Discrete Lusternik–Schnirelmann category

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Introduction

The **Lusternik-Schnirelmann category** of a topological space X, denoted $\mathrm{cat}(X)$, is the least integer m such that there exists open subsets $\{U_0, U_1, \ldots, U_m\}$ covering X with each U_i contractible in X.

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Theorem

(Lusternik, Schnirelmann 1934) If M is a smooth manifold satisfying additional properties and $f : M \to \mathbb{R}$ is smooth with m critical points, then $\mathrm{cat}(M) + 1 \leq m$.

Collapsibility

Recall that if a simplicial complex K contains a pair of simplicies (σ, τ) such that σ is a face of τ and σ has no other cofaces, then K is said to **collapse** onto $K - \{\sigma, \tau\}$. The complex K is said to be **collapsible** if there exits a sequence of collapses $K \to K_1 \to \ldots \to \{v\}$.

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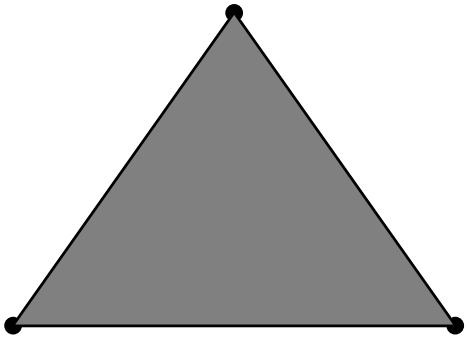

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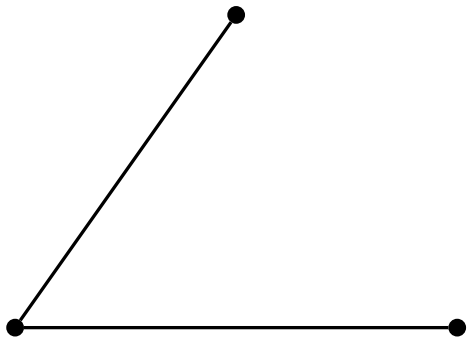

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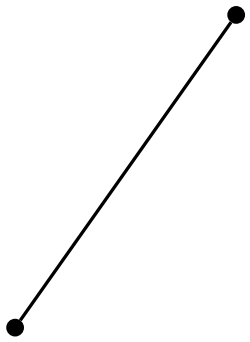

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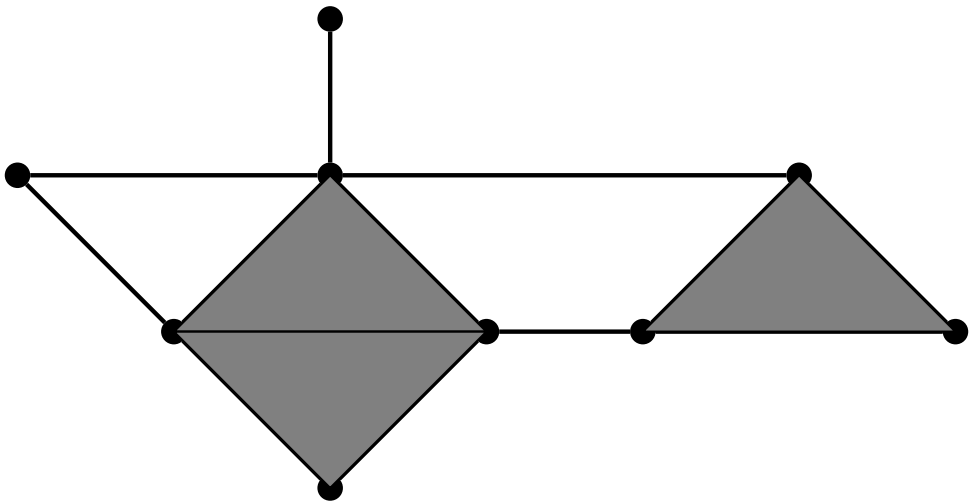

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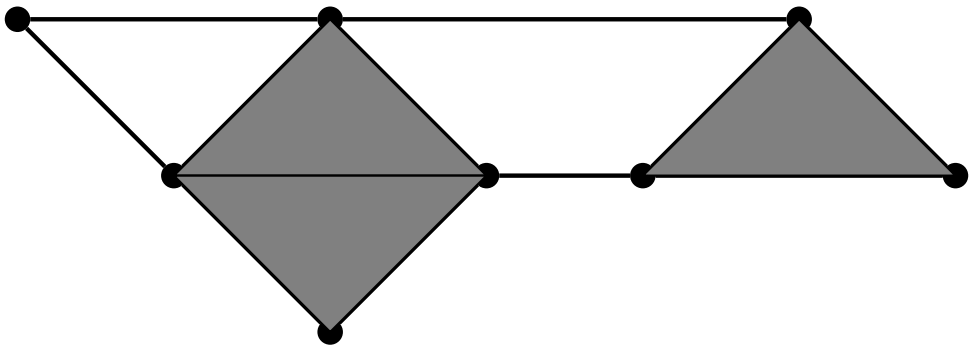

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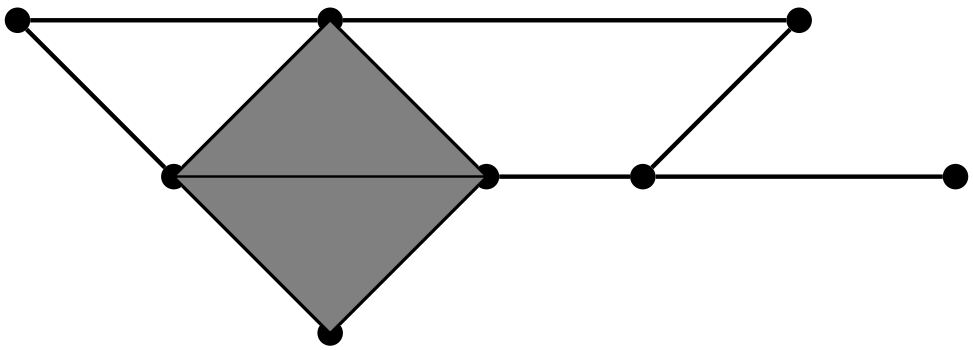

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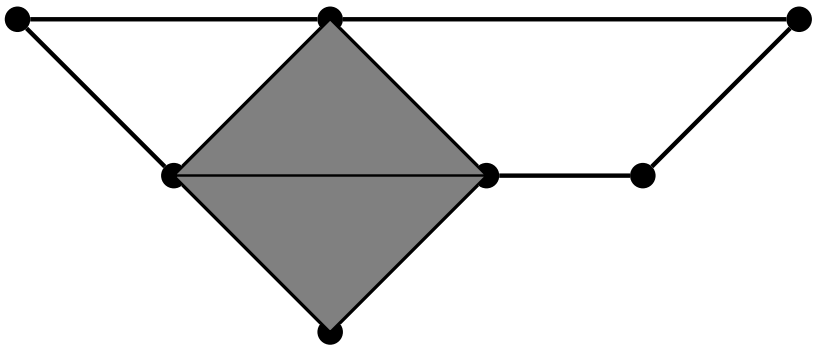

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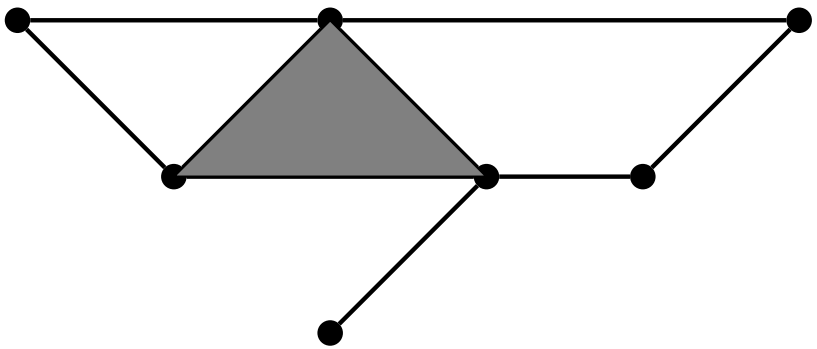

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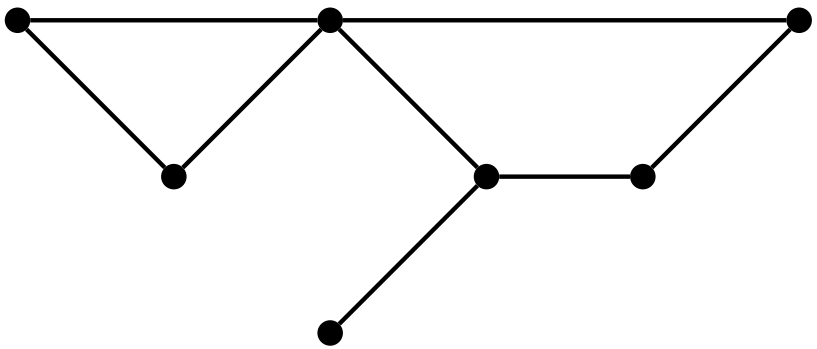

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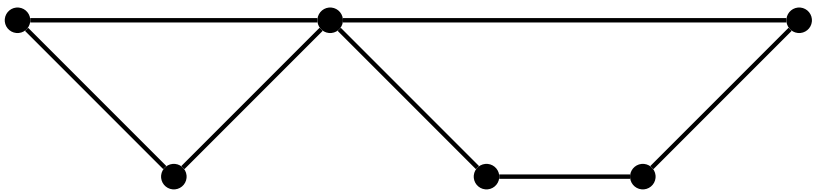

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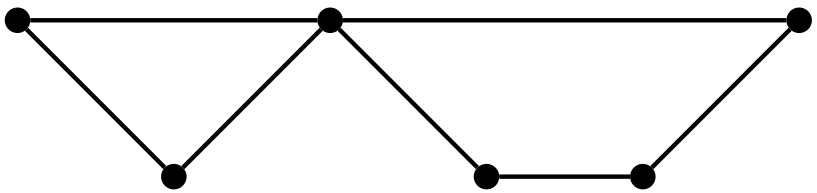

Bing's house with two rooms and the dunce cap are examples of simplicial complexes which are not collpasible, but whose geometric realization is contractible.

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- Allowing elementary expansions (i.e. working up to simple homotopy type) would eliminate Bing's house and dunce cap as interesting examples (and "too close" to homotopy type).
- If $L \subseteq K$ and $L \searrow L'$ is a collapse, we would like $\mathrm{dcat}_K(L) = \mathrm{dcat}_K(L')$.

Discrete LS pre-category

Definition

Let L be a (closed) subcomplex of K. We say that L has **discrete pre-category** m, denoted $\widetilde{\mathrm{dcat}}_K(L) = m$, if m is the least integer such that there exists $m+1$ closed subcomplexes $\{U_0, U_1, \ldots, U_m\}$ of K each of which is collapsible such that $L \subseteq \bigcup_{i=0}^{m} U_i$.

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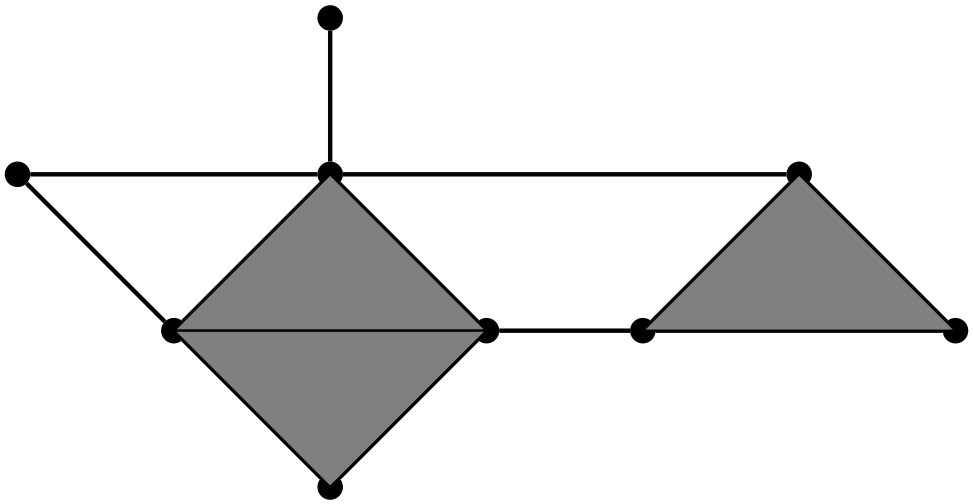
Example

Let B be the collapsible 3-ball of Benedetti and Lutz, $D \subseteq B$ the dunce cap. Since B is collapsible, $\widetilde{\mathrm{dcat}}_B(D) = 0$ while $\widetilde{\mathrm{dcat}}(D) = 1$.

Example

Let $K =$



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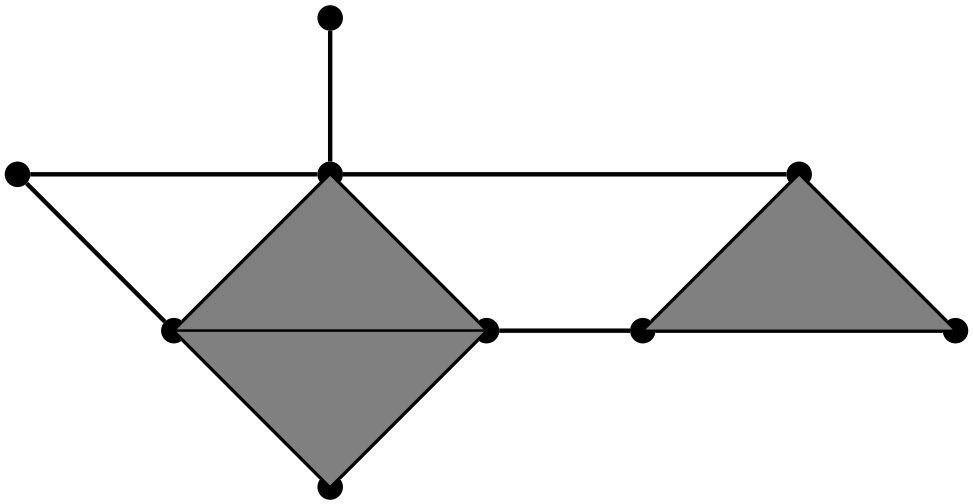
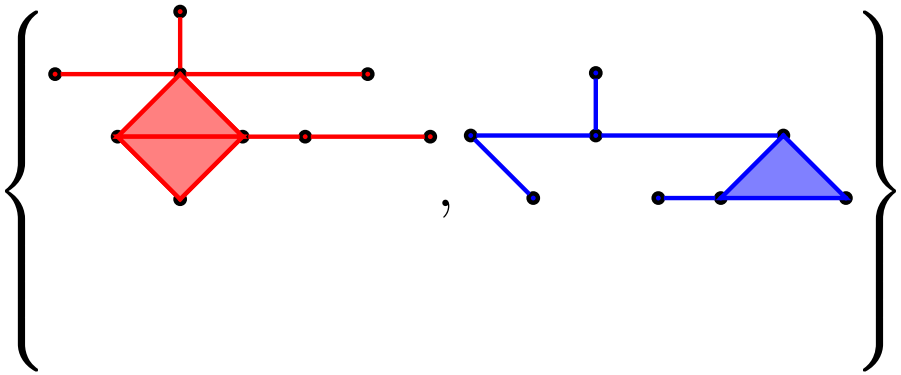


Then



is a collapsible cover of K. Hence $\widetilde{\mathrm{dcat}}(K) \leq 1$.

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If $L \searrow L'$, then $\widetilde{\mathrm{dcat}}_K(L) \geq \widetilde{\mathrm{dcat}}_K(L')$. Can I show that $\widetilde{\mathrm{dcat}}_K(L) \leq \widetilde{\mathrm{dcat}}_K(L')$?

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The **discrete category** of L in K is defined by $\mathrm{dcat}_K(L) = \min\{\widetilde{\mathrm{dcat}}_K(L') : L \text{ collapses to } L' \text{ in } K\}$.

Smooth vs Discrete LS category

Since a collapse corresponds to a deformation retraction, we have that a collapsible complex is contractible. Hence

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$\mathrm{dcat}(CK) = 0$.

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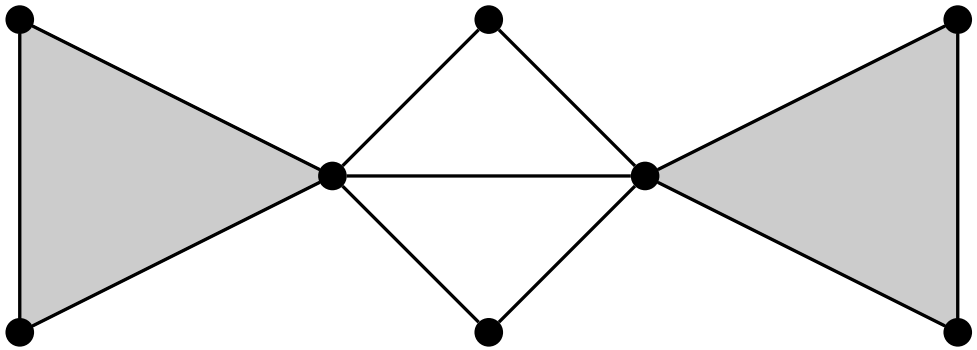

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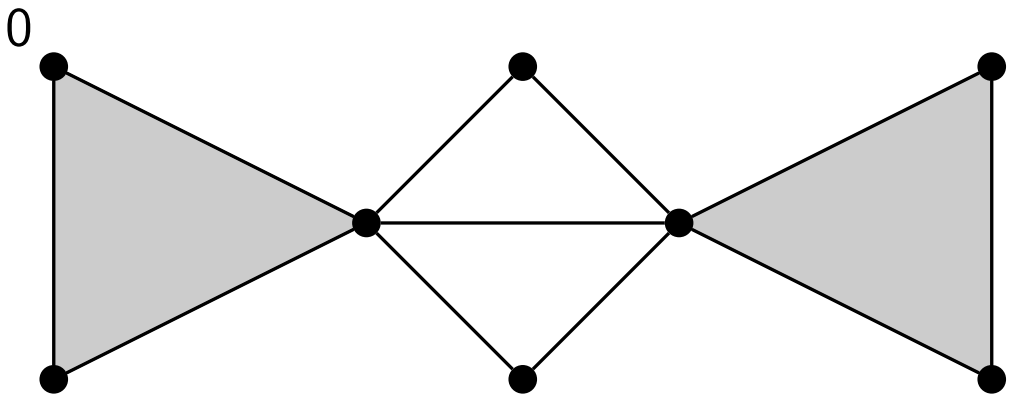

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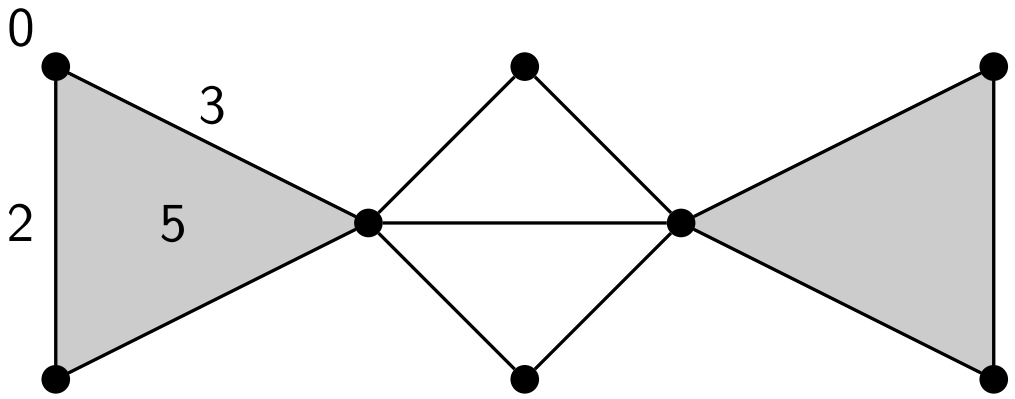

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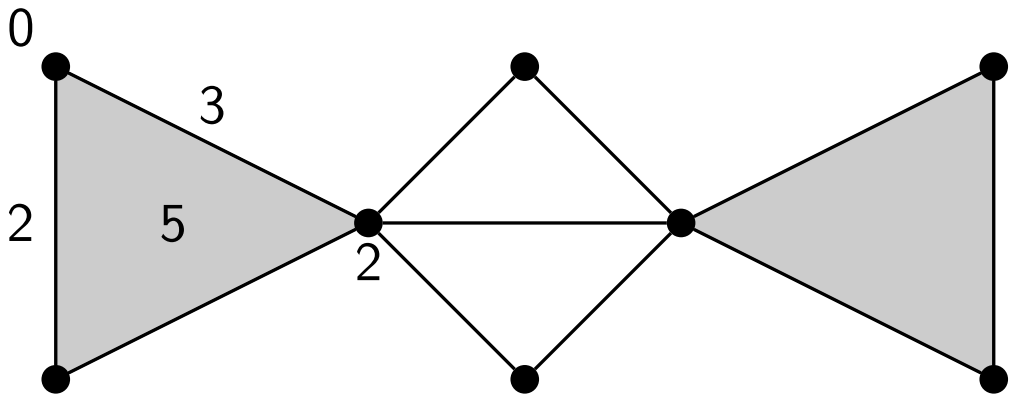

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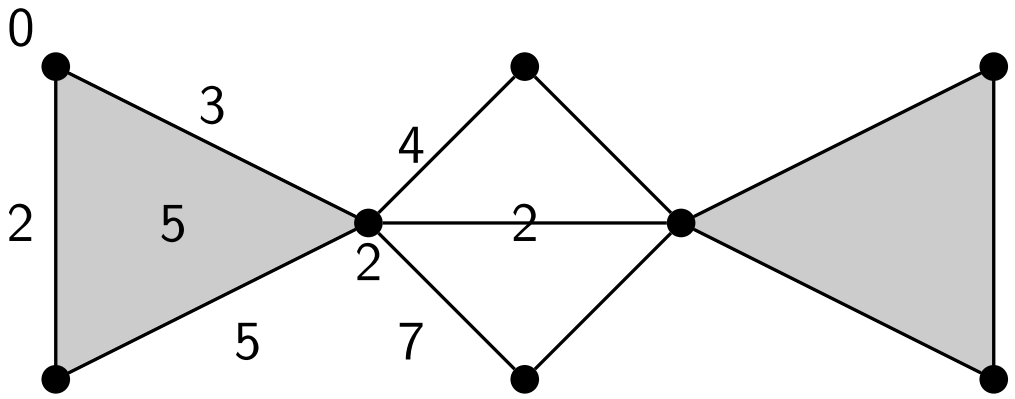

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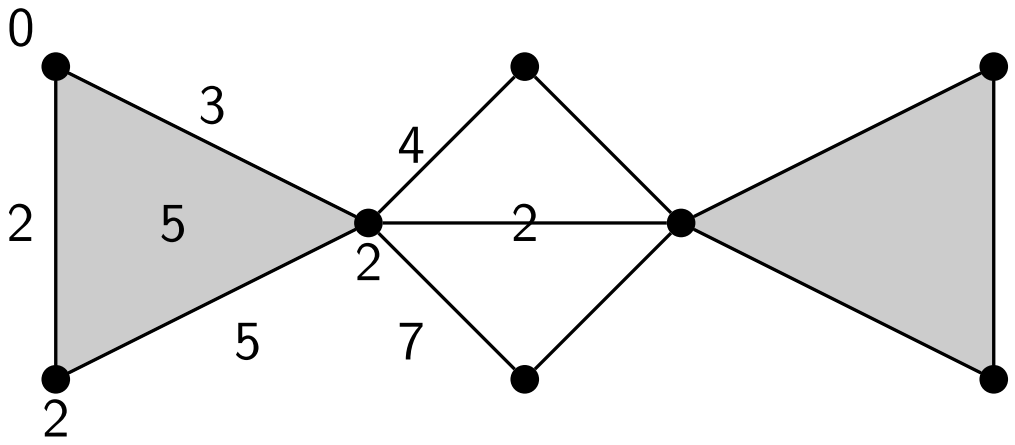

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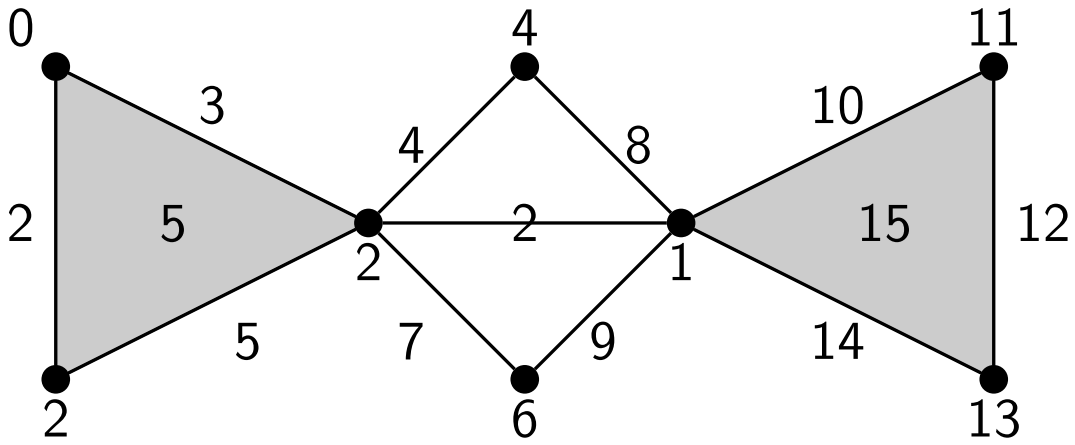

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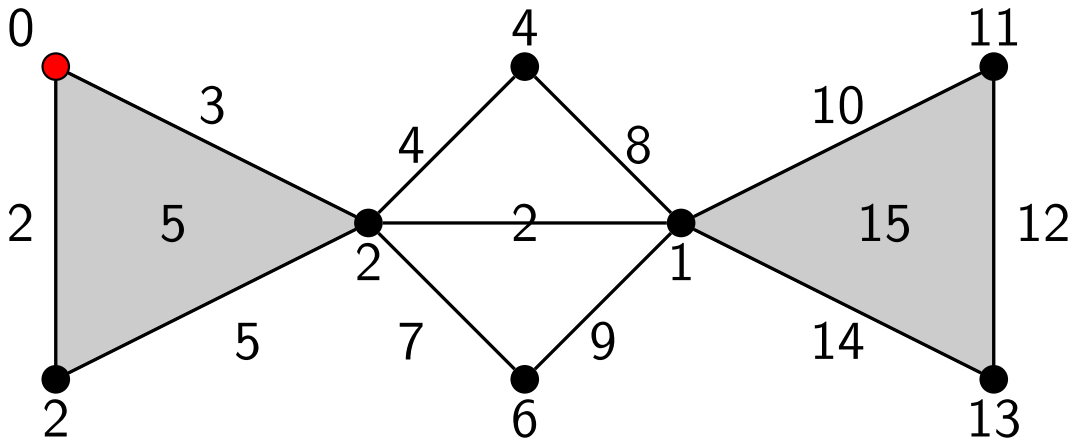

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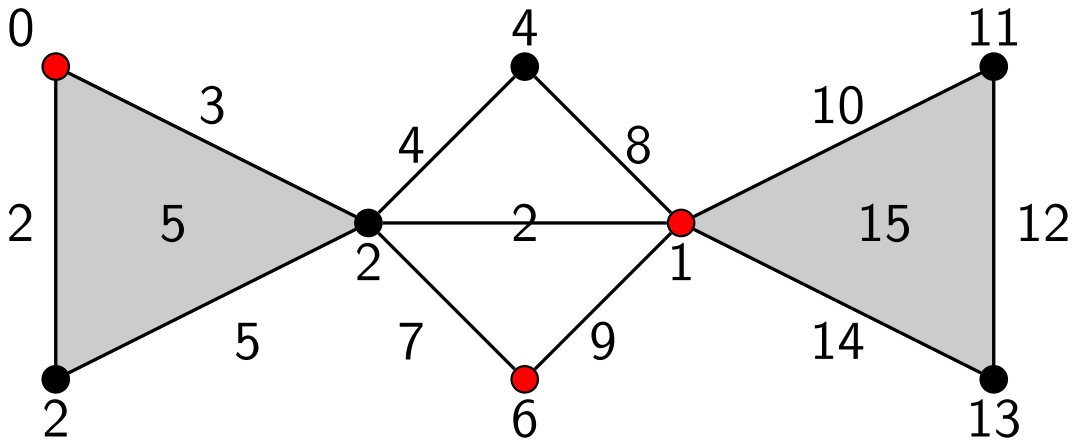

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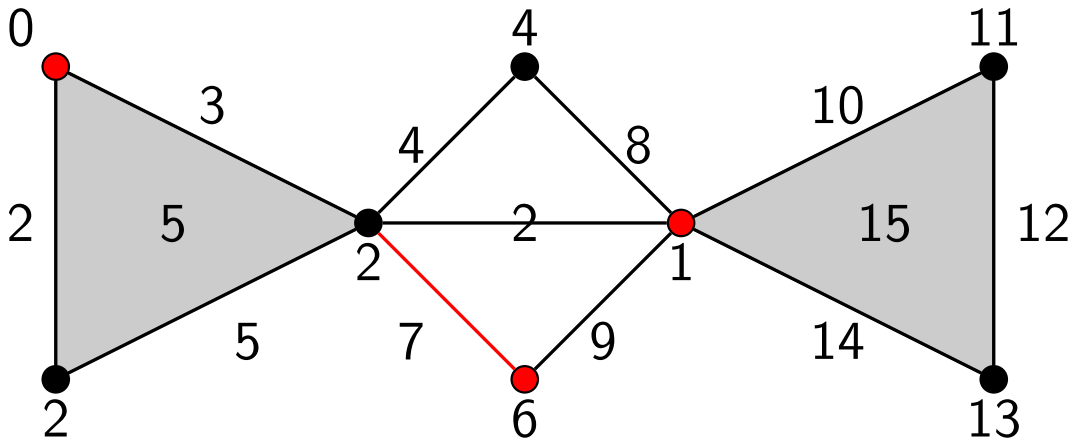

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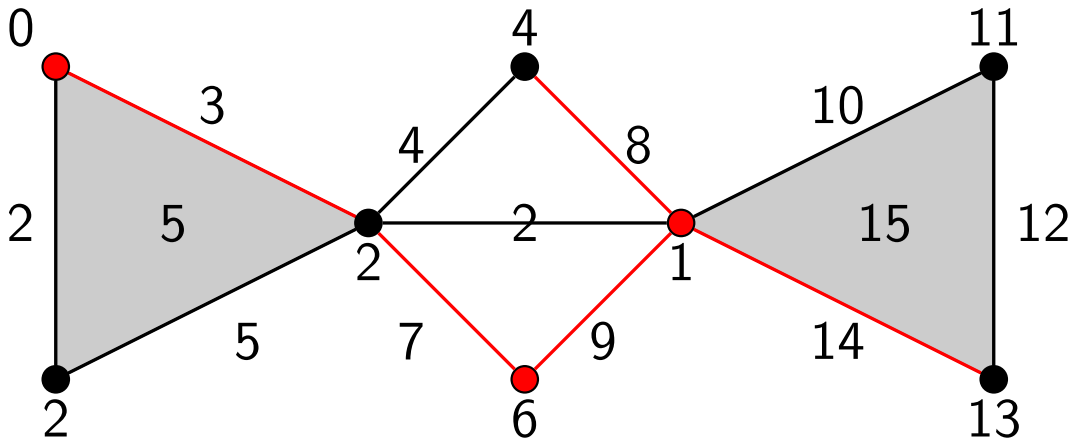

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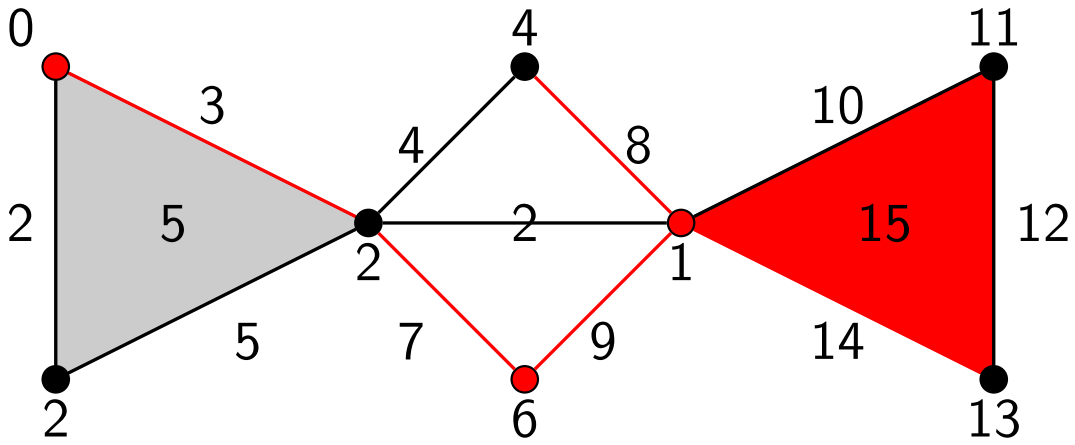

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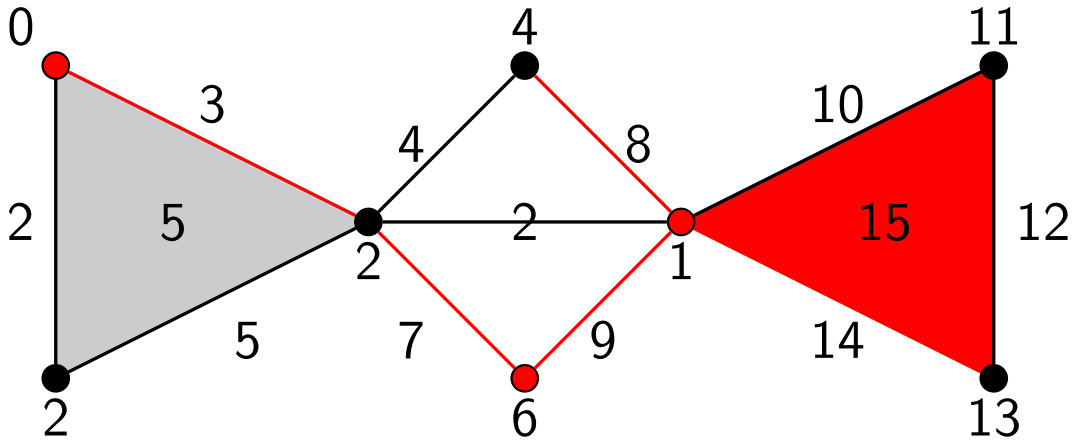

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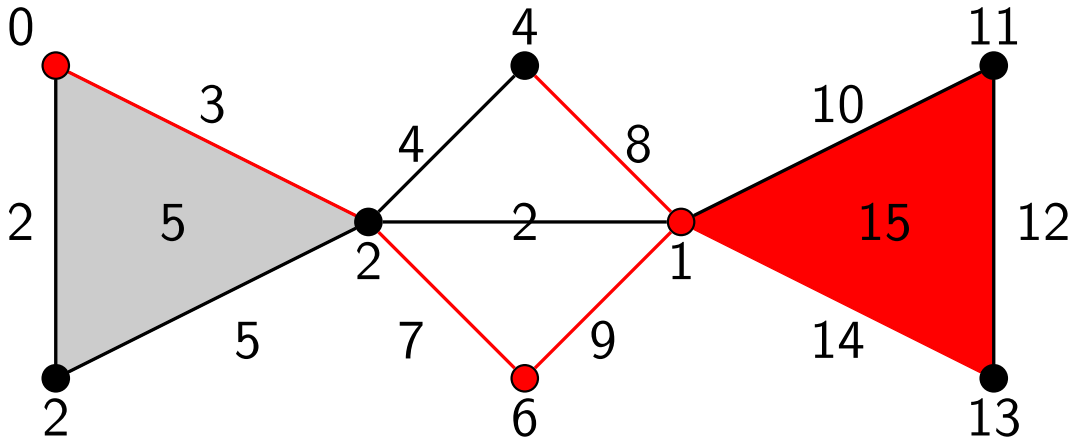

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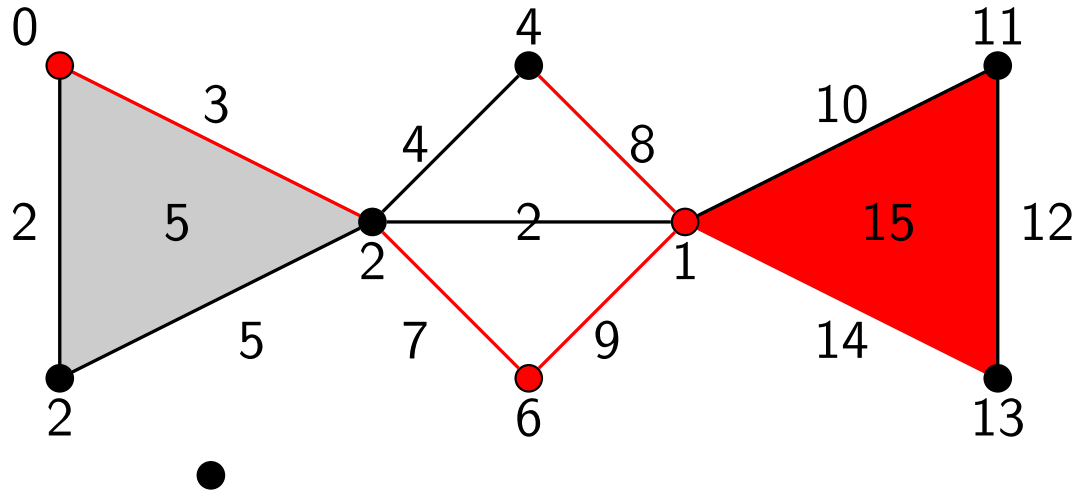

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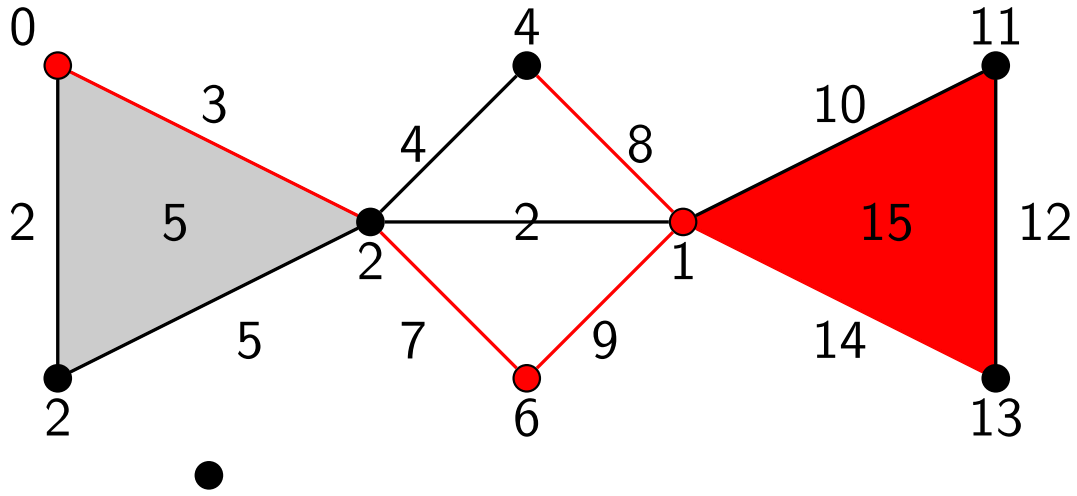

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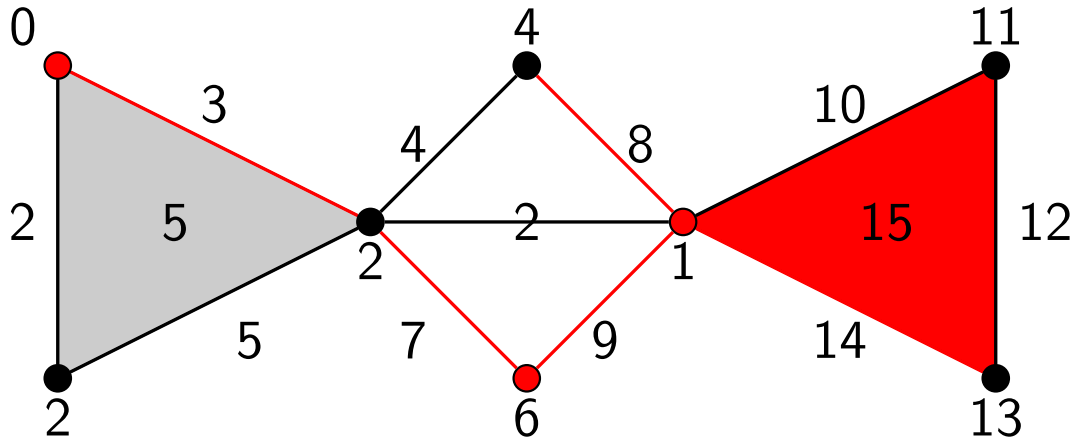
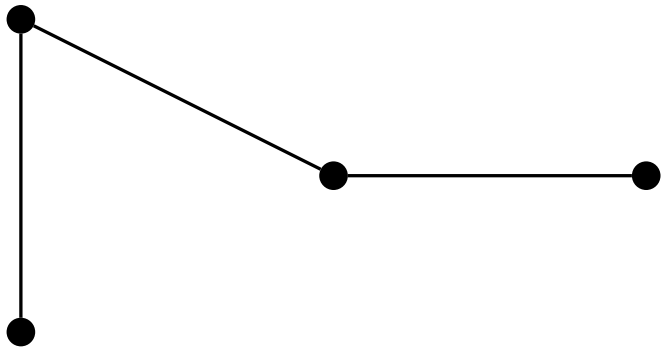


- $K(3) =$



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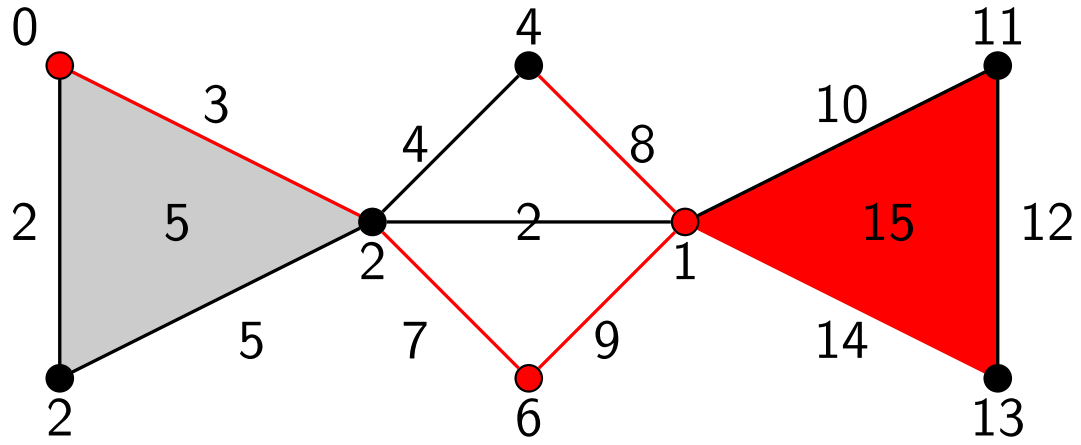
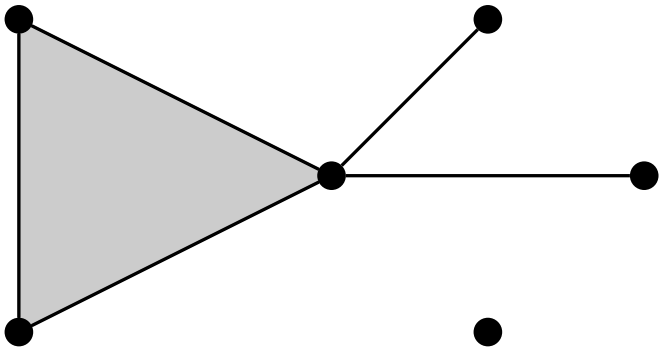


· $K(6) =$



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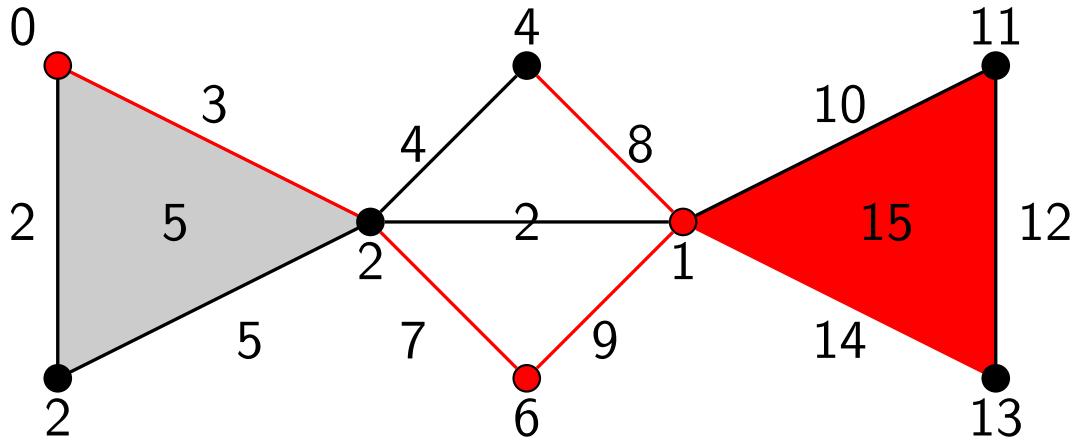
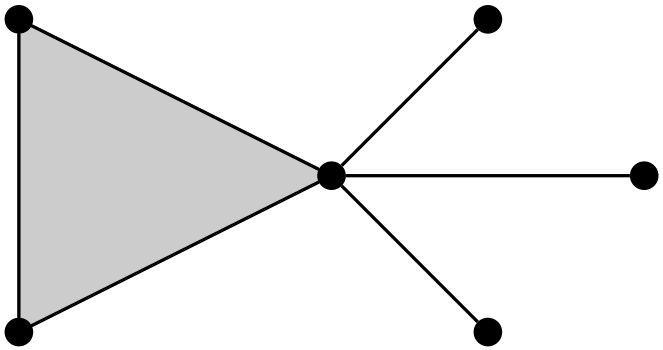


- $K(7) =$



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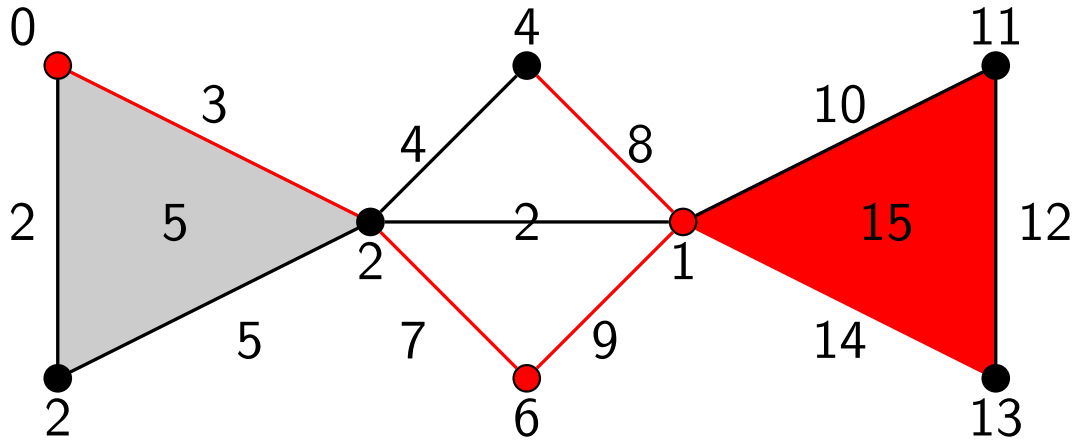
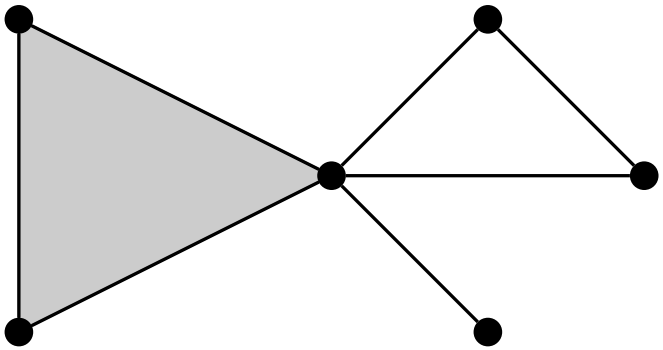


· $K(8) =$



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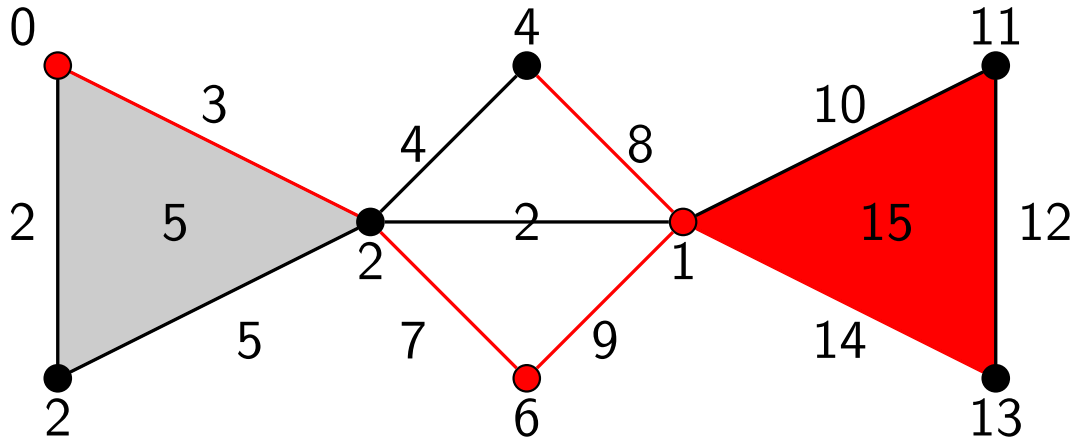
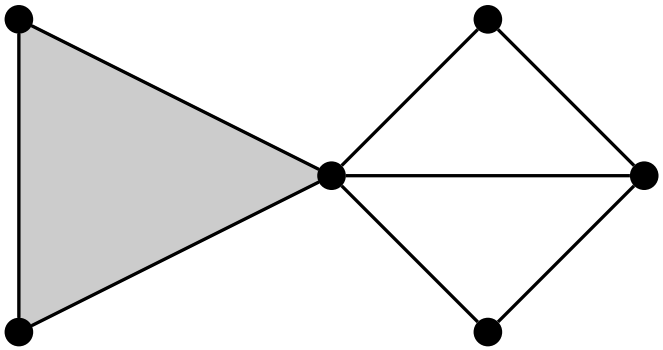


$\cdot\ K(9) =$



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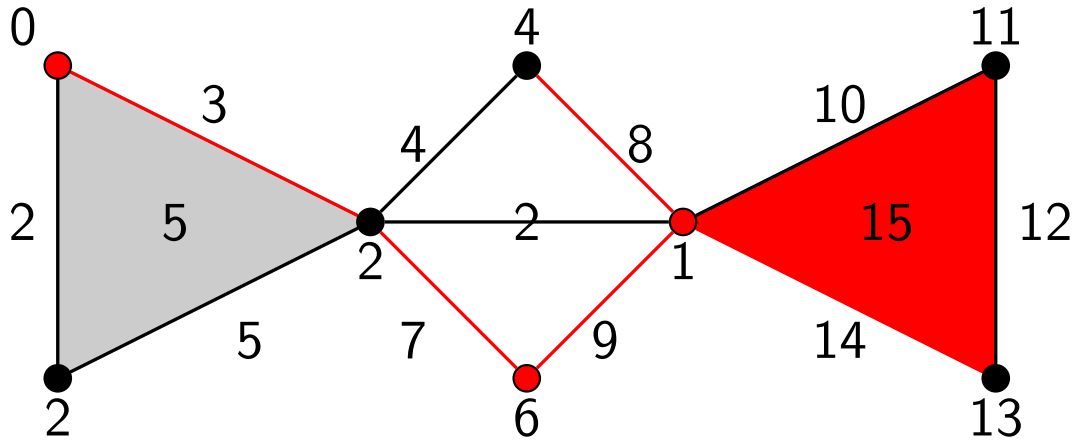
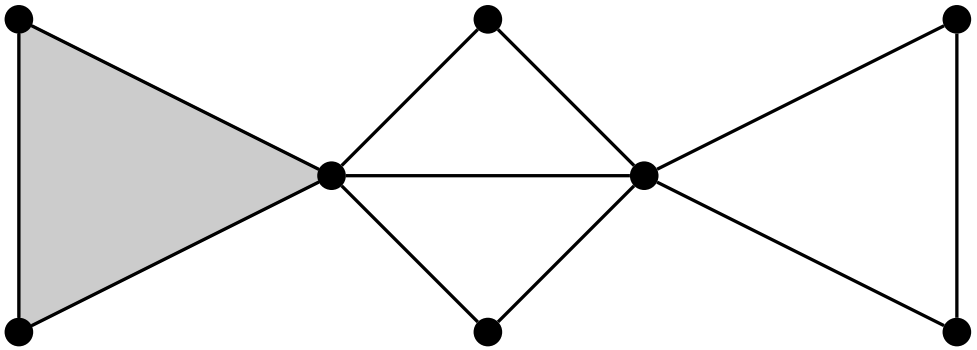


· $K(14) =$



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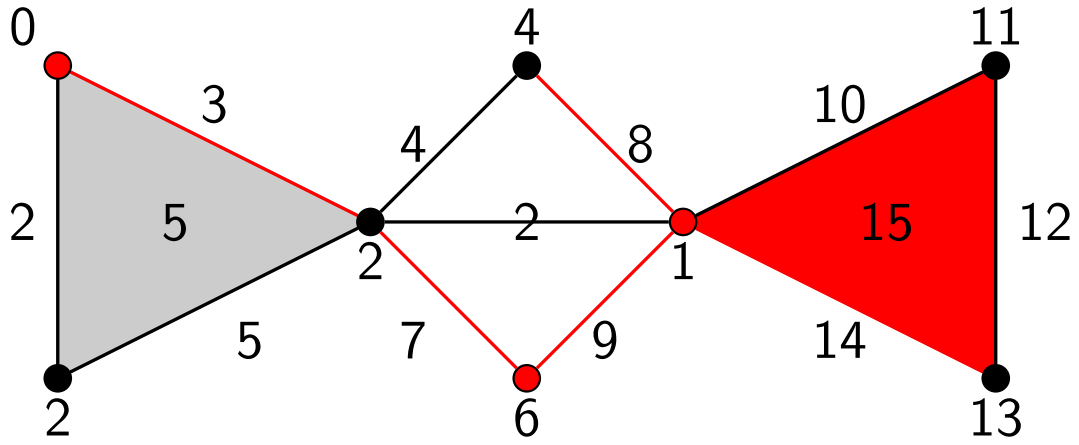
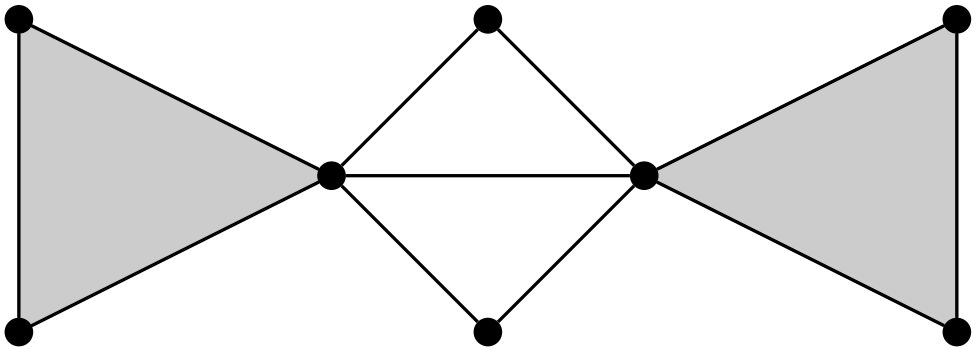


· $K(15) =$



Discrete LS Theorem

Theorem

Let $f : K \to \mathbb{R}$ be a discrete Morse function with m critical points. Then $\mathrm{dcat}(K) + 1 \leq m$.

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Lower bound

Lower bound

Let K be a simplicial complex of dimension n or $n+1$, and let c_i^K denote the number of simplicies of K of dimension i. Let $E(c^K) := c_0^K + c_2^K + \ldots + c_n^K$ and $O(c^K) := c_1^K + c_3^K + \ldots + c_{n\pm 1}^K$.

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Proposition

Let K be a simplicial complex of dimension n with c_i the number of simplicies of K of dimension i. If $E(c^K) - 1 \geq O(c^K)$, then $\left\lceil \frac{E(c^K)-1}{O(c^K)} \right\rceil - 1 \leq \mathrm{dcat}(K)$. If $E(c^K) - 1 \leq O(c^K)$, then $\left\lceil \frac{O(c^K)}{E(c^K)-1} \right\rceil - 1 \leq \mathrm{dcat}(K)$.

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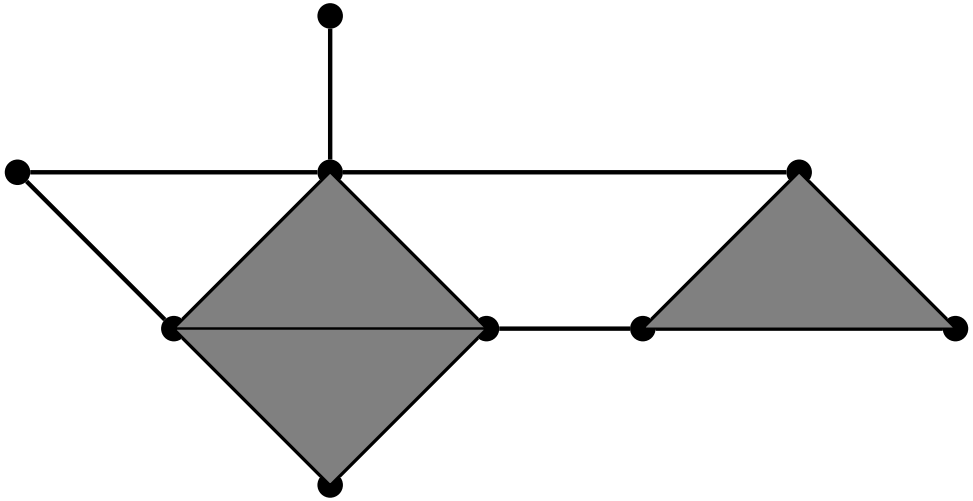
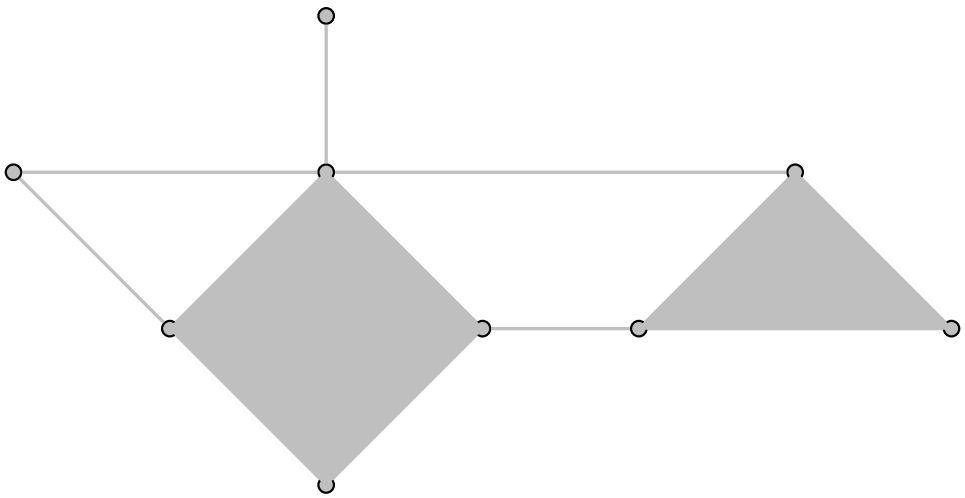
End when $V = \emptyset$.

The set $\mathcal{U}$ obtained in the above algorithm is a collapsible cover of K so that $\mathrm{dcat}(K) \leq |\mathcal{U}| - 1$.

Example



Example



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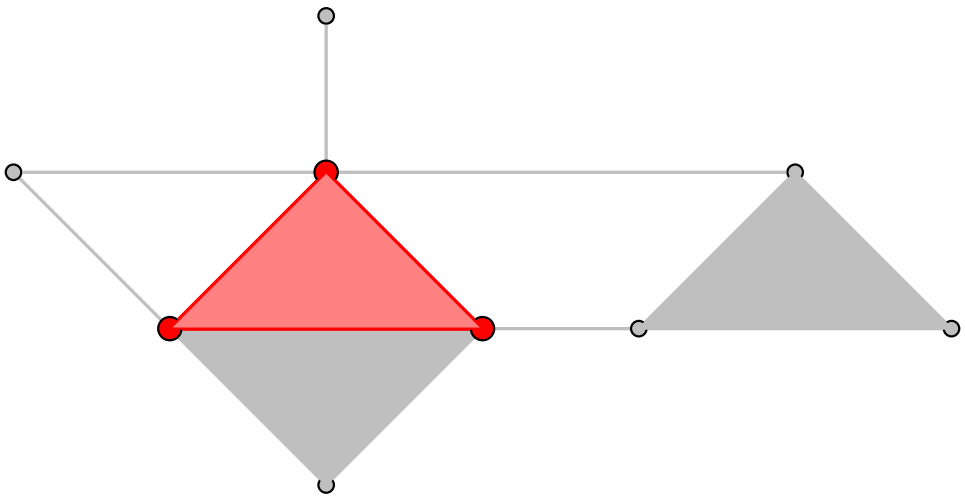

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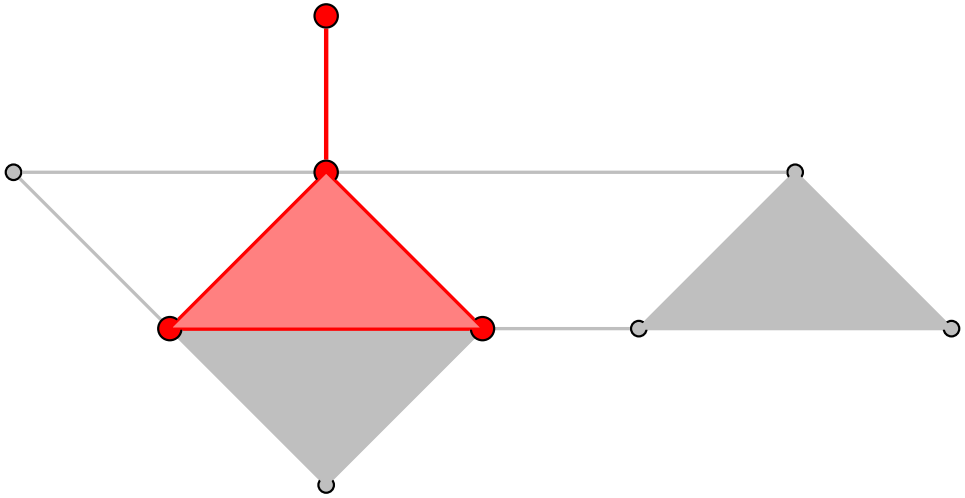

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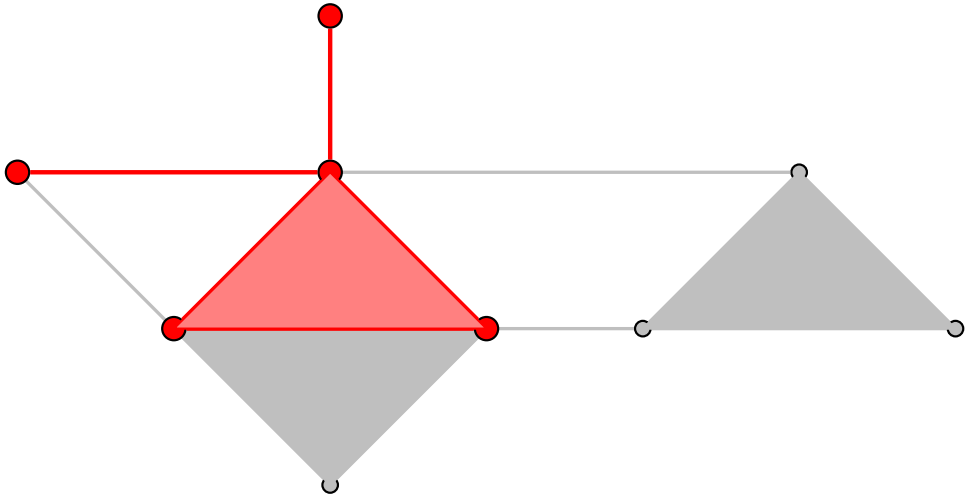

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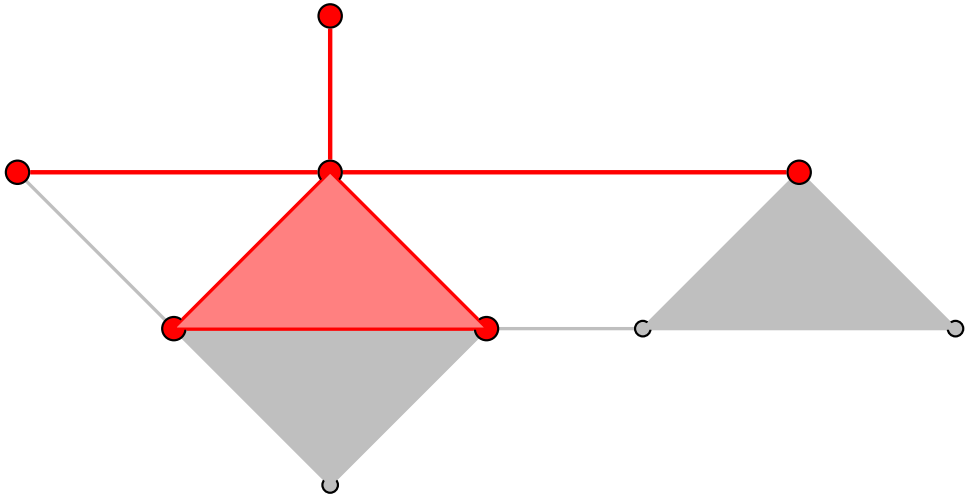

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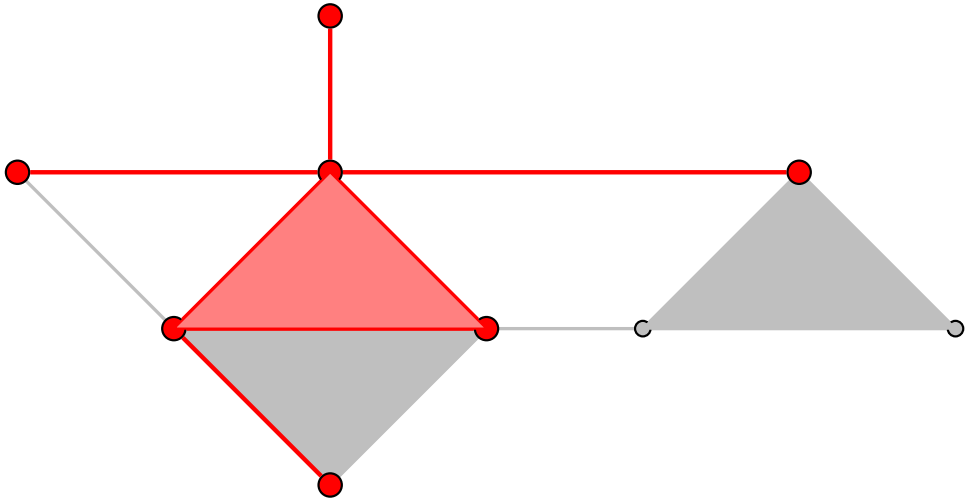

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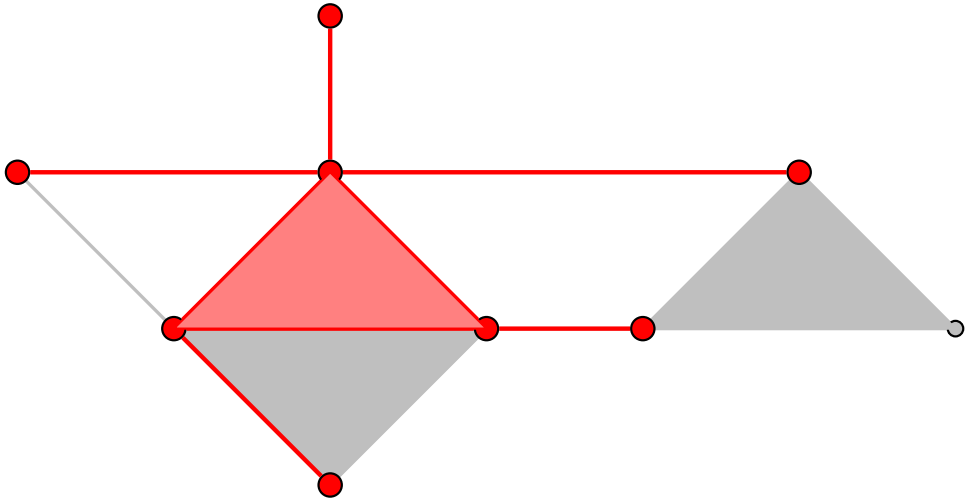

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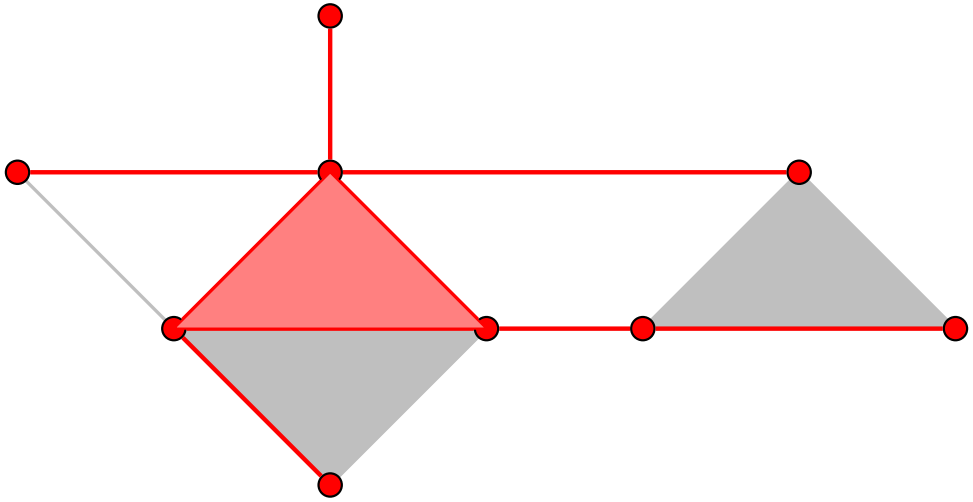

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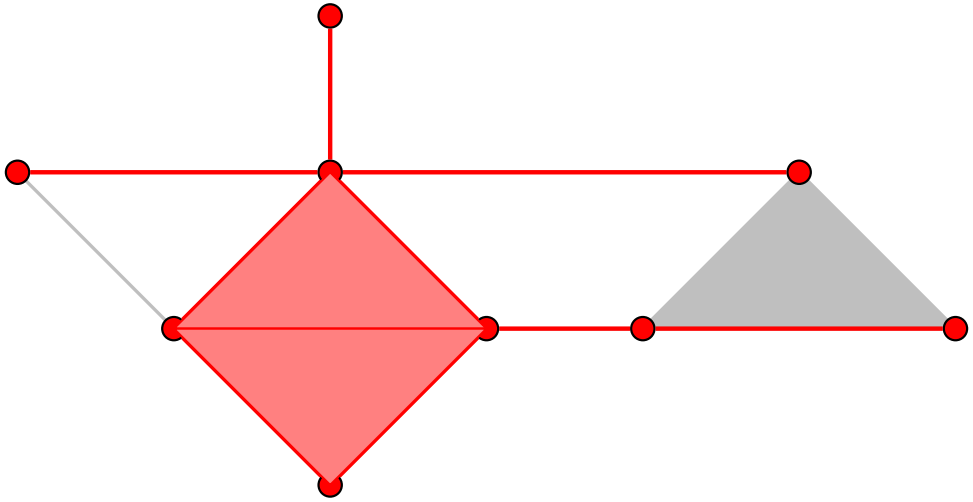

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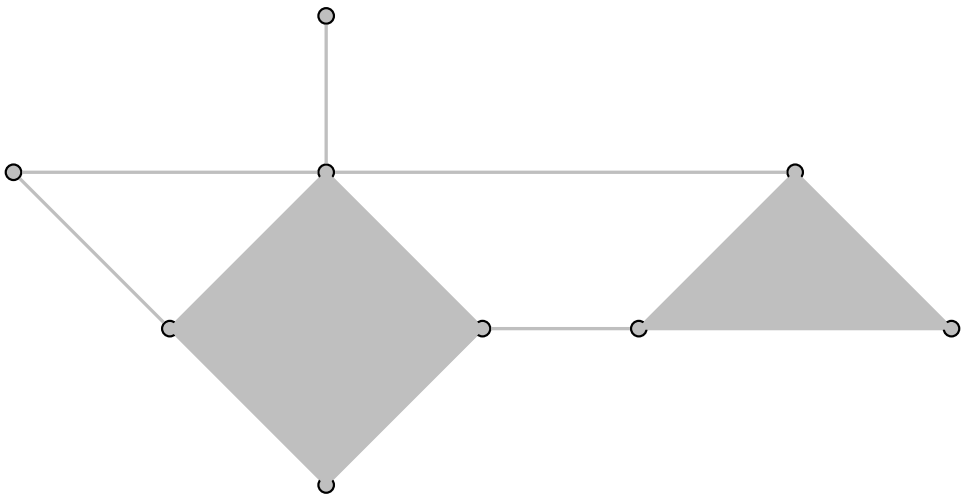

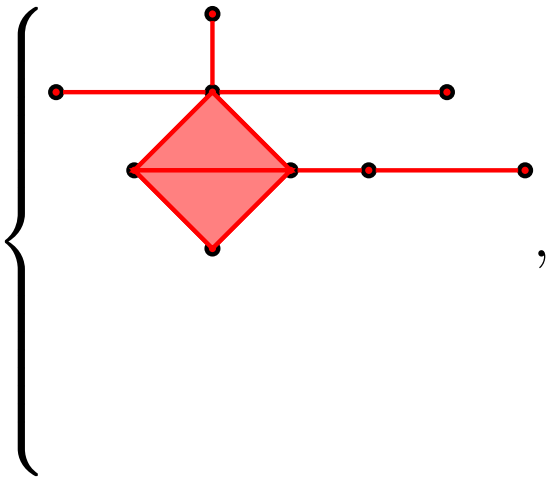

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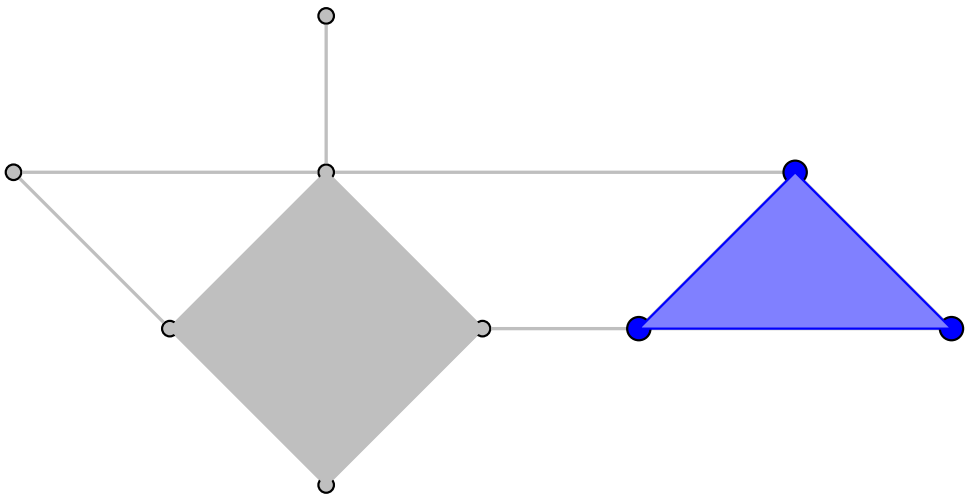

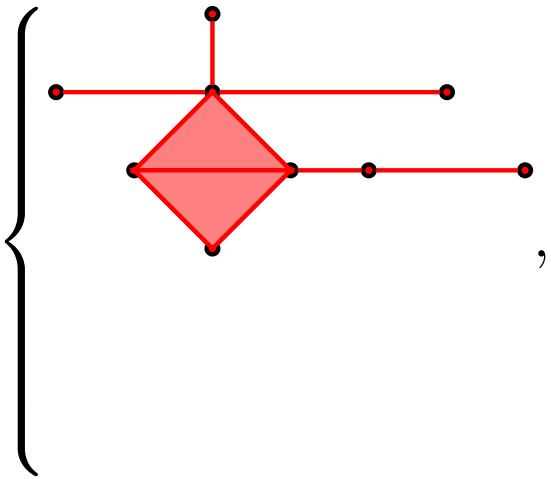

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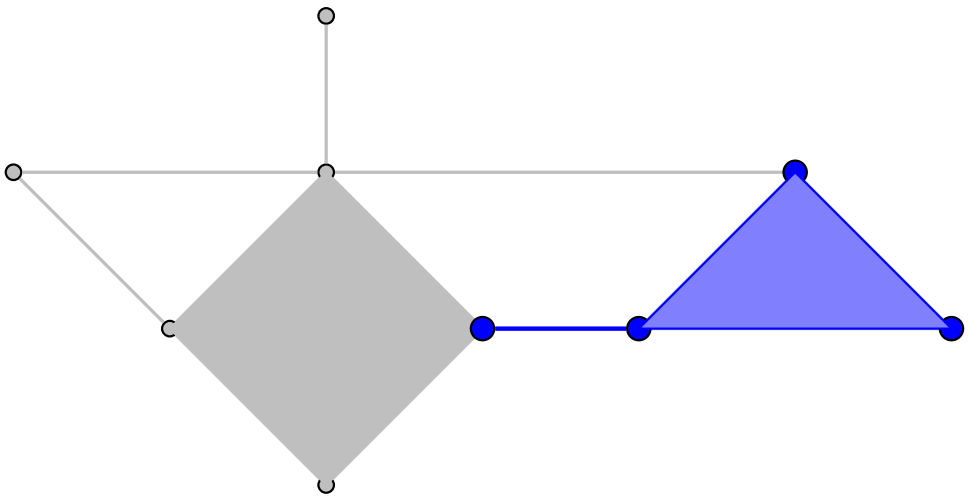

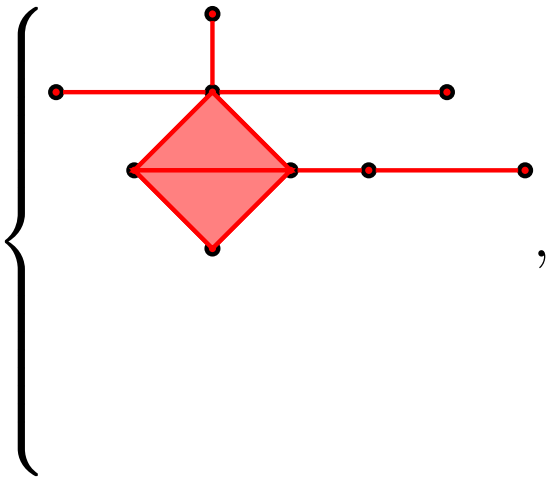

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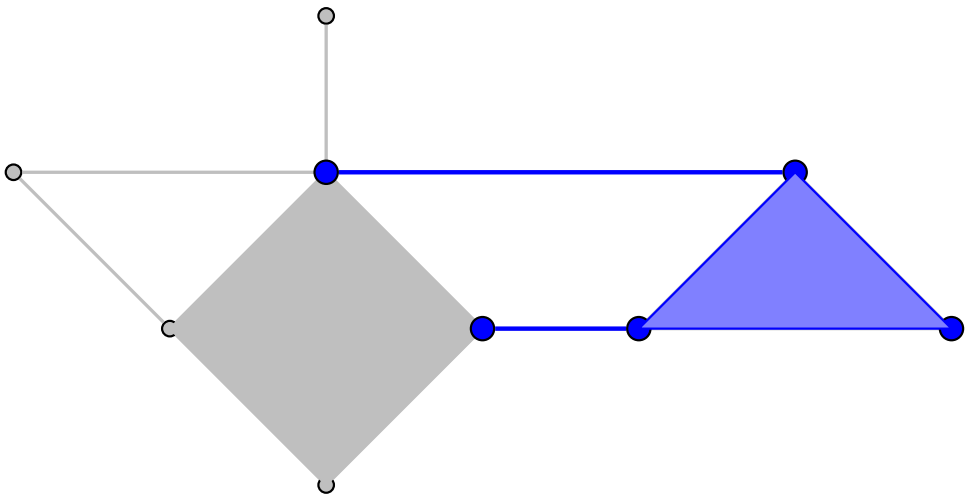

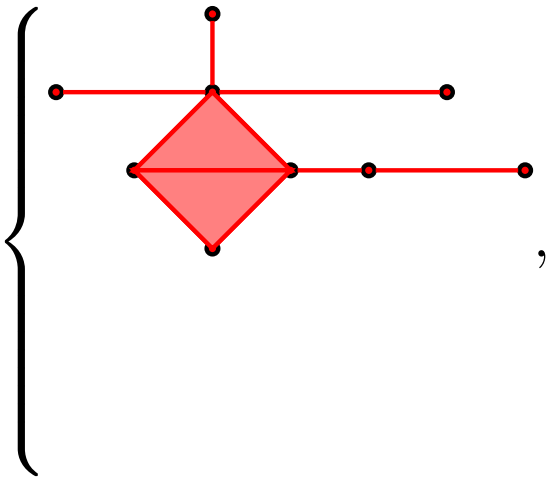

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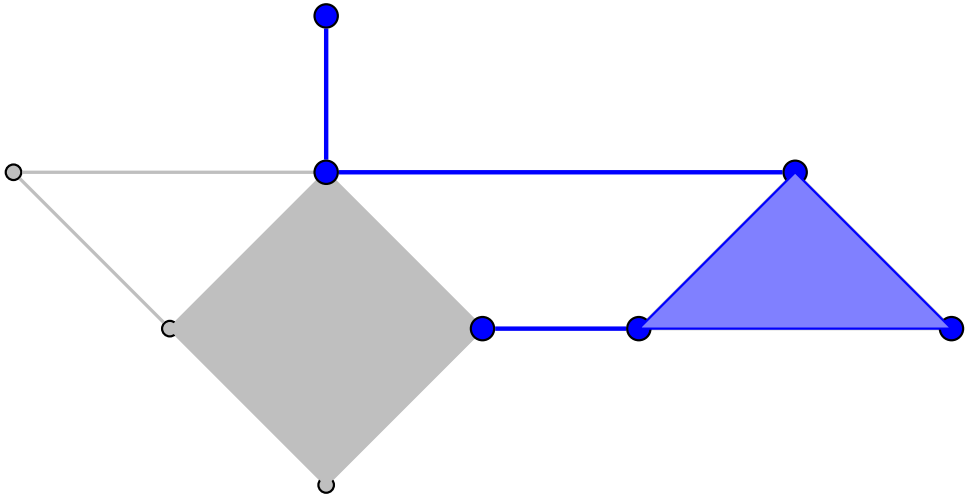

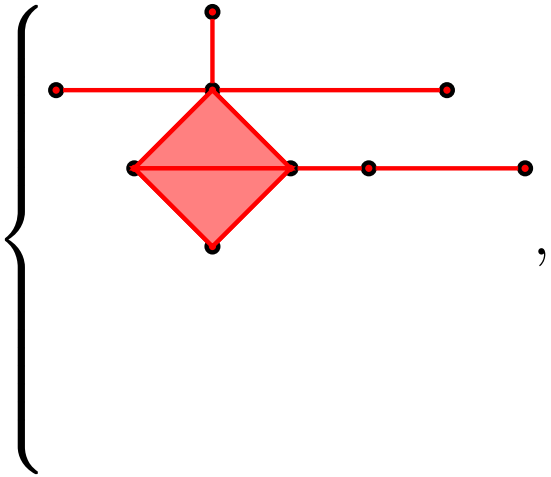

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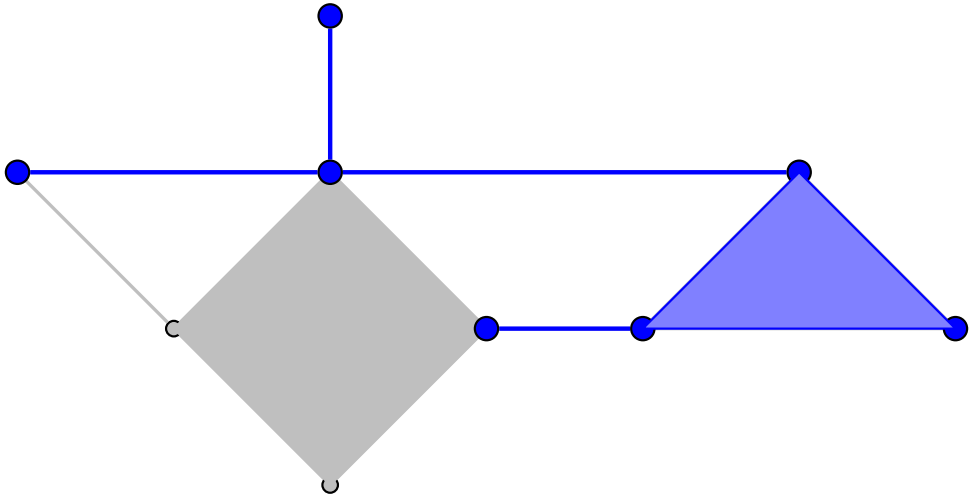

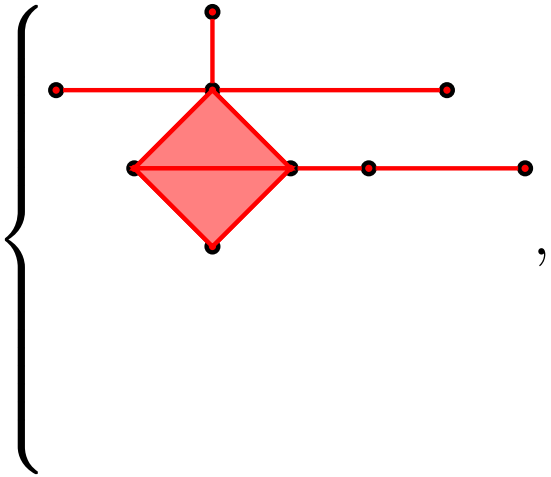

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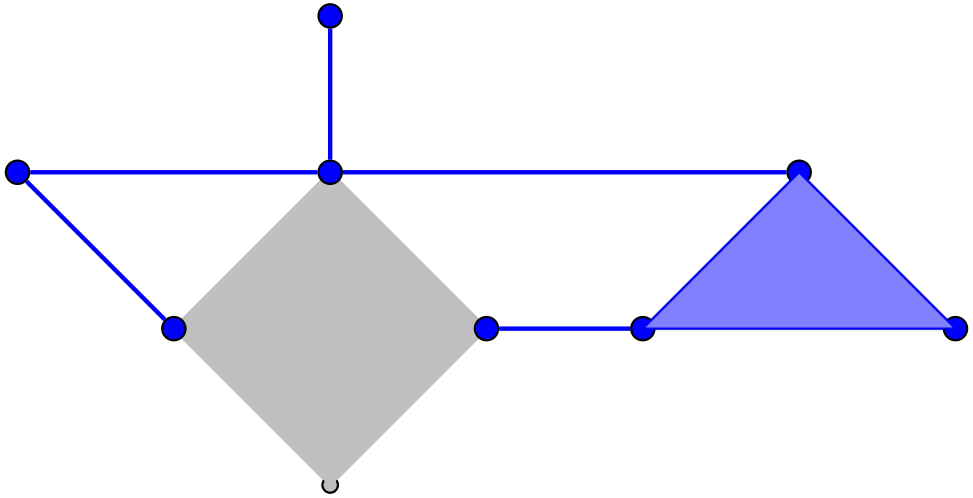

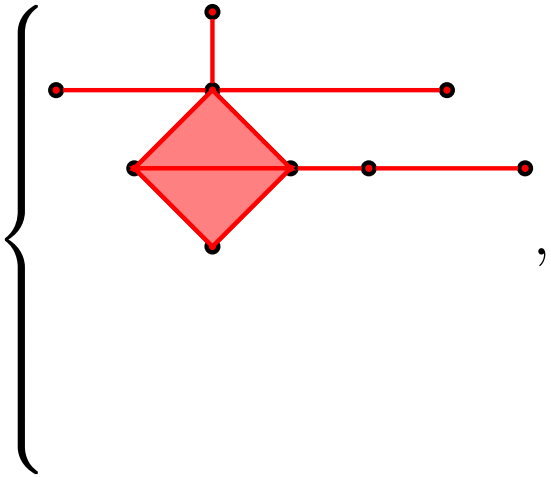

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