ON THE MEAN-VALUE PROPERTY FOR THE DUNKL POLYHARMONIC FUNCTIONS

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1. Introduction

In the lecture we first derive differential relations between Dunkl spherical and solid means of functions. Next we use the relations to give short proof of an analogue of the Beckenbach-Reade theorem stating that equality of spherical and solid means of a continuous function implies its harmonicity. Taking a full advantage of the relations we also give inductive proofs of the Dunkl solid and spherical mean-value properties for the Dunkl polyharmonic functions and their converses in arbitrary dimension.

2. Preliminaries

Recall that for a nonzero vector $\alpha \in \mathbb{R}^n \setminus \{0\}$ the reflection with respect to the orthogonal to α hyperplane H_α is given by

$$\sigma_\alpha(x) = x - \frac{2\langle \alpha, x \rangle}{\|\alpha\|^2}\,\alpha, \qquad x \in \mathbb{R}^n,$$

where $\langle\ ,\ \rangle$ is the euclidian scalar product on $\mathbb{R}^n$ and $\|\cdot\|$ the associated norm. A finite set R of nonzero vectors is called a root system if $R \cap \mathbb{R}\alpha = \{\pm\alpha\}$ and $\sigma_\alpha R = R$ for all $\alpha \in R$. The reflections σ_α with α in a given root system R generate a finite group $W \subset O(n)$, called the reflection group associated

with R. For a fixed $\beta \in \mathbb{R}^n \setminus \bigcup_{\alpha \in R} H_\alpha$ one can decompose $R = R_+ \cup R_-$ where $R_\pm = \{\alpha \in R : \pm\langle \alpha, \beta \rangle > 0\}$; vectors in R_+ are called positive roots.
A function $\kappa : R \to \mathbb{R}$ is called a multiplicity function if it is invariant under the action of the associated reflection group W. Its index γ is defined by

$$\gamma = \sum_{\alpha \in R_+} \kappa(\alpha).$$

Throughout the paper we shall assume that $\kappa \geq 0$ and $\gamma > 0$.

The Dunkl operators T_j, $j = 1, \ldots, n$, associated with a root system R and a multiplicity function κ were introduced by C. Dunkl [6] as

$$T_j f(x) = \frac{\partial}{\partial x_j} f(x) + \sum_{\alpha \in R_+} \kappa(\alpha) \frac{f(x) - f(\sigma_\alpha(x))}{\langle \alpha, x \rangle} \alpha_j \quad \text{for} \quad f \in C^1(\mathbb{R}^n).$$

Clearly, T_j is well defined for $f \in C^1(\Omega)$ where Ω is a W-invariant open subset of $\mathbb{R}^n$ and it reduces to $\frac{\partial}{\partial x_j} f$ if f is W-invariant.

The Dunkl Laplacian Δ_κ is defined as a sum of squares of the operators T_j, $j = 1, \ldots, n$, i.e.,

$$\Delta_\kappa f = \sum_{j=1}^{2} T_j^2 f \quad \text{for} \quad f \in C^2(\Omega).$$

A simple computation leads to

$$\begin{aligned}\Delta_\kappa f(x) &= \Delta f(x) \\ &\quad + \sum_{\alpha \in R_+} \kappa(\alpha) \left(\frac{2\langle \nabla f(x), \alpha \rangle}{\langle \alpha, x \rangle} - \|\alpha\| \, \frac{f(x) - f(\sigma_\alpha(x))}{\langle \alpha, x \rangle^2} \right).\end{aligned}$$

Here Δ and ∇ denote the usual Laplacian and gradient, resp.

The Dunkl intertwining operator V_κ acting on polynomials was defined in [7] by

$$T_j V_\kappa f = V_\kappa \frac{\partial}{\partial x_j} f \quad \text{for} \quad j = 1, \ldots, n \quad \text{and} \quad V_\kappa 1 = 1.$$

The operator V_κ extends to a topological isomorphism of $C^\infty(\mathbb{R}^n)$ onto itself [19]. In general there is no explicite description of V_κ, but M. Rösler has shown [15, Th. 1.2, Cor. 5.3] that for any $x \in \mathbb{R}^n$ there exists a unique probability measure μ_x such that

$$V_\kappa f(x) = \int_{\mathbb{R}^n} f(y) d\mu_x(y). \tag{1}$$

Moreover, the support of μ_x is contained in $\mathrm{ch}(Wx)$ – the convex hull of the set $\{gx : g \in W\}$, $\mu_{rx}(U) = \mu_x(r^{-1}U)$ and $\mu_{gx}(U) = \mu_x(g^{-1}U)$ for $r > 0$, $g \in W$ and a Borel set $U \subset \mathbb{R}^n$. Note that by (1), V_κ can be extended to continuous functions and $|V_\kappa(f)(x)| \le \sup_{y\in\mathrm{ch}(Wx)} |f(y)|$; the extension is a topological isomorphism of $C(\mathbb{R}^n)$.

The Dunkl translation operators τ_x, $x \in \mathbb{R}^n$, are defined on $C(\mathbb{R}^n)$ by

$$\tau_x f(y) = (V_\kappa)_x (V_\kappa)_y \big[V_\kappa^{-1} f(x+y)\big] \quad \text{for} \quad y \in \mathbb{R}^n. \tag{2}$$

A more suggestive notation $f(x *_\kappa y) := \tau_x f(y)$ will be also used. Note that $\tau_0 f = f$ and $\tau_y f(x) = \tau_x f(y)$ for $x, y \in \mathbb{R}^n$.

The Poisson kernel for the Dunkl Laplacian Δ_κ is defined in [8] for $\|x\| < 1$ and $\|y\| \leq 1$ by[1]

$$P_\kappa(x, y) = V_\kappa \left[\frac{1 - \|x\|^2}{(1 - 2\langle x, \cdot \rangle + \|x\|)^{\gamma + n/2}} \right](y), \qquad (3)$$

The kernel $P_\kappa(x, y)$ is non-negative, bounded by 1 and it has the reproducing property for Dunkl harmonic functions on the unit ball. Furthermore it is used as a tool to solve the Dirichlet problem for the Dunkl Laplacian. Namely it holds

[1] $P_\kappa(x, y) = P(h_\kappa^2; y, x)$ where $P(h_\kappa^2; \cdot, \cdot)$ is defined in [8, page 190].

Theorem A [12, Theorem A, Prop. 2.1]. *Let u be a continuous function on the unit sphere $S(0,1)$. For $\|x\|<1$ set*

$$P_\kappa[u](x)=\frac{1}{d_\kappa}\int_{S(0,1)}P_\kappa(x,y)u(y)\,\omega_\kappa(y)dS(y). \qquad (4)$$

Then $P_\kappa[u]$ is Δ_κ-harmonic on $B(0,1)$, extends continuously to $\overline{B}(0,1)$ and $P_\kappa[u]=u$ on $S(0,1)$. Furthermore, $P_\kappa[u]$ is the unique Δ_κ-harmonic function on $B(0,1)$ which extends continuously to u on $S(0,1)$.

Here and in the sequel $\omega_\kappa(y)=\prod_{\alpha\in R_+}|\langle\alpha,y\rangle|^{2\kappa_\alpha}$.

Since $P_\kappa(0,y)=1$ for $\|y\|\leq 1$ for a function u continuous on $\overline{B}(0,1)$ and Dunkl harmonic in $B(0,1)$ we get

$$u(0)=\frac{1}{d_\kappa}\int_{S(0,1)} u(y)\,\omega_\kappa(y)dS(y). \qquad (5)$$

More generally, if a function u is continuous on $\overline{B}(0,1)$ and Δ_κ-harmonic in $B(0,1)$, then for any $x\in B(0,1)$ and $0<$

$R < 1 - \|x\|$ the mean value formula holds [12, Theorem C]

$$u(x) = \frac{1}{d_\kappa} \int_{S(0,1)} \tau_x u(Ry)\, \omega_\kappa(y) dS(y) \tag{6}$$

$$= \frac{1}{d_\kappa R^{2\gamma+n-1}} \int_{S(0,R)} \tau_x u(z)\, \omega_\kappa(z) dS(z). \tag{7}$$

The converse statement was also stated [12, Theorem C] under the assumption that u is a C^2 function. However its proof uses a formula from [14, Corollary 4.18] valid only for C^4 functions. We conjecture that it holds under the assumption of continuity of u.

Conjecture 1 *Let u be a function continuous in the ball $\overline{B}(0,1)$. If for any $x \in B(0,1)$ there is a sequence $r_j > 0$, $j \in \mathbb{N}$, converging to zero such that*

$$u(x) = \frac{1}{d_\kappa} \int_{S(0,1)} \tau_x u(r_j y)\, \omega_\kappa(y) dS(y) \tag{8}$$

$$= \frac{1}{d_\kappa r_j^{2\gamma+n-1}} \int_{S(0,r_j)} \tau_x u(z)\, \omega_\kappa(z) dS(z) \tag{9}$$

for every $j \in \mathbb{N}$, then u is Δ_κ-harmonic in $B(0,1)$.

However it holds

Weil's Lemma for Dunkl laplacian. [13, Theorem 2.1] *If* $u \in L^{\infty}_{loc}(\mathbb{R}^n)$ *and* $\Delta_\kappa(\omega_\kappa u) = 0$ *in* $\mathcal{D}'(\mathbb{R}^n)$*, then there exists a Dunkl harmonic function* v *such that* $u = v$ *a.e. in* $\mathbb{R}^n$*.*

3. Relations between spherical and solid means

Let u be a smooth function on $\Omega \subset \mathbb{R}^n$. For any $x \in \Omega$ and $0 < R < \mathrm{Dist}(\mathring{x}, \partial\Omega)$ we denote by $N^D(u;\, x, R)$ the Dunkl integral mean of u over the Dunkl sphere $S^D(x, R) = \tau_{\mathring{x}} S(0, R)$,

$$N^D(u;\, x, R) = \frac{1}{d_\kappa} \int_{S(0,1)} \tau_{\mathring{x}} u(Ry)\, \omega_\kappa(y) dS(y) \tag{10}$$

$$= \frac{1}{d_\kappa} \int_{S(0,1)} u(x *_\kappa Ry)\, \omega_\kappa(y) dS(y) \tag{11}$$

where

$$d_\kappa = \int_{S(0,1)} \omega_\kappa(y)\, dS(y).$$

It was proved in [16, Theorem 4.1] that the spherical mean operator $u \mapsto N^D(u;\, x, R)$ can be represented in the form

$$N^D(u;\, x, R) = \int_{\mathbb{R}^n} u(y) d\mu^\kappa_{x,R}(y)$$

where $\mu^\kappa_{x,R}$ is a probability measure with support in $\bigcup_{g\in W}\{y \in \mathbb{R}^n : |y - gx| \le R\}$. Hence $N^D(u;\, x, R)$ is well defined for a continuous function u.

Since ω_κ is homogenous of degree 2γ we also have

$$N^D(u, \mathring{x};\ R) = \frac{1}{d_\kappa R^{2\gamma+n-1}} \int_{S(0,R)} \tau_{\mathring{x}} u(z)\, \omega_\kappa(z) dS(z). \quad (12)$$

Note that using the spherical coordinates we get

$$\begin{aligned}
\int_{B(0,1)} \omega_\kappa(x)\, dx &= \int_0^1 \left(\int_{S(0,t)} \omega_\kappa(x) dS(x) \right) dt \\
&= \int_0^1 \left(\int_{S(0,1)} \omega_\kappa(y) dS(y) \right) t^{2\gamma+n-1} dt \\
&= \frac{d_\kappa}{2\gamma+n}.
\end{aligned}$$

So we can define the Dunkl integral mean of u over the closed Dunkl ball $B^D(\mathring{x}, R) = \tau_{\mathring{x}} B(0, R)$ by

$$M^D(u, \mathring{x};\ R) = \frac{2\gamma + n}{d_\kappa} \int_{B(0,1)} \tau_{\mathring{x}} u(Ry)\, \omega_\kappa(y) dy \tag{13}$$

$$= \frac{2\gamma + n}{d_\kappa R^{2\gamma+n}} \int_{B(0,R)} \tau_{\mathring{x}} u(z)\, \omega_\kappa(z) dz. \tag{14}$$

If there is no risk of misunderstanding the notation is shorten to $M^D(u; R)$ and $N^D(u; R)$.

Recall the Green formula for the Dunkl laplacian.

GREEN FORMULA FOR Δ_κ. ([14, Theorem 4.11]). *Let Ω be a bounded W-invariant regular open set in $\mathbb{R}^n$ containing the origin and $u \in C^2(\Omega)$. Then for any closed ball $\overline{B}(0,R) \subset \Omega$ it holds*

$$\int_{B(0,R)} \Delta_\kappa u(x)\, \omega_\kappa(x) dx = \int_{S(0,R)} \frac{\partial u(x)}{\partial \eta}\, \omega_\kappa(x) dS(x) \quad (15)$$

where $\frac{\partial u}{\partial \eta}$ denotes the external normal derivative of u.

The relations between $M^D(u;R)$ and $N^D(u;R)$ are given in the following result:

Lemma 1. *Let u be a continuous function on a domain $\Omega \subset \mathbb{R}^n$. Then for any $\mathring{x} \in \Omega$ and $0 < R < \mathrm{Dist}(\mathring{x}, \partial\Omega)$ it holds*

$$\Big(\frac{R}{2\gamma + n}\frac{\partial}{\partial R} + 1\Big) M^D(u, \mathring{x}; R) = N^D(u, \mathring{x}; R). \tag{16}$$

If we further assume that u has continuous derivatives up to second order, then

$$\frac{2\gamma + n}{R}\frac{\partial}{\partial R} N^D(u, \mathring{x}; R) = M^D(\Delta_\kappa u, \mathring{x}; R). \tag{17}$$

PROOF. Assume $\mathring{x} = 0$. Using (14), the fact that $\tau_0 u = u$, the spherical coordinates and (11) we compute

$$\begin{aligned}
M^D(u,0;R) &= \frac{2\gamma+n}{d_\kappa R^{2\gamma+n}} \int_{B(0,R)} u(x)\,\omega_\kappa(x)dx \\
&= \frac{2\gamma+n}{d_\kappa R^{2\gamma+n}} \int_0^R \left(\int_{S(0,s)} u(x)\,\omega_\kappa(x)dS(x) \right) ds \\
&= \frac{2\gamma+n}{R^{2\gamma+n}} \int_0^R N^D(u,0;s)\, s^{2\gamma+n-1} ds.
\end{aligned}$$

Hence by the Leibniz rule

$$\begin{aligned}
&\frac{\partial}{\partial R} M^D(u,0;R) \\
&= -\frac{(2\gamma+n)^2}{R^{2\gamma+n+1}} \int_0^R N^D(u,0;s)\, s^{2\gamma+n-1} ds \\
&+ \frac{2\gamma+n}{R^{2\gamma+n}} N^D(u,0;R)\, R^{2\gamma+n-1} \\
&= \frac{2\gamma+n}{R}\Big(N^D(u,0;R) - M^D(u,0;R)\Big),
\end{aligned}$$

which proves (16).

To show (17) we differentiate under the integral sign to get

$$\begin{aligned}
\frac{\partial}{\partial R} N^D(u,0;R) &= \frac{\partial}{\partial R} \left(\frac{1}{d_\kappa} \int_{S(0,1)} u(Ry)\, \omega_\kappa(y) dS(y) \right) \\
&= \frac{1}{d_\kappa} \int_{S(0,1)} \langle \nabla u(Ry), y \rangle\, \omega_\kappa(y) dS(y) \\
&= \frac{1}{d_\kappa R^{2\gamma+n-1}} \int_{S(0,R)} \langle \nabla u(z), \tfrac{z}{R} \rangle\, \omega_\kappa(z) dS(z)
\end{aligned}$$

Note that the external normal vector to $S(0,R)$ at a point $z \in S(0,R)$ is $\eta = \frac{z}{R}$ and $\langle \nabla u, \eta \rangle = \frac{\partial u}{\partial \eta}$.

So applying the Green formula (15) we get

$$\begin{aligned}\frac{\partial}{\partial R}N^D(u,0;R) &= \frac{1}{d_\kappa R^{2\gamma+n-1}}\int_{S(0,R)}\frac{\partial u(z)}{\partial \eta}\,\omega_\kappa(z)dS(z)\\ &= \frac{1}{d_\kappa R^{2\gamma+n-1}}\int_{B(0,R)}\Delta_\kappa u(z)\,\omega_\kappa(z)dz\\ &= \frac{R}{2\gamma+n}M^D(\Delta u,0;R)\end{aligned}$$

which implies (17). $\square$

By (16) and (17) it follows

Corollary 1. *Let $u \in C^2(\Omega)$. Then for any $\mathring{x} \in \Omega$ and $0 < R < \operatorname{Dist}(\mathring{x}, \partial\Omega)$ it hold*

$$M^D(\Delta_\kappa u, \mathring{x}; R) = \Big(\frac{\partial^2}{\partial R^2} + \frac{2\gamma + n + 1}{R}\frac{\partial}{\partial R}\Big) M^D(u, \mathring{x}; R) \tag{18}$$

and

$$N^D(\Delta_\kappa u, \mathring{x}; R) = \Big(\frac{\partial^2}{\partial R^2} + \frac{2\gamma + n - 1}{R}\frac{\partial}{\partial R}\Big) N^D(u, \mathring{x}; R). \tag{19}$$

By the first part of Lemma 1 we get an analogue of the Beckenbach-Reade theorem for the Dunkl harmonic functions.

COROLLARY 2 *Let u be smooth on a domain $\Omega \subset \mathbb{R}^d$. If for any $\mathring{x} \in \Omega$ and $0 < R < \mathrm{Dist}(\mathring{x}, \partial\Omega)$ it holds*

$$M^D(u, \mathring{x}; R) = N^D(u, \mathring{x}; R), \tag{20}$$

then u is Dunkl harmonic on Ω.

PROOF. The assumption (20) and (18) imply that $\frac{\partial}{\partial R} M^D(u, \mathring{x}; R) = 0$. So for any $\mathring{x} \in \Omega$, $M^D(u, \mathring{x}; R)$ is a constant equal to $u(\mathring{x})$. The converse to the mean-value property for Dunkl harmonic functions implies that u is Dunkl harmonic. $\square$

3. Mean-value properties for Dunkl polyharmonic functions

Let $m \in \mathbb{N}$. A function $u \in C^{2m}(\Omega)$ defined on a W-invariant open set $\Omega \subset \mathbb{R}^n$ is called m-Dunkl harmonic if it is a solution of the m-times iteration of the Dunkl operator, i.e., $\Delta_\kappa^m u = 0$. One of the most trivial examples is given by an even power of the Euclidean distance from the origin

EXAMPLE. Let $u(x) = r^{2m}(x)$ with $m \in \mathbb{N}_0$, where $r(x) = \left(\sum_{i=1}^n x_i^2\right)^{1/2}$ is the radius function. Since u is W-invariant

$\Delta_\kappa u$ reduces to

$$\Delta_\kappa u(x) = \Delta u(x) + 2 \sum_{\alpha \in R_+} \kappa(\alpha) \frac{\langle \nabla u(x), \alpha \rangle}{\langle \alpha, x \rangle}.$$

Since $\Delta u = 2m(n+2m-2)\, r^{2m-2}$ and $\nabla u = 2m\, x \cdot r^{2m-2}$ we get $\Delta_\kappa u = 2m(n+2m+2\gamma-2)\, r^{2m-2}$. So u is $(m+1)$-Dunkl harmonic, $\Delta^i u(0) = 0$ for $i = 0, 1, ..., m-1$ and

$$\begin{aligned} \Delta_\kappa^m u(0) &= 2m(2m-2)\cdots 2 \times (n+2m+2\gamma-2)\cdots(n+2\gamma)\, r^0(0) \\ &= 4^m \left(\gamma + \frac{n}{2}\right)_m m!, \end{aligned}$$

where for $a \in \mathbb{R}$, $(a)_0 = 1$ and $(a)_i = a(a+1)\cdots(a+i-1)$ for $i \in \mathbb{N}$. On the hand using the spherical coordinates and the fact that ω is homogeneous of degree 2γ we get

$$\begin{aligned} M^D(u,0;\ R) &= \frac{2\gamma+n}{d_\kappa R^{2\gamma+n}} \int_{B(0,R)} u(y)\omega_\kappa(y)dy \\ &= \frac{2\gamma+n}{d_\kappa R^{2\gamma+n}} \int_0^R \int_{S(0,s)} \|y\|^{2m}\omega_\kappa(y)dS(y)ds \\ &= \frac{2\gamma+n}{d_\kappa R^{2\gamma+n}} \int_0^R d_\kappa s^{2m+2\gamma+n-1} ds \\ &= \frac{2\gamma+n}{2m+2\gamma+n} R^{2m}. \end{aligned} \tag{21}$$

Hence

$$M^D(u,0;\ R) = \frac{\Delta_\kappa^m u(0)}{4^m\left(\gamma+\frac{n}{2}+1\right)_m m!}\cdot R^{2m}.$$

The above example suggests a form of an expansion of $M(u,x;\ R)$ for a polyharmonic function u into powers of the radius R of the ball $B(x,R)$.

THEOREM 1 (Mean-value property for solid means). *Let $m \in \mathbb{N}_0$ and let Ω be a W-invariant domain in $\mathbb{R}^n$. If $u \in C^{2m+2}(\Omega)$ and $\Delta_\kappa^{m+1} u = 0$ in Ω, then for any $\mathring{x} \in \Omega$ and $0 < R < \operatorname{Dist}(\mathring{x}, \partial\Omega)$ it holds*

$$M^D(u, \mathring{x};\, R) = \sum_{k=0}^{m} \frac{\Delta_\kappa^k u(\mathring{x})}{4^k \left(\gamma + \frac{n}{2} + 1\right)_k k!} \cdot R^{2k}. \tag{22}$$

PROOF. Clearly, by the mean-value property for Dunkl harmonic functions (22) holds for $m = 0$. Inductively assume that the theorem holds for a fixed $m \in \mathbb{N}_0$. Let $v \in C^{2m+4}(\Omega)$ and $\Delta^{m+2} v = 0$. Then $u = \Delta v \in C^{2m+2}(\Omega)$ satisfies $\Delta^{m+1}u = 0$ and so (22) holds. But by (18)

$$\frac{2\gamma+n}{R}\frac{\partial}{\partial R}\Big(\frac{R}{2\gamma+n}\frac{\partial}{\partial R}+1\Big)M^D(v;R) = M^D(\Delta_\kappa v;R)$$

$$= M^D(u;R) = \sum_{k=0}^{m} \frac{\Delta_\kappa^k u(\mathring{x})}{4^k\big(\gamma+\frac{n}{2}+1\big)_k k!}\cdot R^{2k}.$$

So after one integration

$$\Big(\frac{R}{2\gamma+n}\frac{\partial}{\partial R}+1\Big)M^D(v;R)$$

$$= \sum_{k=0}^{m} \frac{\Delta_\kappa^k u(\mathring{x})}{4^k(2\gamma+n)\big(\gamma+\frac{n}{2}+1\big)_k k!}\cdot\frac{R^{2k+2}}{2k+2}+c. \tag{23}$$

Note that the general solution of $\Big(\frac{R}{2\gamma+n}\frac{\partial}{\partial R}+1\Big)M^D(v;R)=0$ is $CR^{-2\gamma-d}$ and a particular solution of

$$\Big(\frac{R}{2\gamma+n}\frac{\partial}{\partial R}+1\Big)M^D(v;R)=\frac{\Delta_\kappa^k u(\mathring{x})}{4^k(2\gamma+n)\big(\gamma+\frac{n}{2}+1\big)_k k!}\cdot\frac{R^{2k+2}}{2k+2}.$$

is A_kR^{2k+2}, where
$A_k\big(\frac{2k+2}{2\gamma+n}+1\big)=\Delta_\kappa^k u(\mathring{x})\cdot\big[4^k(2\gamma+n)(2k+2)\big(\gamma+\frac{n}{2}+1\big)_k k!\big]^{-1}$.
So

$$A_k=\frac{\Delta_\kappa^k u(\mathring{x})}{4^{k+1}\big(\gamma+\frac{n}{2}+1\big)_{k+1}(k+1)!}.$$

Hence the general solution of (23) is

$$M^D(v;R) = CR^{-2\gamma-n} + \sum_{k=0}^{m} \frac{\Delta_\kappa^k u(\mathring{x})}{4^{k+1}\left(\gamma+\frac{n}{2}+1\right)_{k+1}(k+1)!} \cdot R^{2k+2} + c.$$

Finally note that $\lim_{R\to 0} M^D(v,R) = v(\mathring{x})$ and $\lim_{R\to 0} R^{2\gamma+n} M^D(v,R) = 0$. So $c = v(\mathring{x})$, $C = 0$ and

$$M^D(v;R) = v(\mathring{x}) + \sum_{k=0}^{m} \frac{\Delta^{k+1} v(\mathring{x})}{4^{k+1}\left(\gamma+\frac{n}{2}+1\right)_{k+1}(k+1)!} \cdot R^{2k+2}$$

which proves Theorem 1. $\square$

By Theorem 1 and relation (16) we get

COROLLARY 3 (Mean-value property for spherical means). *Under the assumptions of Theorem 1 for any* $\mathring{x} \in \Omega$ *and* $0 < R < \operatorname{dist}(\mathring{x}, \partial\Omega)$ *it holds*

$$N^D(u, \mathring{x}; R) = \sum_{k=0}^{m} \frac{\Delta_\kappa^k u(\mathring{x})}{4^k \left(\gamma + \frac{n}{2}\right)_k k!} \cdot R^{2k}. \tag{24}$$

THEOREM 2 (Converse to the mean value property for spherical means). *Let $m \in \mathbb{N}_0$ and Ω be a domain in $\mathbb{R}^d$. If $u \in C^{2m+2}(\Omega)$ and for all $\mathring{x} \in \Omega$ and R small enough (24) holds, then $\Delta^{m+1}u = 0$ in Ω.*

PROOF. Clearly the theorem holds for $m = 0$ (see [2, Theorem 1.20]). Fix $n \in \mathbb{N}$ and assume that the theorem holds for $m < n$. We shall prove that it holds for $m = n$. To this end take $v \in C^{2n+2}(\Omega)$ and assume that for any $\mathring{x} \in \Omega$ and R small enough (24) holds with $m = n$ and $u = v$. Set $u = \Delta v$. Then $u \in C^{2n}(\Omega)$. By (17) and (24) with $m = n$

and $u = v$ we get

$$\begin{aligned}
M^D(u, \mathring{x}; R) &= \frac{2\gamma + n}{R} \frac{\partial}{\partial R} N^D(v, \mathring{x}; R) \\
&= \sum_{k=1}^{n} \frac{(2\gamma + n) 2k \Delta_\kappa^k v(\mathring{x})}{4^k \left(\gamma + \frac{n}{2}\right)_k k!} \cdot R^{2k-2} \\
&= \sum_{k=0}^{n-1} \frac{\Delta_\kappa^k u(\mathring{x})}{4^k \left(\gamma + \frac{n}{2} + 1\right)_k k!} \cdot R^{2k}.
\end{aligned}$$

So for any $\mathring{x} \in \Omega$ and R small enough

$$\begin{aligned}
N^D(u, \mathring{x}; R) &= \Big(\frac{R}{2\gamma + n}\frac{\partial}{\partial R} + 1\Big) M^D(u, \mathring{x}; R) \\
&= \sum_{k=0}^{n-1} \Big(\frac{2k}{2\gamma + n} + 1\Big) \frac{\Delta_\kappa^k u(\mathring{x})}{4^k \big(\gamma + \frac{n}{2} + 1\big)_k k!} \cdot R^{2k} \\
&= \sum_{k=0}^{n-1} \frac{\Delta_\kappa^k u(\mathring{x})}{4^k \big(\gamma + \frac{n}{2}\big)_k k!} \cdot R^{2k}.
\end{aligned}$$

Hence by the inductive assumption $\Delta_\kappa^n u = \Delta_\kappa^{n+1} v = 0$. $\square$

By Theorem 2 and relation (17) we get

Corollary 4 (Converse to the mean value property for solid means). *Under the assumptions of Theorem 2 if* $u \in C^{2m+2}(\Omega)$ *and for all* $\mathring{x} \in \Omega$ *and* R *small enough it holds*

$$M^D(u, \mathring{x};\, R) = \sum_{k=0}^{m} \frac{\Delta_\kappa^k u(\mathring{x})}{4^k \left(\gamma + \frac{n}{2} + 1\right)_k k!} \cdot R^{2k},$$

then $\Delta^{m+1} u = 0$ *in* Ω.

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