Semiclassical orthogonal polynomials: q-Laguerre and little q-Laguerre

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Plan of the talk:

- Orthogonal polynomials: a quick introduction (classical and semi-classical polynomials, ladder operators, examples)

- q-Laguerre orthogonal polynomials and generalizations

  We consider a semi-classical variation of the weight related to the q-Laguerre polynomials and study their recurrence coefficients. In particular, we obtain a second degree second order discrete equation which in particular cases can be reduced to either the qP_V or the qP_{III} equation.

- Little q-Laguerre orthogonal polynomials and generalizations

  We consider a semi-classical variation of the weight related to the little q-Laguerre polynomials and obtain a second order second degree discrete equation for the recurrence coefficients in the three-term recurrence relation.

- Perspectives and open problems

Orthogonal polynomials

For a sequence $(P_n)_{n\geq 0}$ of monic polynomials (of degree n in x)

$$P_n(x) = x^n + \gamma_n x^{n-1} + \dots . \tag{1}$$

orthogonal with respect to a positive measure μ with support on the real line

$$\int P_n(x)P_m(x)d\mu(x) = \zeta_n \delta_{n,m}, \quad \zeta_n > 0, \quad n,m = 0,1,2,\dots, \tag{2}$$

where $\delta_{n,m}$ is the Kronecker delta, the three-term recurrence relation takes the following form:

$$xP_n(x) = P_{n+1}(x) + \alpha_n P_n(x) + \beta_n P_{n-1}(x) \tag{3}$$

with the recurrence coefficients given by the following integrals

$$\alpha_n = \frac{1}{\zeta_n}\int xP_n^2(x)\,d\mu(x), \qquad \beta_n = \frac{1}{\zeta_{n-1}}\int xP_n(x)P_{n-1}(x)\,d\mu(x) \tag{4}$$

with $\beta_0 P_{-1} = 0$ and $P_0 = 1$.

For classical orthogonal polynomials (Hermite, Laguerre, Jacobi) one knows these recurrence coefficients explicitly in contrast to non-classical weights. The recurrence coefficients of semi-classical weights obey nonlinear recurrence relations, which, in many cases, can be identified as discrete Painlevé equations.

One of the useful characterizations of classical polynomials is a Pearson equation

$$[\sigma(x)w(x)]' = \tau(x)w(x),$$

where σ and τ are polynomials satisfying $\deg\sigma \leq 2$ and $\deg\tau = 1$ and $w(x)$ is the weight ($\int p_n(x)p_m(x)w(x)dx = \delta_{m,n}$).

In case of discrete polynomials or q-polynomials, the derivative in the Pearson equation is replaced by difference or q-difference operator.

Semi-classical orthogonal polynomials are defined as orthogonal polynomials for which the weight function $w(x)$ satisfies a Pearson equation with $\deg\sigma > 2$ or $\deg\tau \neq 1$.

Examples of discrete Painlevé equations

dP_I:

$$y_{n+1} + y_n + y_{n-1} = \frac{an+b}{y_n} + c;$$

dP_{II}:

$$y_{n+1} + y_{n-1} = \frac{(an+b)y_n + c}{1 - y_n^2};$$

qP_{III}

$$y_{n+1}y_{n-1} = \frac{cd(y_n - aq^n)(y_n - bq^n)}{(y_n - c)(y_n - d)};$$

dP_{IV}:

$$(y_{n+1} + y_n)(y_n + y_{n-1}) = \frac{(y_n^2 - \kappa^2)(y_n^2 - \mu^2)}{(x_n + z_n)^2 - \gamma^2}, \quad z_n = \alpha n + \beta.$$

Examples of semi-classical weights for orthogonal polynomials giving rise to discrete or q-discrete Painlevé equations

- $w(x) = |x|^\rho e^{-x^4}$ on $\mathbb{R}$ is related to dP_I;
- $w(k) = \frac{a^k}{(k!)^2}$ on $\mathbb{N}$ is related to dP_{II};
- $w(x) = x^\alpha e^{-x^2}$ on $\mathbb{R}^+$ is related to dP_{IV};
- $w(x) = (q^4x^4; q^4)_\infty$ on $\{\pm q^k | k \in \mathbb{N}\}$ is related to qP_I;
- $w(x) = |x|^\alpha (q^2x^2; q^2)_\infty (cq^2x^2; q^2)_\infty$ on $\{\pm q^k | k \in \mathbb{N}\}$ is related to αqP_V.

Ladder operators

[M.E.H. Ismail et al].

Let the weight function w be on the positive half line, so that the orthogonality condition is

$$\int_0^\infty P_n(x)P_m(x)w(x)dx = \zeta_n \delta_{n,m}. \tag{5}$$

Let us define the function u, called the potential, by the following formula:

$$u(x) = -\frac{D_{q^{-1}}w(x)}{w(x)}. \tag{6}$$

The q-difference operator is given by

$$(D_q f)(x) = \begin{cases} \frac{f(x)-f(qx)}{x(1-q)}, & x \neq 0, \\ f'(0), & x = 0. \end{cases}$$

Then the polynomials satisfy the following lowering equation:

$$D_q P_n(x) = A_n(x)P_{n-1}(x) - B_n(x)P_n(x),$$

where the functions $A_n(x)$ and $B_n(x)$ are given by

$$A_n(x) = \frac{1}{\zeta_n}\int_0^\infty \frac{u(qx)-u(y)}{qx-y} P_n(y)P_n(y/q)w(y)dy, \tag{7}$$

$$B_n(x) = \frac{1}{\zeta_{n-1}}\int_0^\infty \frac{u(qx)-u(y)}{qx-y} P_n(y)P_{n-1}(y/q)w(y)dy. \tag{8}$$

Furthermore, the following relations (compatibility conditions) hold:

$$B_{n+1}(x) + B_n(x) = (x-\alpha_n)A_n(x) + x(q-1)\sum_{j=0}^{n} A_j(x) - u(qx), \tag{9}$$

$$1 + (x-\alpha_n)B_{n+1}(x) - (qx-\alpha_n)B_n(x) = \beta_{n+1}A_{n+1}(x) - \beta_n A_{n-1}(x). \tag{10}$$

q-Laguerre orthogonal polynomials and generalizations

Let

$$(a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k).$$

The classical q-Laguerre weight is given by

$$w(x) = \frac{x^\alpha}{(-x;q)_\infty}.$$

We consider the recurrence coefficients of the generalized q-Laguerre polynomials for the weight function

$$w(x) = \frac{x^\alpha(-p_1/x;q)_\infty(-p_2/x;q)_\infty}{(-x^2;q^2)_\infty(-q^2/x^2;q^2)_\infty}, \tag{11}$$

where $x \in (0,+\infty)$, $|q| < 1$, $p_1 > 0$, $p_2 > 0$, $p_1p_2 < q^{2-\alpha}$, $\alpha \geq 0$. The case $p_1 + p_2 = 0$, $p_1p_2 = -p$ was considered by L. Boelen. It was shown that the recurrence coefficients are related to the q-discrete Painlevé equation qP_V. The proof was based on the compatibility relations for the ladder operators for orthogonal polynomials [Ismail et al].

Theorem*

The recurrence coefficients α_n and β_n in the three-term recurrence relation for monic polynomials

$$xP_n(x) = P_{n+1}(x) + \alpha_n P_n(x) + \beta_n P_{n-1}(x)$$

for the weight

$$w(x) = \frac{x^\alpha(-p_1/x;q)_\infty(-p_2/x;q)_\infty}{(-x^2;q^2)_\infty(-q^2/x^2;q^2)_\infty}$$

with $x \in (0,+\infty)$, $|q|<1$, $p_1>0$, $p_2>0$, $p_1p_2<q^{2-\alpha}$, $\alpha \geq 0$, can be expressed in terms of the function y_n, which satisfies the second order second degree discrete equation

$$(c_n^2 - b_nb_{n-1})^2 - a_na_{n-1}c_n(c_n^2+b_nb_{n-1}) - c_n^2(a_n^2b_{n-1}+a_{n-1}^2b_n) = 0,$$

with

$$\begin{aligned} a_n &= q^{n-1}(p_1+p_2), \\ b_n &= q^{2n+\alpha}(y_{n+1}y_n - p_1p_2q^{-\alpha-2}), \\ c_n &= q^{n-1}\frac{(y_n+q^{-\alpha})(y_n+p_1p_2q^{-2})}{y_n+q^{-n-\alpha}}, \end{aligned}$$

*[GF, C. Smet, *On the recurrence coefficients for generalized q-Laguerre polynomials*, accepted in JNMP].

where

$$\beta_n = q^{1-n}(y_n + q^{-n-\alpha})$$

and

$$q^{2\alpha+2n+2}(q^{\alpha+2}y_n y_{n+1} - p_1 p_2)\alpha_n^2 + (p_1+p_2)q^{\alpha+n+1}(q^2 + q^\alpha(p_1p_2 + q^2(y_n + y_{n+1})))\alpha_n$$
$$+(q^2 + q^\alpha(p_1p_2 + q^2(y_n + y_{n+1})))^2 = 0.$$

Remark. In particular, if $p_1 + p_2 = 0$ and $p = -p_1^2$, all a_i are zero and we obtain $c_n^2 = b_n b_{n-1}$, or in terms of y_n:

$$(y_n y_{n-1} - pq^{-\alpha-2})(y_n y_{n+1} - pq^{-\alpha-2}) = \frac{(y_n + q^{-\alpha})^2(y_n + pq^{-2})^2}{(q^{\alpha+n}y_n + 1)^2}.$$

This case was considered by L. Boelen and it was shown to be a particular case of qP_V after some change of variables.

If we take $p_1 = p_2 = 0$, i.e. a special case of the previous one, we get the equation

$$y_{n-1}y_{n+1} = \frac{(y_n + q^{-\alpha})^2}{(q^{n+\alpha}y_n + 1)^2}.$$

This is the q-discrete Painlevé equation qP_{III}.

Remarks on the proof

We use the technique of ladder operators. In this case

$$w(x/q) = \frac{q^{2-\alpha}}{(x+p_1)(x+p_2)}\,w(x),$$

The potential (17) is given by

$$u(x) = \frac{q}{1-q}\frac{1}{x} - \frac{q^{3-\alpha}}{1-q}\frac{1}{x(x+p_1)(x+p_2)}.$$

The functions $A_n(x)$ and $B_n(x)$ in, respectively, (15) and (16) are given by

$$A_n(x) = \frac{q^2}{1-q}\frac{T_n}{x(qx+p_1)(qx+p_2)} + \frac{q^{n+2}}{1-q}\frac{1}{(qx+p_1)(qx+p_2)},$$

where

$$T_n = q^{n-1}(p_1+p_2+\gamma_n - q\gamma_{n+1}) \tag{12}$$

$$\sum_{j=0}^{n} T_j = \frac{(p_1+p_2)}{q}\frac{1-q^{n+1}}{1-q} - q^n\gamma_{n+1}. \tag{13}$$

Similarly,

$$B_n(x) = -\frac{1}{x}\frac{1-q^n}{1-q} + \frac{q^2}{1-q}\frac{r_n}{(qx+p_1)(qx+p_2)} + \frac{q^2}{1-q}\frac{t_n}{x(qx+p_1)(qx+p_2)}$$

with

$$r_n = \frac{1}{\zeta_{n-1}}\int_0^\infty P_n(qu)P_{n-1}(u)w(u)du,$$

$$t_n = \frac{p_1+p_2}{q\zeta_{n-1}}\int_0^\infty P_n(qu)P_{n-1}(u)w(u)du + \frac{1}{\zeta_{n-1}}\int_0^\infty uP_n(qu)P_{n-1}(u)w(u)du.$$

Clearly, $r_0 = t_0 = 0$ and

$$r_n = (1-q)\gamma_n q^{n-1}. \tag{14}$$

Next one uses compatibility conditions and some tricks to reduce the number of unknowns.

Discrete q-orthogonal polynomials on an exponential lattice

The orthogonality relation is given by

$$\int_a^b p_k(x)p_n(x)w(x)d_qx = \delta_{k,n},$$

where $\delta_{k,n}$ is the Kronecker delta and the q-integral is defined by

$$\int_a^b f(x)d_qx = b(1-q)\sum_{n=0}^{\infty} q^n f(bq^n) - a(1-q)\sum_{n=0}^{\infty} q^n f(aq^n).$$

Here the weight function w is supported on the exponential lattice

$$\{aq^n,\ bq^n \,|\, n \in \mathbb{N}_0\}.$$

The classical examples include little q-Laguerre polynomials, which are orthogonal on the exponential lattice $\{q^k \,|\, k \in \mathbb{N}_0\}$ with respect to the weight function

$$w(x) = x^\alpha (qx;q)_\infty, \ \ \alpha > -1, \ \ q \in (0,1).$$

They can be written in terms of the basic hypergeometric function ${}_2\phi_1$ given by

$$ {}_2\phi_1(a_1, a_2; b_1; q; z) = \sum_{\ell=0}^{\infty} \frac{(a_1;q)_\ell (a_2;q)_\ell}{(b_1;q)_\ell} \frac{z^\ell}{(q;q)_\ell}. $$

Ladder operators

[Ismail et al].

Consider a weight function w on the exponential lattice $\{aq^n,\ bq^n \,|\, n \in \mathbb{N}_0\}$, such that $w(a/q) = w(b/q) = 0$ and the sequence of orthonormal polynomials $\{p_n\}$ of degree n with respect to this weight. Then the polynomials satisfy the following relation:

$$ D_q p_n(x) = A_n(x) p_{n-1}(x) - B_n(x) p_n(x) $$

with

$$ A_n(x) = a_n \int_a^b \frac{u(qx) - u(y)}{qx - y} p_n(y) p_n(y/q) w(y) d_q y, \tag{15} $$

$$ B_n(x) = a_n \int_a^b \frac{u(qx) - u(y)}{qx - y} p_n(y) p_{n-1}(y/q) w(y) d_q y. \tag{16} $$

Here the function u, called the potential, is defined by the following formula:

$$-u(qx)w(qx) = D_q w(x). \tag{17}$$

Furthermore, the following relations (compatibility conditions) hold:

$$B_n + B_{n+1} = (x - b_n)\frac{A_n}{a_n} + (q-1)x\sum_{j=0}^{n}\frac{A_j}{a_j} - u(qx), \tag{18}$$

$$a_{n+1}A_{n+1} - a_n^2\frac{A_{n-1}}{a_{n-1}} = (x - b_n)B_{n+1} - (qx - b_n)B_n + 1. \tag{19}$$

Relations (18), (19) are important in deriving nonlinear discrete equations for the recurrence coefficients, which in some cases can be further reduced to (q-)discrete Painlevé equations.

Main results [†]

- We study the recurrence coefficients for the weight functions supported on the exponential lattice $\{q^k \,|\, k \in \mathbb{N}_0\}$ and satisfying the q-difference equation (17) with

$$u(x) = \frac{k_1 q}{1-q}\frac{1}{x} + \frac{k_2 x + k_3}{1-q}, \quad k_1 \neq 0, \quad k_2 \neq 0, \tag{20}$$

  and conditions $w(0) = w(1/q) = 0$.

- To find out which weights can give rise to a potential of the form (20), we notice that it is sufficient if

$$\frac{w(x/q)}{w(x)} = Ax^2 + Bx + C$$

  for certain constants A, B, C, since an easy calculation shows that in that case the potential is given by (20) with $k_1 = 1 - C$, $k_2 = -Aq$ and $k_3 = -Bq$.

[†]GF, C. Smet, *On the recurrence coefficients of the generalized little q-Laguerre polynomials*, submitted.

If we define

$$
\begin{aligned}
v_1^\alpha(x) = x^\alpha, \quad v_2^c(x) &= (cx;q)_\infty, \quad v_3^c(x) = (cx^2;q^2)_\infty, \\
v_4^c(x) &= (c/x;q)_\infty, \quad v_5^c(x) = (c/x^2;q^2)_\infty,
\end{aligned}
$$

then

$$
\begin{aligned}
\frac{v_1^\alpha(x/q)}{v_1^\alpha(x)} = q^{-\alpha}, \quad \frac{v_2^c(x/q)}{v_2^c(x)} &= \frac{q-cx}{q}, \quad \frac{v_3^c(x/q)}{v_3^c(x)} = \frac{q^2-cx^2}{q^2}, \\
\frac{v_4^c(x/q)}{v_4^c(x)} &= \frac{x}{x-c}, \quad \frac{v_5^c(x/q)}{v_5^c(x)} = \frac{x^2}{x^2-c}.
\end{aligned}
$$

Hence it is clear which products of v_i lead to a weight for which the potential satisfies (20). These include, among others, the little q-Laguerre weight, products of rational functions and the weights above, the weights in the following examples and others.

- In the following we assume that the sequence of polynomials $\{p_n\}$ is orthonormal with respect to the weight function with potential (20) and hence the orthogonality relation takes the form

$$\int_0^1 p_m(x)p_n(x)w(x)d_qx = \delta_{m,n}.$$

Hence, the expressions (15) and (16) can be computed as follows:

$$A_n(x) = \frac{a_n R_n}{x(1-q)} + \frac{a_n k_2 q^{-n}}{1-q},$$

$$B_n(x) = \frac{r_n}{(1-q)x},$$

where

$$R_n = -k_1 \int_0^1 p_n(y)p_n(y/q)\frac{w(y)}{y}d_qy, \quad r_n = -a_n k_1 \int_0^1 p_n(y)p_{n-1}(y/q)\frac{w(y)}{y}d_qy.$$

The compatibility conditions give that the recurrence coefficients a_n, b_n appearing in the three-term recurrence relation for the weight supported on the exponential lattice $\{q^k \,|\, k \in \mathbb{N}_0\}$ with the potential satisfying (20) and conditions $w(0) = w(1/q) = 0$ can be expressed in terms of r_n, which is a solution of a complicated second order second degree discrete equation (with respect to n).

- If $k_3 = 0$ in (20) then the variable $x_n = (1 + r_n)(1 - k_1)^{-1/2}$ satisfies qP_V given by

$$(x_n x_{n-1} - 1)(x_n x_{n+1} - 1) = \frac{\gamma\delta q^{2n}(x_n - \alpha)(x_n - 1/\alpha)(x_n - \beta)(x_n - 1/\beta)}{(x_n - \gamma q^n)(x_n - \delta q^n)}, \tag{21}$$

with

$$\alpha = \beta = \gamma = \delta = \frac{1}{p}, \ \ p = \sqrt{1 - k_1}.$$

The initial conditions are given by

$$x_0 = \frac{1}{p} \qquad \text{and} \qquad x_1 = p - \frac{k_2}{qp}\left(\frac{\mu_1}{\mu_0}\right)^2.$$

The function x_n is related to the recurrence coefficients a_n and b_n of the orthogonal polynomials by

$$a_n^2 = \frac{q^n}{k_2}\left(p x_n - q^n\right) \tag{22}$$

and

$$b_n^2 = -\frac{q^{2n+1}(px_n + px_{n+1} - 1 - p^2)^2}{k_2 p^2 (x_n x_{n+1} - 1)}. \tag{23}$$

Example 1. In this example we consider the semi-classical little q-Laguerre weight

$$w(x) = x^{\alpha}(qx; q)_{\infty}(cqx; q)_{\infty}, \quad \alpha > 0, \tag{24}$$

on the positive exponential lattice $\{q^n \mid n \in \mathbb{N}_0\}$. The case $c = -1$ was considered by L. Boelen. The case $c = 0$ gives the little q-Laguerre weight (and, hence, the recurrence coefficients are known explicitly). We observe that $w(0) = w(1/q) = 0$.

The potential (17) is given by

$$u(x) = \frac{1}{1-q}\left(\frac{q}{x} - \frac{q^{1-\alpha}}{x} + q^{1-\alpha}(1+c) - cq^{1-\alpha}x\right)$$

and, hence, $k_1 = 1 - q^{-\alpha}$, $k_2 = -cq^{1-\alpha}$, $k_3 = (1+c)q^{1-\alpha}$ in (20). We assume that $c \neq 0$.

Since $k_3 = 0$ if and only if $c = -1$, we get that in this case the variable $x_n = q^{\alpha/2}(r_n + 1)$ satisfies

$$(x_n x_{n-1} - 1)(x_n x_{n+1} - 1) = \frac{q^{2n+\alpha}(x_n - q^{\alpha/2})^2(x_n - q^{-\alpha/2})^2}{(x_n - q^{n+\alpha/2})^2}, \tag{25}$$

which is a particular case of qP_V (21) (with $\alpha = \beta = \gamma = \delta = q^{\alpha/2}$).

Next we study initial conditions for the recurrence coefficients of the weight (24) for general c. We have

$$a_n^2 = \frac{1}{c}\, q^{n-1+\alpha/2} \left(q^{n+\alpha/2} - x_n \right)$$

and for $n \geq 1$

$$c b_n^2 (x_n x_{n+1} - 1) = q^{2n} \left(1 + q^{\alpha} - q^{\alpha/2}(x_n + x_{n+1}) \right)^2 .$$

We also get that

$$b_0^2 = \frac{1}{c} \left(q^{\alpha/2} x_1 - 1 \right) .$$

Recalling that $b_0 = \mu_1/\mu_0$, we immediately get that the initial values are given by $x_0 = q^{\alpha/2}$ (since $r_0 = 0$) and

$$x_1 = q^{-\alpha/2}\left(1 + c\frac{\mu_1^2}{\mu_0^2}\right), \tag{26}$$

where μ_k is the k-th moment of the weight (24). In fact, we can also calculate μ_k by definition and get

$$\mu_k = (1-q)(q;q)_\infty (cq;q)_\infty \, {}_2\phi_1(0,0;cq;q;q^{\alpha+k+1}),$$

where ${}_2\phi_1$ is the basic hypergeometric function. Note that for $c = q^\nu$ the last expression (up to a factor) can be written in terms of the modified q-Bessel function

$$I_\nu^{(1)}(z,q) = \frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty}(z/2)^\nu {}_2\phi_1(0,0;q^{\nu+1};q;z^2/4)$$

with $z = 2q^{(\alpha+k+1)/2}$.

Example 2. In this example we consider another semi-classical generalization of the little q-Laguerre weight:

$$w(x) = x^\alpha \frac{(qx;q)_\infty \left(\frac{c_1}{x};q\right)_\infty \left(\frac{qx}{c_1};q\right)_\infty}{\left(\frac{c_2}{x};q\right)_\infty}, \;\; \alpha > 0, \; c_1 < 0, \; c_2 < 0 \tag{27}$$

on the positive exponential lattice $\{q^n \mid n \in \mathbb{N}_0\}$. The case where $c_1 = c_2 = 1/c$ gives the weight from the previous example. Again, it is clear that $w(0) = w(1/q) = 0$. It is easy to calculate that for this weight we get

$$k_1 = 1 - \frac{c_2}{c_1} q^{-\alpha}, \qquad k_2 = -\frac{q^{1-\alpha}}{c_1} \qquad \text{and} \qquad k_3 = \frac{c_2 + 1}{c_1} q^{1-\alpha}.$$

As mentioned earlier, to obtain a discrete Painlevé equation, we need that $k_3 = 0$, hence $c_2 = -1$. We have

$$x_n = \sqrt{-c_1} q^{\alpha/2}(r_n + 1) \qquad \text{and} \qquad a_n^2 = -c_1 q^{n+\alpha-1}(1 - q^n + r_n)$$

where x_n satisfies (21) with

$$\alpha = \beta = \gamma = \delta = \sqrt{-c_1} q^{\frac{\alpha}{2}}.$$

As for the initial conditions, we find that

$$x_0 = \sqrt{-c_1} q^{\frac{\alpha}{2}}$$

and

$$x_1 = \sqrt{-c_1} q^{\frac{\alpha}{2}} \left(-1 + \frac{c_2}{c_1} q^{-\alpha} + \frac{b_0^2}{c_1 q^{\alpha}} \right) + 1.$$

Moreover, $b_0 = \mu_1/\mu_0$, and it is easily seen that the k'th moment of this weight is given by

$$\mu_k = (1-q)(q;q)_\infty \left(\frac{q}{c_1};q\right)_\infty \frac{(c_1;q)_\infty}{(c_2;q)_\infty} {}_2\phi_1\left(0,0;\frac{q}{c_2};q;\frac{c_1}{c_2}q^{\alpha+1+k}\right).$$

Conclusions and open problems

We have shown that it is possible to study recurrence coefficients in the three-term recurrence relation for simultaneously a large class of weights by using the technique of ladder operators. The crucial point is to consider the potential (20) with parameters. This allows us to obtain a second degree second order discrete equation, which in some particular cases, can be further reduced to the discrete Painlevé equation.

It is an interesting open problem to try to classify the weights which lead to the appearance of the discrete Painlevé equations for the recurrence coefficients.

Thank you very much for your attention!