{U_0, U_1, \ldots, U_m\}$ of $K$ each of which is collapsible such that $L \subseteq \bigcup_{i=0}^{m} U_i$.

**Example**

Let $B$ be the collapsible 3-ball of Benedetti and Lutz, $D \subseteq B$ the dunce cap. Since $B$ is collapsible, $\widetilde{\text{dcat}}_B(D) = 0$ while $\widetilde{\text{dcat}}(D) = 1$. 

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Example

Let $K =$

\[
\begin{array}{c}
\end{array}
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\[
\begin{cases}
    \text{a collapsible cover of } K. \quad \text{Hence } \tilde{\text{dcat}}(K) \leq 1.
\end{cases}
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Discrete LS category

If $L \downarrow L'$, then $\tilde{dcat}_K(L) \geq \tilde{dcat}_K(L')$. Can I show that $\tilde{dcat}_K(L) \leq \tilde{dcat}_K(L')$?
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Definition

The **discrete category** of $L$ in $K$ is defined by

$$\text{dcat}_K(L) = \min \{ \text{dcat}_K(L') : L \text{ collapses to } L' \text{ in } K \}.$$
Smooth vs Discrete LS category

Since a collapse corresponds to a deformation retraction, we have that a collapsible complex is contractible. Hence
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In the smooth case, it is well-known that $\text{cat}(X) \leq \text{dim}(X)$. This is not true in the discrete case, even for a 1-dimensional complex.
\[ \text{dcat}(K) \not\leq \dim(K) \]

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**Proposition**

Let \( K \) be a 1-dimensional complex with \( v \) vertices and \( e \) edges. Then \( \left\lceil \frac{e}{v-1} \right\rceil - 1 \leq \text{dcat}(K) \).
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Let $K$ be a 1-dimensional complex with $v$ vertices and $e$ edges. Then $\left\lceil \frac{e}{v-1} \right\rceil - 1 \leq \text{dcat}(K)$.

This is a special case of a more general combinatorial lower bound which we will discuss later.
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Let $K = K_n$, the complete graph on $n$ vertices. Then 

$\left\lceil \frac{e}{v-1} \right\rceil - 1 = \left\lceil \frac{n}{2} \right\rceil - 1 \leq \text{dcat}(K)$. 
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**Proposition**

Let $K$ be a 1-dimensional complex with $v$ vertices and $e$ edges. Then $\left\lceil \frac{e}{v-1} \right\rceil - 1 \leq d\text{cat}(K)$.

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Let $K = K_n$, the complete graph on $n$ vertices. Then $\left\lceil \frac{e}{v-1} \right\rceil - 1 = \left\lceil \frac{n}{2} \right\rceil - 1 \leq d\text{cat}(K)$. A collapsible cover of size $\left\lceil \frac{n}{2} \right\rceil$ can be constructed for $K_n$. Thus $d\text{cat}(K) = \left\lceil \frac{n}{2} \right\rceil - 1 \not\leq 1 = \dim(K)$. 

$d\text{cat}(K) \not\leq \dim(K)$
Discrete Morse theory
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![Diagram of a graph]

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![Graph]

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\[ \begin{array}{c}
0 \\
3 \\
5 \\
2 \\
2 \\
\end{array} \]
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![Graph diagram with nodes and edges labeled with numbers]
Discrete Morse theory

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![Diagram of a graph with vertices labeled 0, 2, 4, and 7, and edges connecting them. The edges and vertices are numbered 3, 5, 7, and 2 respectively.]}
Discrete Morse theory

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![Graph diagram showing a discrete Morse complex with labeled vertices and edges.](image-url)
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![Diagram of a discrete Morse complex]

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![Diagram of a graph with labeled vertices and edges]
Building a complex

The critical values are 0, 1, 3, 6, 7, 8, 9, 14, and 15.
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Theorem

Let $f : K \to \mathbb{R}$ be a discrete Morse function with $m$ critical points. Then $d_{\text{cat}}(K) + 1 \leq m$. 

Proof.

Let $c_n := \min \{ a \in \mathbb{R} : \exists L(a) \subseteq K \text{ s.t. } d_{\text{cat}}(L(a)) \geq n - 1 \}$. If $c_n$ is a regular value of $f$, then there is an $\epsilon > 0$ such that $K(c_n + \epsilon) \downarrow K(c_n - \epsilon)$ in $K$ by Forman. Hence $d_{\text{cat}}(K(c_n + \epsilon)) = d_{\text{cat}}(K(c_n - \epsilon)) \geq n - 1$ so that $c_n > c_n - \epsilon \in \{ a \in \mathbb{R} : \exists L(a) \subseteq K \text{ s.t. } d_{\text{cat}}(L(a)) \geq n - 1 \}$ contradicting the fact that $c_n$ is minimum. Thus each $c_n$ is a critical value of $f$.

A straightforward induction argument now shows that if $c_1 < c_2 < \ldots < c_{d_{\text{cat}}(K) + 1}$ are the critical values, then $K(c_{d_{\text{cat}}(K) + 1})$ contains at least $d_{\text{cat}}(K) + 1$ critical simplices. Thus $d_{\text{cat}}(K) + 1 \leq m$.
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Let $K$ be a simplicial complex of dimension $n$ or $n+1$, and let $c_K^i$ denote the number of simplicies of $K$ of dimension $i$. Let $E(c_K) := c_K^0 + c_K^2 + \ldots + c_K^n$ and $O(c_K) := c_K^1 + c_K^3 + \ldots + c_K^{n+1}$.

Proposition

Let $K$ be a simplicial complex of dimension $n$ with $c_i$ the number of simplicies of $K$ of dimension $i$. If $E(c_K) - 1 \geq O(c_K)$, then $\lceil E(c_K) - 1 \rceil - 1 \leq \text{dcat}(K)$. If $E(c_K) - 1 \leq O(c_K)$, then $\lceil O(c_K) E(c_K) - 1 \rceil - 1 \leq \text{dcat}(K)$.
Let $K$ be a simplicial complex of dimension $n$ or $n+1$, and let $c^K_i$ denote the number of simplicies of $K$ of dimension $i$. Let $E(c^K) := c^K_0 + c^K_2 + \ldots + c^K_n$ and $O(c^K) := c^K_1 + c^K_3 + \ldots + c^K_{n\pm1}$. 

Proposition

Let $K$ be a simplicial complex of dimension $n$ with $c^K_i$ the number of simplicies of $K$ of dimension $i$. If $E(c^K) - 1 \geq O(c^K)$, then $\lceil E(c^K) - 1 \rceil - 1 \leq d_{\text{cat}}(K)$. If $E(c^K) - 1 \leq O(c^K)$, then $\lceil O(c^K) E(c^K) - 1 \rceil - 1 \leq d_{\text{cat}}(K)$. 

Discrete Lusternik–Schnirelmann category

Scoville, Aaronson, Green
Lower bound

Let $K$ be a simplicial complex of dimension $n$ or $n+1$, and let $c^K_i$ denote the number of simplices of $K$ of dimension $i$. Let $E(c^K) := c^K_0 + c^K_2 + \ldots + c^K_n$ and $O(c^K) := c^K_1 + c^K_3 + \ldots + c^K_{n\pm 1}$.

Proposition

Let $K$ be a simplicial complex of dimension $n$ with $c_i$ the number of simplices of $K$ of dimension $i$. If $E(c^K) - 1 \geq O(c^K)$, then

$$\left\lceil \frac{E(c^K) - 1}{O(c^K)} \right\rceil - 1 \leq \text{dcat}(K).$$

If $E(c^K) - 1 \leq O(c^K)$, then

$$\left\lceil \frac{O(c^K)}{E(c^K) - 1} \right\rceil - 1 \leq \text{dcat}(K).$$
Computation

Let $K$ be a simplicial complex.
Computation

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1. Initialize $V := K$ and $U := \emptyset$.
Computation

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1. Initialize $V := K$ and $U := \emptyset$.
2. Let $U = \emptyset$. Pick random top dimensional subcomplex $\Delta$ and add to $U$, remove from $V$.

The set $U$ obtained in the above algorithm is a collapsible cover of $K$ so that $d_{\text{cat}}(K) \leq |U| - 1$. 
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Let $K$ be a simplicial complex.

1. Initialize $V := K$ and $U := \emptyset$.
2. Let $U = \emptyset$. Pick random top dimensional subcomplex $\Delta$ and add to $U$, remove from $V$.
3. Expand along 0-simplicies, then 1-simplicies, 2-simplicies, etc. Add to $U$, remove from $V$.

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