Module category weight of compact Lie groups

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First of all I would like to thank the organizers of this wonderful meeting

and

I also would like to give special congratulations to Prof. Yuli Rudyak on his 65th birthday.

Def: Lusternik–Schnirelmann category

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The Lusternik–Schnirelmann category of a space X, $\mathrm{cat}(X)$, is defined to be minimal number n such that there exists an open covering $\{U_1, \ldots, U_{n+1}\}$ of X with each U_i contractible in X.

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Projective spaces

Every space X has a filtration given by the X-projective k-space $P^k(\Omega X)$ of its loop space ΩX. Then there is a sequence of quasi-fibration

$$\{p_k : E^k(\Omega X) \to P^{k-1}(\Omega X); k \geq 1\}$$

with the fibre ΩX such that E^k has the homotopy type of the k-fold join of ΩX and P^k has the homotopy type of the mapping cone of p_k.

Remark: The space $P^k(\Omega X)$ is homotopy equivalent to the k-th Ganea space $G_k(X)$.

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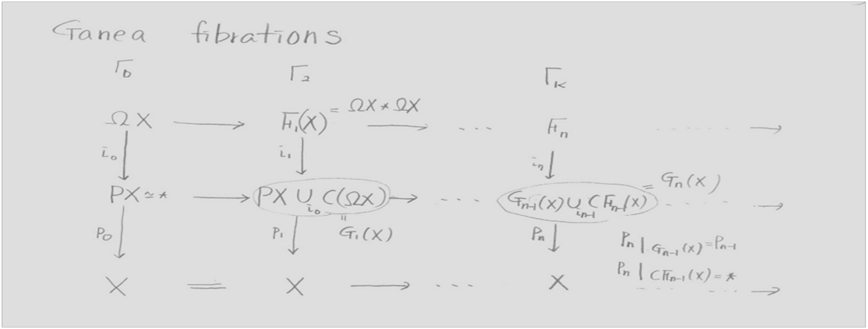
Ganea fibration



The Rothenberg–Steenrod spectral sequence associated with the filtration of $P^{\infty}(\Omega X) \simeq X$ given by $\{P^m(\Omega X) |\ m \geq 0\}$ coincides with the Rothenberg–Steenrod spectral sequence associated with that of $G_{\infty}(X) \simeq X$ given by $\{G_m(X) |\ m \geq 0\}$.

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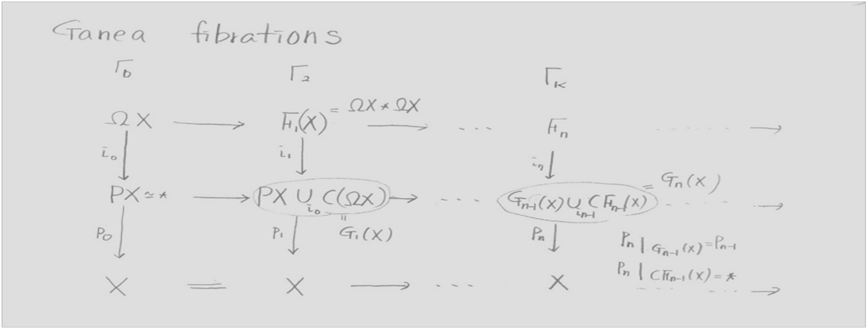

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Projective spaces

Let $e_m : P^m(\Omega X) \to P^\infty(\Omega X) \simeq X$ be the inclusion map,then we can pose the following problems:

1. Find the minimal number m such that $(e_m)^* : H^*(X; \boldsymbol{F}_p) \to H^*(P^m(\Omega X); \boldsymbol{F}_p)$ is a monomorphism.
2. Find the minimal number m such that $(e_m)^* : H^*(X; \boldsymbol{F}_p) \to H^*(P^m(\Omega X); \boldsymbol{F}_p)$ is a split monomorphism of modules over the Steenrod algebra, that is, there is a epimorphism $\phi_m : H^*(P^m(\Omega X); \boldsymbol{F}_p) \to H^*(X; \boldsymbol{F}_p)$ which preserves all Steenrod actions and $\phi_m \circ (e_m)^* \cong 1_{H^*(X;\boldsymbol{F}_p)}$.
3. Find the minimal number m such that there is a map $\sigma : X \to P^m(\Omega X)$ such that $e_m \circ \sigma \simeq 1_X$.

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Another homotopy invariants and their relation

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category weight $wgt(X; \boldsymbol{F}_p)$,

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[Ganea]

Let X be a connected space. Then $cat(X) \leq m$ if and only if there is a map $\sigma : X \to P^m(\Omega X)$ such that $e_m \circ \sigma \simeq 1_X$.

Then, we have the following relation:
$cup(X; \boldsymbol{F}_p) \leq wgt(X; \boldsymbol{F}_p) \leq Mwgt(X; \boldsymbol{F}_p) \leq cat(X)$.

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The aim of this talk is to compute the module category weight of simply connected compact simple Lie groups to give a lower bound for the Lusternik–Schnirelmann category of them.

However, the classical types are not so interesting except the case of $Spin(n)$ with $\boldsymbol{F}_2$ coefficients. Here we will explain exceptional Lie groups cases, G_2, F_4, E_6, E_7, E_8 and mention the result of $Spin(n)$ with $\boldsymbol{F}_2$ coefficients without explanation.

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Remark 1: Toomer calculated the difference $cup(X; \boldsymbol{F}_p) - wgt(X; \boldsymbol{F}_p)$ of any simply connected compact simple Lie group. In fact, it is precisely F_4, E_6, E_7, E_8 which yield a positive difference.

Remark 2: On the other hands, Iwase and Kono determined $cat(Spin(9)) = 8$ by computing the lower bound of the difference between the category weight and the module category weight of $Spin(9)$, which is $Mwgt(Spin(9); \boldsymbol{F}_2) - wgt(Spin(9); \boldsymbol{F}_2) \geq 2$.

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Rothenberg–Steenrod spectral sequence

For a simply connected space X and given a path–loop fibration, $\Omega X \to PX \to X$,
we consider the Rothenberg–Steenrod spectral sequence $\{E_r^{*,*}, d_r\}$ converging to $H^*(X; \boldsymbol{F}_p)$ with

$$E_2 \cong \mathrm{Cotor}_{H^*(\Omega X;\boldsymbol{F}_p)}(\boldsymbol{F}_p, \boldsymbol{F}_p)$$
$$E_\infty^{s,t} \cong F^s H^{s+t}(X; \boldsymbol{F}_p)/F^{s+1} H^{s+t}(X; \boldsymbol{F}_p)$$

where
$F^{q+1} H^n(X; \boldsymbol{F}_p) \cong \ker\{(e_q)^* : H^n(X; \boldsymbol{F}_p) \to H^n(P^q(\Omega X); \boldsymbol{F}_p)\}$

Hence for all $s \geq m+1$,

$$E_\infty^{s,*} = 0 \quad \Leftrightarrow \quad F^s H^*(X; \boldsymbol{F}_p) = F^{s+1} H^*(X; \boldsymbol{F}_p)$$
$$\Leftrightarrow \quad \ker\,(e_{s-1})^* = \ker\,(e_s)^*.$$

Since $wgt(X; \boldsymbol{F}_p)$ is the minimum number m such that $\ker\,(e_m)^* = 0$, $wgt(X; \boldsymbol{F}_p)$ can be defined as the minimal number m such that $E_\infty^{s,*} = 0$ for all $s \geq m+1$.
Hence $wgt(X; \boldsymbol{F}_p)$ is $\mathrm{f}_p(X)$, which is called the $\boldsymbol{F}_p$-filtration length of X.

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The coalgebra structure of $H^*(\Omega G)$

We analyze the Rothenberg–Steenrod spectral sequence converging to $H^*(G)$ with $E_2^{*,*} \cong \mathrm{Cotor}_{H^*(\Omega G)}(\boldsymbol{F}_2, \boldsymbol{F}_2)$ in order to get module category weight of exceptional Lie groups G. This requires understanding of the coalgebra structure of $H^*(\Omega G)$.

Theorem

The coalgebra structure of the mod 2 cohomology of the loop spaces of exceptional Lie groups are as follows.

$$\begin{aligned}
H^*(\Omega G_2;\mathbb{F}_2) &\cong E(a_2)\otimes\Gamma(a_4,b_{10})\\
H^*(\Omega F_4;\mathbb{F}_2) &\cong E(a_2)\otimes\Gamma(a_4,b_{10},a_{14},a_{16},a_{22})\\
H^*(\Omega E_6;\mathbb{F}_2) &\cong E(a_2)\otimes\Gamma(a_4,a_8,b_{10},a_{14},a_{16},a_{22})\\
H^*(\Omega E_7;\mathbb{F}_2) &\cong E(a_2,a_4,a_8)\otimes\Gamma(b_{10},a_{14},a_{16},b_{18},a_{22},a_{26},b_{34})\\
H^*(\Omega E_8;\mathbb{F}_2) &\cong E(a_2,a_4,a_8,a_{14})\otimes\Gamma(a_{16},a_{22},a_{26},a_{28},b_{34},b_{38},b_{46},b_{58})
\end{aligned}$$

especially we have $Sq^4 b_{10} = a_{14}$ and $Sq^8 b_{18} = a_{26}$.

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$E_2 = \mathrm{Cotor}_{H^*(\Omega G;\boldsymbol{F}_2)}(\boldsymbol{F}_2, \boldsymbol{F}_2)$

Theorem

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$$\begin{aligned}
\mathrm{Cotor}_{H^*(\Omega G_2;\boldsymbol{F}_2)}(\boldsymbol{F}_2, \boldsymbol{F}_2) &\cong \mathbb{F}_2[x_3] \otimes E(x_5, z_{11}) \\
\mathrm{Cotor}_{H^*(\Omega F_4;\boldsymbol{F}_2)}(\boldsymbol{F}_2, \boldsymbol{F}_2) &\cong \mathbb{F}_2[x_3] \otimes E(x_5, z_{11}, x_{15}, x_{23}) \\
\mathrm{Cotor}_{H^*(\Omega E_6;\boldsymbol{F}_2)}(\boldsymbol{F}_2, \boldsymbol{F}_2) &\cong \mathbb{F}_2[x_3] \otimes E(x_5, x_9, z_{11}, x_{15}, x_{17}, x_{23}) \\
\mathrm{Cotor}_{H^*(\Omega E_7;\boldsymbol{F}_2)}(\boldsymbol{F}_2, \boldsymbol{F}_2) &\cong \mathbb{F}_2[x_3, x_5, x_9] \otimes E(z_{11}, x_{15}, x_{17}, z_{19}, x_{23}, x_{27}, z_{35}) \\
\mathrm{Cotor}_{H^*(\Omega E_8;\boldsymbol{F}_2)}(\boldsymbol{F}_2, \boldsymbol{F}_2) &\cong \mathbb{F}_2[x_3, x_5, x_9, x_{15}] \otimes E(x_{17}, x_{23}, x_{27}, x_{29}, z_{35}, z_{39}, z_{47}, z_{59})
\end{aligned}$$

especially we have $Sq^4 z_{11} = x_{15}$ and $Sq^8 z_{19} = x_{27}$.

Differentials

Then from information of $H^*(G; \boldsymbol{F}_2)$, we can analyze non trivial differentials of the Rothenberg–Steenrod spectral sequence converging to $H^*(G; \boldsymbol{F}_2)$ as follows:

$$
\begin{aligned}
d_3(z_{11}) &= x_3^4 && \text{for } G = G_2, F_4, E_6, E_7 \\
d_3(z_{19}) &= x_5^4 && \text{for } G = E_7 \\
d_3(z_{35}) &= x_9^4 && \text{for } G = E_7, E_8 \\
d_7(z_{39}) &= x_5^8 && \text{for } G = E_8 \\
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$H^*(P^m(\Omega G); \boldsymbol{F}_2)$

Consider the spectral sequence of Stasheff's type converging to $H^*(P^m(\Omega G); \boldsymbol{F}_2)$. Let $A = H^*(G; \boldsymbol{F}_2)$. Then for low m such as $1 \leq m \leq 3$, we have the following:

$$H^*(P^m(\Omega G); \boldsymbol{F}_2) = A^{[m]} \oplus \sum_i z_{4i+3} \cdot A^{[m-1]} \oplus S_m, \begin{cases} i = 3, \text{ for } G = G_2, F_4, E_6 \\ i = 3, 4, 8, \text{ for } G = E_7 \\ i = 8, 9, 11, 14, \text{ for } G = E_8 \end{cases}$$

as modules where $A^{[m]}, (m \geq 0)$ denotes the quotient module $A/D^{m+1}(A)$ of A by the submodule $D^{m+1}(A) \subseteq A$ generated by all the products of $m+1$ elements in positive dimensions in A, $z_{4i+3} \cdot A^{[m-1]}$ denotes a submodule corresponding to a submodule in $A \otimes E(z_{4i+3})$ and S_m satisfies $S_m \cdot \tilde{H}^*(P^m(\Omega G); \boldsymbol{F}_2) = 0$ and $S_m|_{P^{m-1}(\Omega G)} = 0$.

Module Category Weight for F_2 coefficients

Theorem

The module category weights with respect to $\boldsymbol{F}_2$ coefficients are as follows:

$$
\begin{aligned}
Mwgt(G_2;\boldsymbol{F}_2) &\geq 4,\\
Mwgt(F_4;\boldsymbol{F}_2) &\geq 8,\\
Mwgt(E_6;\boldsymbol{F}_2) &\geq 10,\\
Mwgt(E_7;\boldsymbol{F}_2) &\geq 15,\\
Mwgt(E_8;\boldsymbol{F}_2) &\geq 32.
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Sketch of a proof for the F_4 case

- The case of F_4

$$H^*(F_4 ; \mathbb{F}_2) = \mathbb{F}_2[x_3]/(x_3^4) \otimes E(Sq^2 x_3, x_{15}, Sq^8 x_{15})$$

$$E_2 = \Big\downarrow \ \mathrm{Tor}^{*,*}_{H^*(F_4 ; \mathbb{F}_2)}(\mathbb{F}_2, \mathbb{F}_2) = E_\infty$$

$$H^*(\Omega F_4 ; \mathbb{F}_2) = E(a_2) \otimes \Gamma(a_4, b_{10}, a_{14}, a_{16}, a_{22})$$

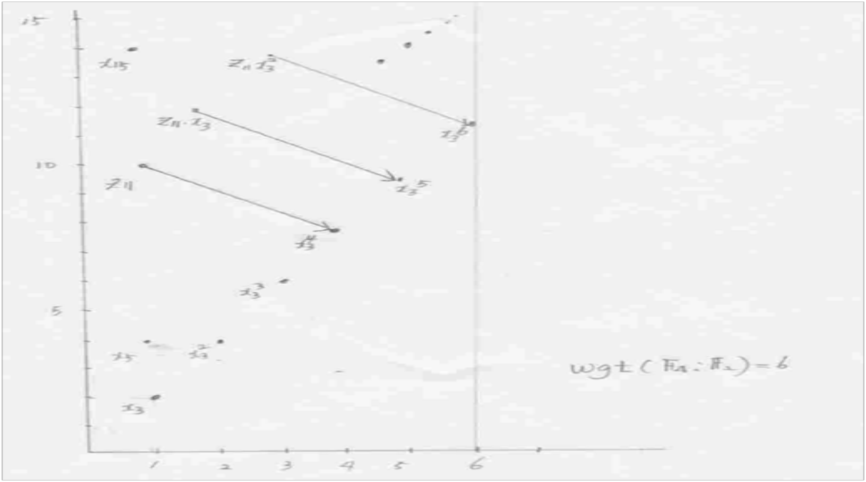
$$Sq^4 b_{10} = a_{14}$$

$$E_2 \Big\downarrow = \mathrm{Cotor}^{*,*}_{H^*(\Omega F_4 ; \mathbb{F}_2)}(\mathbb{F}_2, \mathbb{F}_2)$$

$$E_2 = \mathbb{F}_2[x_3] \otimes E(x_5, z_{11}, x_{15}, x_{23}) \Longrightarrow H^*(F_4 ; \mathbb{F}_2)$$

$$Sq^4 z_{11} = x_{15}.$$

Sketch of a proof for the F_4 case



Sketch of a proof for the F_4 case

$H^*(P^j(\Omega F_4) : \mathbb{F}_2)$: Spectral sequence of Stasheff's type

$$Sq^4 z_{11} = x_{15} \quad \text{in } H^*(P^1(\Omega F_4) : \mathbb{F}_2)$$

$$Sq^4 z_{11} = x_{15} \quad \text{in } H^*(P^2(\Omega F_4) : \mathbb{F}_2)$$

$$Sq^4 z_{11} = x_{15} + w \quad \text{in } H^*(P^3(\Omega F_4) : \mathbb{F}_2)$$

Suppose that $\exists$ epimorphism

$$\phi_7 : H^*(P^7(\Omega F_4) : \mathbb{F}_2) \longrightarrow H^*(F_4 : \mathbb{F}_2).$$

Sketch of a proof for the F_4 case

Then consider

$$
\begin{array}{ccc}
H^*(P^7(\Omega F_4); \mathbb{F}_2) & \longrightarrow & H^*(F_4; \mathbb{F}_2) \\
x_3^3 x_5 x_{15} x_{23} & \longrightarrow & x_3^3 x_5 x_{15} x_{23} \\
Sq^4 \uparrow & & \uparrow Sq^4 \\
x_3^3 x_5 z_{15} x_{23} & \longrightarrow & 0
\end{array}
$$

#

$\therefore \ \mathrm{Mwgt}(F_4; \mathbb{F}_2) \geq 8$

Summarizing above results for F_2 coefficients

X	$wgt(X; \mathbf{F}_2)$	$Mwgt(X; \mathbf{F}_2)$	$cat(X)$
G_2	4	$\geq$ 4	4
F_4	6	$\geq$ 8	?
E_6	8	$\geq$ 10	?
E_7	13	$\geq$ 15	?
E_8	32	$\geq$ 32	?

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The coalgebra structure of $H^*(\Omega G; \mathbb{F}_3)$

Theorem

The coalgebra structure of the mod 3 cohomology of the loop spaces of exceptional Lie groups are as follows.

$$
\begin{aligned}
H^*(\Omega G_2;\mathbb{F}_3) &\cong \Gamma(a_2, a_{10}) \\
H^*(\Omega F_4;\mathbb{F}_3) &\cong \mathbb{F}_3[a_2]/(a_2^3)\otimes\Gamma(a_6, a_{10}, a_{14}, b_{22}) \\
H^*(\Omega E_6;\mathbb{F}_3) &\cong \mathbb{F}_3[a_2]/(a_2^3)\otimes\Gamma(a_6, a_8, a_{10}, a_{14}, a_{16}, b_{22}) \\
H^*(\Omega E_7;\mathbb{F}_3) &\cong \mathbb{F}_3[a_2]/(a_2^3)\otimes\Gamma(a_6, a_{10}, a_{14}, a_{18}, b_{22}, a_{26}, a_{34}) \\
H^*(\Omega E_8;\mathbb{F}_3) &\cong \mathbb{F}_3[a_2]/(a_2^3)\otimes\mathbb{F}_3[a_6]/(a_6^3)\otimes\Gamma(a_{14}, a_{18}, b_{22}, a_{26}, a_{34}, a_{38}, a_{46}, b_{58})
\end{aligned}
$$

especially we have $\mathcal{P}^1 b_{22} = a_{26}$.

$E_2 = \mathrm{Cotor}_{H^*(\Omega G;\boldsymbol{F}_3)}(\boldsymbol{F}_3, \boldsymbol{F}_3)$

Theorem

$\mathrm{Cotor}_{H^*(\Omega G;\boldsymbol{F}_3)}(\boldsymbol{F}_3, \boldsymbol{F}_3)$ *of the exceptional Lie groups G are as follows.*

$$\mathrm{Cotor}_{H^*(\Omega G_2;\boldsymbol{F}_3)}(\boldsymbol{F}_3, \boldsymbol{F}_3) \cong E(x_3, x_{11})$$

$$\mathrm{Cotor}_{H^*(\Omega F_4;\boldsymbol{F}_3)}(\boldsymbol{F}_3, \boldsymbol{F}_3) \cong E(x_3) \otimes \mathbb{F}_3[\beta\mathcal{P}^1 x_3] \otimes E(\mathcal{P}^1 x_3, x_{11}, \mathcal{P}^1 x_{11}, z_{23})$$

$$\mathrm{Cotor}_{H^*(\Omega E_6;\boldsymbol{F}_3)}(\boldsymbol{F}_3, \boldsymbol{F}_3) \cong E(x_3) \otimes \mathbb{F}_3[\beta\mathcal{P}^1 x_3] \otimes E(\mathcal{P}^1 x_3, x_9, x_{11}, \mathcal{P}^1 x_{11}, x_{17}, z_{23})$$

$$\mathrm{Cotor}_{H^*(\Omega E_7;\boldsymbol{F}_3)}(\boldsymbol{F}_3, \boldsymbol{F}_3) \cong E(x_3) \otimes \mathbb{F}_3[\beta\mathcal{P}^1 x_3] \otimes E(\mathcal{P}^1 x_3, x_{11}, \mathcal{P}^1 x_{11}, x_{19}, z_{23}, x_{27}, x_{35}$$

$$\mathrm{Cotor}_{H^*(\Omega E_8;\boldsymbol{F}_3)}(\boldsymbol{F}_3, \boldsymbol{F}_3) \cong E(x_3) \otimes \mathbb{F}_3[\beta\mathcal{P}^1 x_3] \otimes E(\mathcal{P}^1 x_3) \otimes \mathbb{F}_3[\beta\mathcal{P}^3\mathcal{P}^1 x_3] \otimes E(x_{15}, x_{19}, z_{23}, x_{27}, x_{35}, x_{39}, x_{47}, z_{59})$$

especially we have $\mathcal{P}^1 z_{23} = x_{27}$.

Differentials and $H^*(P^m(\Omega G); \boldsymbol{F}_3)$

Then from information of $H^*(G; \boldsymbol{F}_3)$, we can analyze non trivial differentials of the Rothenberg–Steenrod spectral sequence converging to $H^*(G; \boldsymbol{F}_3)$ as follows:

$$
\begin{aligned}
d_3(z_{23}) &= (\beta\mathcal{P}^1 x_3)^3, \text{ for } G = F_4, E_6, E_7 \\
d_3(z_{59}) &= (\beta\mathcal{P}^3\mathcal{P}^1 x_3)^3, \text{ for } G = E_8
\end{aligned}
$$

Let $A = H^*(G; \boldsymbol{F}_3)$. Then for low m such as $1 \leq m \leq 3$, we have the following:

$$
H^*(P^m(\Omega G); \boldsymbol{F}_3) = A^{[m]} \oplus \sum_i z_{4i+3} \cdot A^{[m-1]} \oplus S_m \begin{cases} i = 5, \text{ for } G = F_4, E_6, E_7 \\ i = 5, 14, \text{ for } G = E_8 \end{cases}
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Module Category Weight for F_3 coefficients

Theorem

The module category weights with respect to $\mathbf{F}_3$ coefficients are as follows:

$$\begin{aligned} Mwgt(G_2; \mathbf{F}_3) &\geq 2, \\ Mwgt(F_4; \mathbf{F}_3) &\geq 8, \\ Mwgt(E_6; \mathbf{F}_3) &\geq 10. \\ Mwgt(E_7; \mathbf{F}_3) &\geq 13, \\ Mwgt(E_8; \mathbf{F}_3) &\geq 18. \end{aligned}$$

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Sketch of a proof for the E_7 case

From above Theorem, $\mathcal{P}^1 z_{23} = x_{27}$ in $H^*(P^1(\Omega E_7); \boldsymbol{F}_3)$.
Then $\mathcal{P}^1 z_{23} = x_{27}$ modulo S_2 in $H^*(P^2(\Omega E_7); \boldsymbol{F}_3)$.
Since S_2 is even-dimensional, the modulo S_2 is trivial and

$$\mathcal{P}^1 z_{23} = x_{27} \text{ in } H^*(P^2(\Omega E_7); \boldsymbol{F}_3).$$

So in $H^*(P^{12}(\Omega E_7); \boldsymbol{F}_3)$, we have

$$\mathcal{P}^1((\beta\mathcal{P}^1 x_3)^2 x_3 x_7 x_{11} x_{15} x_{19} z_{23} x_{35}) = (\beta\mathcal{P}^1 x_3)^2 x_3 x_7 x_{11} x_{15} x_{19} x_{27} x_{35}$$

Note that the filtration lengths of $\beta\mathcal{P}^1 x_3$ are 2.

Let $\phi_m : H^*(P^m(\Omega E_7); \boldsymbol{F}_p) \to H^*(E_7; \boldsymbol{F}_p)$ be a epimorphism which preserves all Steenrod actions and $\phi_m \circ (e_m)^* \cong 1_{H^*(E_7; \boldsymbol{F}_p)}$. Suppose that there are epimorphisms

$$\phi_{12} : H^*(P^{12}(\Omega E_7); \boldsymbol{F}_3) \to H^*(E_7; \boldsymbol{F}_3).$$

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H^*(P^{12}(\Omega E_7);\mathbf{F}_3) & \xrightarrow{\phi_{12}} & H^*(E_7;\mathbf{F}_3) \\
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\end{array}$$

Obviously this is a contradiction. So ϕ_{12} is not epimorphisms. This means that $(e_{12})^*$, can not be split monomorphisms of all Steenrod algebra module.
Hence we obtain that

$$Mwgt(E_7;\mathbf{F}_3) \geq 13.$$

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Combined with Toomer's result, we have the following conclusion:

G	$wgt(G;\boldsymbol{F}_3) - cup(G;\boldsymbol{F}_3)$		$Mwgt(G;\boldsymbol{F}_2) - wgt(G;\boldsymbol{F}_2)$		$Mwgt(G;\boldsymbol{F}_3) - wgt(G;\boldsymbol{F}_3)$
G_2	0	$\geq$	0	$\geq$	0
F_4	2	$\geq$	2	$\geq$	0
E_6	2	$\geq$	2	$\geq$	0
E_7	2	$\geq$	2	$\geq$	2
E_8	4	$\geq$	0	$\geq$	2

Combined with Toomer's result, we have the following conclusion:

G	$wgt(G; \mathbf{F}_3) - cup(G; \mathbf{F}_3)$	$Mwgt(G; \mathbf{F}_2) - wgt(G; \mathbf{F}_2)$	$Mwgt(G; \mathbf{F}_3) - wgt(G; \mathbf{F}_3)$
G_2	0	$\geq$ 0	$\geq$ 0
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E_6	2	$\geq$ 2	$\geq$ 0
E_7	2	$\geq$ 2	$\geq$ 2
E_8	4	$\geq$ 0	$\geq$ 2

Module Category Weight for *Spin*(n)

Theorem

The category weight of Spin(n) *with* $\boldsymbol{F}_2$ *coefficients is as follows:*

(a) $wgt(Spin(2^n); \boldsymbol{F}_2) = wgt(Spin(2^n + 1); \boldsymbol{F}_2) = 2^{n-1} \times n - 2^n + 2$,

(b) $wgt(Spin(2^n + 2k + 2); \boldsymbol{F}_2) = wgt(Spin(2^n + 2k + 1); \boldsymbol{F}_2) + 1$ *for all integer* $k \geq 0$,

(c) $wgt(Spin(2^n + 2k + 1); \boldsymbol{F}_2) = 2^{n-1} \times n - 2^n + 2 + \sum_{i=1}^{k} 2^{\nu_i} - k$ *for* $1 \leq k \leq 2^{n-1} - 1$ *where* ν_i, $i = 1, \cdots, k$, *is the positive integer such that* $2k + 2 \leq 2^{\nu_i}(2i - 1) \leq 4k$.

Theorem

For $n \geq 3$, *the module category weight is as follows:*

(a) $Mwgt(Spin(2^n + k); \boldsymbol{F}_2) \geq wgt(Spin(2^n + k); \boldsymbol{F}_2) + 2$ *for* $1 \leq k \leq 2^{n-1}$,

(b) $Mwgt(Spin(2^n + k); \boldsymbol{F}_2) = wgt(Spin(2^n + k); \boldsymbol{F}_2)$ *for* $2^{n-1} + 1 \leq k \leq 2^n$.

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Module Category Weight for $Spin(n)$

G	$wgt(G;\mathbf{F}_2)$	$Mwgt(G;\mathbf{F}_2)$	$cat(G)$
$Spin(3)$	1	1	1
$Spin(4)$	2	2	2
$Spin(5)$	2	2	3
$Spin(6)$	3	3	3
$Spin(7)$	5	5	5
$Spin(8)$	6	6	6
$Spin(9)$	6	8	8
$Spin(10)$	7	≥ 9	?
$Spin(11)$	9	≥ 11	?
$\vdots$	$\vdots$	$\vdots$	?

The End

Thank you!

The End

Thank you!