

Some fundamental lemmas for linear ordinary differential equations

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§ Weyl algebra, polynomial ring, Linear ODE

$W[x]$: Weyl algebra of one variable

generated by x and $\partial = \frac{d}{dx}$ with $[\partial, x] = 1$

$W(x) := \mathbb{C}(x) \otimes W[x]$

Linear differential equations: finitely generated $W(x)$ -modules

$W^{(\epsilon)}[x], W^{(\epsilon)}(x) : [\partial, x] = \epsilon \in \mathbb{C}$

Finitely generated $W^{(0)}(x)$ -module \Rightarrow Theory of elementary divisors

$W^{(\epsilon)}(x)$: Euclidian ring

$$P = a_m(x)\partial^m + a_{m-1}(x)\partial^{m-1} + \cdots + a_0(x)$$

$$Q = b_n(x)\partial^n + b_{n-1}(x)\partial^{n-1} + \cdots + b_0(x), b_n \neq 0.$$

$$m \geq n \Rightarrow \text{ord}(P - \frac{a_m(x)}{b_n(x)}\partial^{m-n}Q) \leq m - 1$$

$$P = SQ + R \quad (\text{ord } R < \text{ord } Q) \quad \text{division}$$

Euclid algorithm \Rightarrow GCD, LCM

$$\begin{aligned}
\begin{pmatrix} P_1 \\ P_2 \end{pmatrix} &= \begin{pmatrix} S_1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} P_2 \\ P_3 \end{pmatrix} = \begin{pmatrix} S_1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} S_2 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} P_3 \\ P_4 \end{pmatrix} \\
&= \begin{pmatrix} S_1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} S_2 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} S_{N-1} & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} P_N \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} P_N \\ 0 \end{pmatrix} \\
\begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} &:= \begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & -S_{N-1} \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ 1 & -S_1 \end{pmatrix}
\end{aligned}$$

$$P_1 = AP_N, \quad P_2 = CP_N, \quad A'P_1 + B'P_2 = P_N, \quad T := C'P_2 = -D'P_2$$

$$W^{(\epsilon)}(x)P_1 + W^{(\epsilon)}(x)P_2 = W^{(\epsilon)}(x)P_N \quad (\text{GCD})$$

$$W^{(\epsilon)}(x)P_1 \cap W^{(\epsilon)}(x)P_2 = W^{(\epsilon)}(x)T \quad (\text{LCM})$$

$$\operatorname{ord} P_1 + \operatorname{ord} P_2 = \operatorname{ord} P_N + \operatorname{ord} T$$

$$P = \begin{pmatrix} P_{1,1} & \cdots & P_{1,n} \\ \vdots & \dots & \vdots \\ P_{m,1} & \cdots & P_{m,n} \end{pmatrix} \in M(m, n; W^{(\epsilon)}(x))$$

By elementary transformations of rows and columns

$$\epsilon = 0 : P \rightsquigarrow \begin{pmatrix} Q_1 & & & \\ & \ddots & & \\ & & Q_k & \\ & & & 0 \end{pmatrix} \quad (Q_{j+1} \in W^{(0)}Q_j, \ j = 0, 1, \dots)$$

\Rightarrow Elementary divisors

Theorem. If $\epsilon \neq 0$, then $P \rightsquigarrow \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & Q \end{pmatrix}$

Corollary. Any finitely generated left $W(x)$ -module is isomorphic to $W(x)^\ell \oplus (W(x)/W(x)Q)$.

In particular, any determined system has a cyclic vector.

Proof. As in the case when $\epsilon = 0$,

$$P \rightsquigarrow \begin{pmatrix} Q_1 & 0 & \cdots & 0 \\ 0 & Q_{2,2} & \cdots & Q_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & Q_{m,2} & \cdots & Q_{m,n} \end{pmatrix} \quad (\text{ord } Q_1 \leq \text{ord } Q_{i,j} \text{ or } Q_{i,j} = 0)$$

If $\exists Q_{i,j} \neq 0$, we may assume $Q_{2,2} \neq 0$.

$$Q_{2,2} = \partial^k + a_{k-1}(x)\partial^{k-1} + \cdots + a_0(x)$$

$$\begin{pmatrix} Q_1 & 0 & \cdots & 0 \\ 0 & Q_{2,2} & \cdots & Q_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & Q_{m,2} & \cdots & Q_{m,n} \end{pmatrix} \rightsquigarrow \begin{pmatrix} Q_1 & 0 & \cdots & 0 \\ Q_{2,2}x^\nu & Q_{2,2} & \cdots & Q_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ Q_{m,2}x^\nu & Q_{m,2} & \cdots & Q_{m,n} \end{pmatrix}$$

$$\rightsquigarrow \begin{pmatrix} Q_1 & 0 & \cdots & 0 \\ Q_{2,2}x^\nu - SQ_1 & Q_{2,2} & \cdots & Q_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ Q_{m,2}x^\nu & Q_{m,2} & \cdots & Q_{m,n} \end{pmatrix}$$

$$\Rightarrow x^{k-\nu}Q_{2,2}x^\nu = R_k\nu^k + R_{k-1}\nu^{k-1} + \cdots + R_1\nu + R_0 \in W(x)Q_1$$

$$(x^{k-\nu}\partial^\ell x^\nu = x^{k-\ell} \prod_{j=0}^{\ell-1} (\vartheta + \nu - j) = x^{k-\ell} (\nu^\ell + r_{1,\ell}(\vartheta)\nu^{\ell-1} + \cdots))$$

$$\Rightarrow R_k (= 1) \in W(x)Q_1 \Rightarrow Q_1 = 1. \quad (\vartheta = x\partial)$$

Hukuhara-Turrittin ($W^{(\epsilon)}(x)$)

$P = a(x)(\vartheta - \lambda_1(x)) \cdots (\vartheta - \lambda_n(x))$ with $\lambda_j(x) \in \mathbb{C}((x^{\frac{1}{q}}))$

$\vartheta := x\partial$

A single linear differential equation of order n is isomorphic to

$$(\vartheta - \lambda'_\nu(x))^{m_\nu} u_\nu = 0 \quad (\nu = 1, \dots, k, m_1 + \cdots + m_k = n)$$

$$\epsilon = 0 \Rightarrow \lambda_j(x) \in \mathbb{C}((x^{\frac{1}{q}})) \quad \text{as } \overline{\mathcal{R}_0} \otimes W[x]\text{-modules}$$

$$\epsilon \neq 0 \Rightarrow \lambda_j(x) \in \mathbb{C}[x^{-\frac{1}{q}}] \quad \text{as } \overline{\mathbb{C}((x))} \otimes W[x]\text{-modules}$$

Poincaré-Hukuhara, ...

\exists Solutions with asymptotics corresponding to formal solutions

Stokes coefficients (Birkoff, ...)

Generalized Riemann scheme, Fuchs relation,

Deligne-Simpson problem, ...

Riemann scheme when $\epsilon = 0$:

$$y := \partial$$

$a_n(x)y^n + a_{n-1}(x)y^{n-1} + \cdots + a_0(x) = 0$: a plane algebraic curve

$y = \lambda_j(x) = c_{j,0}x^{r_{j,0}} + c_{j,1}x^{r_{j,1}} + \cdots$ ($j = 1, \dots, n$) at $x = 0$

$$r_{j,0} < r_{j,1} < \cdots$$

$\bar{\lambda}_j(x) = \sum_{r_{j,\nu} < 0} c_{j,\nu}x^{r_{j,\nu}}$: characteristic exponents

$x = 0$: Singular point (w.r.t. y) $\Leftrightarrow \exists j$ such that $\bar{\lambda}_j \neq 0$

$r_{j,0} \geq -1 \Leftrightarrow$ regular singularity

Generalized Riemann scheme, Spectral type,

Deligne-Simpson problem, Kac-Moody root system, ...

Fuchsian case \Rightarrow can be solved as in the case $\epsilon = 0$

Irregular singularities, Ramifications

§ Connection coefficients and middle convolutions

A key lemma for Regular singularities

$$(I_c^\mu(u))(x) := \frac{1}{\Gamma(\mu)} \int_c^x u(t)(x-t)^{\mu-1} dt.$$

Lemma. Suppose $u(x) \in C([0, 1])$, $\operatorname{Re} a \geq 0$ and $\operatorname{Re} \mu > 0$.

$$(T_{a,b}^\mu u)(x) := x^{-a-\mu}(1-x)^{-b-\mu} I_0^\mu x^a (1-x)^b u(x),$$

$$(S_{a,b}^\mu u)(x) := x^{-a-\mu} I_0^\mu x^a (1-x)^b u(x)$$

$$\operatorname{Re} b + \operatorname{Re} \mu < 0 \Rightarrow T_{a,b}^\mu u \in C([0, 1]), \quad T_{a,b}^\mu(u)(0) = \frac{\Gamma(a+1)}{\Gamma(a+\mu+1)} u(0)$$

$$\frac{T_{a,b}^\mu(u)(1)}{T_{a,b}^\mu(u)(0)} = \frac{u(1)}{u(0)} \frac{\Gamma(a+\mu+1)\Gamma(-\mu-b)}{\Gamma(a+1)\Gamma(-b)}$$

$$\operatorname{Re} b + \operatorname{Re} \mu > 0 \Rightarrow S_{a,b}^\mu u \in C([0, 1]), \quad S_{a,b}^\mu(u)(0) = \frac{\Gamma(a+1)}{\Gamma(a+\mu+1)} u(0)$$

$$\frac{S_{a,b}^\mu(u)(1)}{S_{a,b}^\mu(u)(0)} = \frac{1}{u(0)} \frac{\Gamma(a+\mu+1)}{\Gamma(\mu)\Gamma(a+1)} \int_0^1 t^a (1-t)^{b+\mu-1} u(t) dt$$

Applying the lemma to the solution

$$u_0^{\lambda_0+\mu}(x) = \int_0^x t^{\lambda_0} (1-t)^{\lambda_1} \left(\prod_{j=2}^{p-1} \left(1 - \frac{t}{c_j}\right)^{\lambda_j} \right) (x-t)^{\mu-1} dt$$

of the Jordan-Pochhammer equation with the Riemann scheme

$$\left\{ \begin{array}{ccccccc} x = 0 & c_1 = 1 & \cdots & c_j & \cdots & c_p = \infty \\ [0]_{(p-1)} & [0]_{(p-1)} & \cdots & [0]_{(p-1)} & \cdots & [1-\mu]_{(p-1)} \\ \lambda_0 + \mu & \lambda_1 + \mu & \cdots & \lambda_j + \mu & \cdots & -\sum_{\nu=0}^{p-1} \lambda_\nu - \mu \end{array} \right\}$$

($p = 2 \Rightarrow$ Gauss hypergeometric), we have

$$c(0:\lambda_0 + \mu \rightsquigarrow 1:\lambda_1 + \mu) = \frac{\Gamma(\lambda_0 + \mu + 1)\Gamma(-\lambda_1 - \mu)}{\Gamma(\lambda_0 + 1)\Gamma(-\lambda_1)} \prod_{j=2}^{p-1} \left(1 - \frac{1}{c_j}\right)^{\lambda_j}$$

$$c(0:\lambda_0 + \mu \rightsquigarrow 1 : 0) = \frac{\Gamma(\lambda_0 + \mu + 1)}{\Gamma(\mu)\Gamma(\lambda_0 + 1)} \int_0^1 t^{\lambda_0} (1-t)^{\lambda_1 + \mu - 1} \prod_{j=2}^{p-1} \left(1 - \frac{t}{c_j}\right)^{\lambda_j} dt$$

Suppose $\operatorname{Re} a \geq 0$ and $0 < \operatorname{Re} \mu < -\operatorname{Re} b$. Then

$$\begin{aligned}
& \Gamma(\mu) T_{a,b}^\mu(u)(x) \\
&= x^{-a-\mu} (1-x)^{-b-\mu} \int_0^x t^a (1-t)^b (x-t)^{\mu-1} u(t) dt \quad (t = xs_1, 0 < x < 1) \\
&= (1-x)^{-b-\mu} \int_0^1 s_1^a (1-s_1)^{\mu-1} (1-xs_1)^b u(xs_1) ds_1 \\
&= \int_0^1 s_1^a \left(\frac{1-s_1}{1-x} \right)^\mu \left(\frac{1-xs_1}{1-x} \right)^b u(xs_1) \frac{ds}{1-s_1} \\
&= \int_0^1 (1-s_2)^a \left(\frac{s_2}{1-x} \right)^\mu \left(1 + \frac{xs_2}{1-x} \right)^b u(x-xs_2) \frac{ds_2}{s_2} \quad (s_1 = 1-s_2) \\
&= \int_0^{\frac{1}{1-x}} (1-s(1-x))^a s^\mu (1+xs)^b u(x-x(1-x)s) \frac{ds}{s} \quad (s_2 = (1-x)s).
\end{aligned}$$

When $0 \leq s_1 < 1$ and $0 \leq x \leq \frac{2}{3}$,

$$\left| s_1^a (1-s_1)^{\mu-1} (1-xs_1)^b u(xs_1) \right| \leq \max\{(1-s_1)^{\operatorname{Re} \mu - 1}, 1\} 3^{-\operatorname{Re} b} \max_{0 \leq t \leq 1} |u(t)|$$

$$\Rightarrow T_{a,b}^\mu(u)(x) \in C([0, \frac{2}{3}]).$$

Suppose $\operatorname{Re} a \geq 0$ and $0 < \operatorname{Re} \mu < -\operatorname{Re} b$. Then

$$\begin{aligned}
& \Gamma(\mu) T_{a,b}^\mu(u)(x) \\
&= x^{-a-\mu} (1-x)^{-b-\mu} \int_0^x t^a (1-t)^b (x-t)^{\mu-1} u(t) dt \quad (t = xs_1, 0 < x < 1) \\
&= (1-x)^{-b-\mu} \int_0^1 s_1^a (1-s_1)^{\mu-1} (1-xs_1)^b u(xs_1) ds_1 \\
&= \int_0^1 s_1^a \left(\frac{1-s_1}{1-x} \right)^\mu \left(\frac{1-xs_1}{1-x} \right)^b u(xs_1) \frac{ds}{1-s_1} \\
&= \int_0^1 (1-s_2)^a \left(\frac{s_2}{1-x} \right)^\mu \left(1 + \frac{xs_2}{1-x} \right)^b u(x-xs_2) \frac{ds_2}{s_2} \quad (s_1 = 1-s_2) \\
&= \int_0^{\frac{1}{1-x}} (1-s(1-x))^a s^\mu (1+xs)^b u(x-x(1-x)s) \frac{ds}{s} \quad (s_2 = (1-x)s).
\end{aligned}$$

When $\frac{1}{2} \leq x \leq 1$ and $0 < s \leq \frac{1}{1-x}$,

$$\left| (1-s(1-x))^a s^{\mu-1} (1+xs)^b u(x-x(1-x)s) \right| \leq s^{\operatorname{Re} \mu - 1} (1 + \frac{s}{2})^{\operatorname{Re} b} \max_{0 \leq t \leq 1} |u(t)|$$

$$\Rightarrow T_{a,b}^\mu(u)(x) \in C([\frac{1}{2}, 1])$$

Suppose $\operatorname{Re} a \geq 0$ and $0 < \operatorname{Re} \mu < -\operatorname{Re} b$. Then

$$\begin{aligned}
& \Gamma(\mu) T_{a,b}^\mu(u)(x) \\
&= x^{-a-\mu} (1-x)^{-b-\mu} \int_0^x t^a (1-t)^b (x-t)^{\mu-1} \textcolor{blue}{u(t)} dt \quad (t = xs_1, \ 0 < x < 1) \\
&= (1-x)^{-b-\mu} \int_0^1 s_1^a (1-s_1)^{\mu-1} (1-xs_1)^b u(xs_1) ds_1 \\
&= \int_0^1 s_1^a \left(\frac{1-s_1}{1-x} \right)^\mu \left(\frac{1-xs_1}{1-x} \right)^b u(xs_1) \frac{ds}{1-s_1} \\
&= \int_0^1 (1-s_2)^a \left(\frac{s_2}{1-x} \right)^\mu \left(1 + \frac{xs_2}{1-x} \right)^b u(x-xs_2) \frac{ds_2}{s_2} \quad (s_1 = 1-s_2) \\
&= \int_0^{\frac{1}{1-x}} (1-s(1-x))^a s^\mu (1+xs)^b u(x-x(1-x)s) \frac{ds}{s} \quad (s_2 = (1-x)s).
\end{aligned}$$

$$T_{a,b}^\mu(u)(0) = \frac{1}{\Gamma(\mu)} \int_0^1 (1-s_2)^a s_2^\mu u(0) \frac{ds_2}{s_2} = \frac{\Gamma(a+1)}{\Gamma(a+\mu+1)} u(0),$$

$$T_{a,b}^\mu(u)(1) = \frac{1}{\Gamma(\mu)} \int_0^\infty s^\mu (1+s)^b u(1) \frac{ds}{s} = \frac{\Gamma(-\mu-b)}{\Gamma(-b)} u(1).$$

Similarly (and more easily) we have the claim for $S_{a,b}^\mu$

$$u(x) \mapsto x^{\lambda_{0,1}} \prod_{j=1}^{p-1} \left(1 - \frac{x}{c_j}\right)^{\lambda_{j,1}} \int_{c_0}^x s^{-\lambda_{0,1}} \prod_{j=1}^{p-1} \left(1 - \frac{s}{c_j}\right)^{-\lambda_{j,1}} (x-s)^{\sum_{k=1}^p \lambda_{k,1}} u(s) ds$$

$$\begin{aligned} \{\lambda_{\mathbf{m}}\} &= \{[\lambda_{j,\nu}]_{(m_{j,\nu})}\}_{\substack{0 \leq j \leq p \\ 1 \leq \nu \leq n_j}} = \begin{cases} x = c_j & (j = 0, \dots, p-1) \\ & [\lambda_{j,1}]_{(m_{j,1})} \\ & [\lambda_{j,\nu}]_{(m_{j,\nu})} \end{cases} \begin{cases} \infty \\ [\lambda_{p,1}]_{(m_{p,1})} \\ [\lambda_{p,\nu}]_{(m_{p,\nu})} \end{cases} \\ &\mapsto \{\lambda'_{\mathbf{m}'}\} = \{[\lambda'_{j,\nu}]_{(m'_{j,\nu})}\}_{\substack{0 \leq j \leq p \\ 1 \leq \nu \leq n_j}} \\ &= \begin{cases} x = c_j & (j = 0, \dots, p-1) \\ & [\lambda_{j,1}]_{(m_{j,1}-d)} \\ & [\lambda_{j,\nu} + \sum_{k=0}^p \lambda_{k,1} - 1]_{(m_{j,\nu})} \end{cases} \begin{cases} \infty \\ [\lambda_{p,1} - 2 \sum_{k=0}^p \lambda_{k,1} + 2]_{(m_{p,1}-d)} \\ [\lambda_{p,\nu} - \sum_{k=0}^p \lambda_{k,1} + 1]_{(m_{p,\nu})} \end{cases} \\ &d = m_{0,1} + \dots + m_{p,1} - (p-1) \operatorname{ord} \mathbf{m} \end{aligned}$$

If $m_{0,n_0} = 1$ and $n_0 > 1$ and $n_1 > 1$, then

$$\frac{c'(c_0 : \lambda'_{0,n_0} \rightsquigarrow c_1 : \lambda'_{1,n_1})}{\Gamma(\lambda'_{0,n_0} - \lambda'_{0,1} + 1) \cdot \Gamma(\lambda'_{1,1} - \lambda'_{1,n_1})} = \frac{c(c_0 : \lambda_{0,n_0} \rightsquigarrow c_1 : \lambda_{1,n_1})}{\Gamma(\lambda_{0,n_0} - \lambda_{0,1} + 1) \cdot \Gamma(\lambda_{1,1} - \lambda_{1,n_1})}$$

+ Weyl group of Kac-Moody root space \Rightarrow Connection coefficients