Singularities of
first order nonlinear ordinary differential equations
and contact geometry

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Example. In a neighborhood of the origin $x = 0$,

$$u - \frac{1}{4}\left(\frac{du}{dx}\right)^2 = f(x)$$

We may assume $u(0) = 0$ by $u \mapsto u - u(0)$.

$C := u'(0)$. By $u \mapsto u - Cx$, we may assume $u(0) = u'(0) = 0$

$$u - \frac{C}{2}\frac{du}{dx} - \frac{1}{4}\left(\frac{du}{dx}\right)^2 = h(x) \ (:= f(x) - Cx + \frac{C^2}{4})$$

- $C \neq 0 \ \Rightarrow$ non-characteristic (Cauchy-Kowalevsky)
- $C = 0 \ \Rightarrow \ f(0) = f'(0) = 0$,

  $f(x) = f_2 x^2 + f_3 x^3 + \cdots, \quad u(x) = u_2 x^2 + u_3 x^3 + \cdots$

  $u_2 - u_2^2 = f_2 \ \Rightarrow \ u_2 = \frac{1}{2}(1 \pm \sqrt{1 - 4f_2})$ ($\exists\, 2$ u_2's)

  $f_2 \neq 0$ or $u_2 \neq 0 \ \Rightarrow$ Regular singularity (formal sol. converges).

  Generic u_2 (char. exp. : $\frac{1}{u_2} \notin \mathbb{Z}_{>1}$) $\Rightarrow \ \exists_1$ sol.

Generic $f_2 \ \Rightarrow$ two (formal and covergent) solutions

$u_2 = 0 \qquad \Rightarrow$ Irregular singularity

First order partial differential equations with one unknown function

Hamilton-Jacobi, Cauchy-Kowalevsky

Singular (characteristic) cases:

[O, 1973] : linear differential equations (Cauchy-Kowalevsky)

Miyake, Shiraiwa, Ouchi, . . .

: nonlinear differential equations $\dashrightarrow$ (multi-)summability

Perturbations of linear differential equations

[O, 1974] : nonlinear differential equation (Hamilton-Jacobi)

$\dashrightarrow$ (multi-)summability ?

$\dashrightarrow$ real category (ex, C^∞) ?

$\dashrightarrow$ Boundary value porblems with regular singularities

[Kashiwara-O, 1977]

$$f_k(\frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_n}, x_1, \ldots, x_n, u) = 0 \quad (k = 1, \ldots, N)$$

Put $z := u$, $p_j := \frac{\partial u}{\partial x_j} \Rightarrow$

$$\mathcal{M} : f_k(p, x, z) = 0 \quad (k = 1, \ldots, N)$$

f_k : analytic at the origin of $\mathbb{C}^{2n+1}$.

Problem : Study solutions of $\mathcal{M}$ through the origin of $\mathbb{C}^{2n+1}$

$0 \in V_f := \{(p, x, z) \mid f_k(p, x, z) = 0 \quad (k = 1, \ldots, N)\} \subset \mathbb{C}^{2n+1}$

$0 \in S_{z(x)} := \{(p, x, z) \mid p_j(x) = \frac{\partial z(x)}{\partial x_j}, \ z = z(x), \ x \in \mathbb{C}^n\} \subset V_f$

(p, x, z)-space : a contact manifold with the 1-form

$$\omega := dz - p_1 dx_1 - p_2 dx_2 - \cdots - p_n dx_n$$

$\omega|_{S_{z(x)}} (= d(z(x)) - p_1(x) dx_1 - \cdots) = 0$ and $\dim S_{z(x)} = n$

$\Rightarrow S_{z(x)}$ is an Lagrangean submanifold

$\pi : \mathbb{C}^{2n+1} \ni (p, x, z) \mapsto x \in \mathbb{C}^n$

$Sol := \{L \subset V_f \mid 0 \in L :$ Lagrangean and π_L are submersions$\}$

$= \{S_{z(x)} \mid z(x)$ are solutions of $\mathcal{M}$ with $z(0) = \frac{dz}{dx_j}(0) = 0\}$

Compativility condititions are given by Lagrange bracket

$$[f, g] := \sum_{j=1}^{n} \Big(\frac{\partial f}{\partial x_j} + p_j \frac{\partial f}{\partial z}\Big) \frac{\partial g}{\partial x_j} - \sum_{j=1}^{n} \Big(\frac{\partial g}{\partial x_j} + p_j \frac{\partial g}{\partial z}\Big) \frac{\partial f}{\partial x_j}$$

$z(x)$ is a solution $\;\Rightarrow\;$ $[f_k, f_\ell](p, x, z) = 0$

$\bar{V}$: the maximal subset of V_f such that

$$g_1|_{V_f} = g_2|_{V_f} = 0 \Rightarrow [g_1, g_2]|_{V_f} = 0.$$

$I_0 = \langle f_k \rangle \subset I_1 \subset I_2 \subset \cdots \quad I_{\ell+1} := \sqrt{\langle g_1,\ [g_1, g_2] \mid g_k \in I_\ell \rangle}$

$\bar{V} \not\ni 0 \Rightarrow Sol = \emptyset$ (incompatible).

Suppose $\bar{V}$ is non-singular.

$D(V) := \{(p,x,z)\,;\, \omega|_V(p,x,z) = 0\}$: degenerate points

$D(V) = \emptyset \Rightarrow$ By a suitable contact transformation $\mathcal{T}$ (classical theory)

$$\mathcal{T} : V \mapsto \{p_1 = \cdots = p_d = 0\} \quad (\mathcal{T}^*\omega = \phi(x,p,z)\omega,\ \ \phi(0) \neq 0)$$

Theorem [O '74]. Assume $D(V) \neq \emptyset \Rightarrow$

$$\exists \mathcal{T} : V \mapsto \{p_1 = \cdots = p_{d-1} = z + h(p_d, \ldots, p_n, x_d, \ldots, x_n) = 0\},$$

$$V \underset{\text{contact}}{\sim} V' \Leftrightarrow \omega|_V \underset{\text{coord.}}{\sim} \omega|_{V'}$$

Then we may assume $d = 1$ (a single equation).

$$h(p,x) = h_1(p,x) + h_2(p,x) + \cdots \quad (h_j(p,x) : \text{homog. of degree } j)$$

$\operatorname{grad}_p h_1 \neq 0 \Rightarrow$ non-charactersitic

We assume $\operatorname{grad}_p h_1 = 0$. Then if $\operatorname{grad}_x h_1 \neq 0$, $Sol = \emptyset$.

Hence we assume $h_1 = 0$.

Theorem [O '74]. Assume $h_1 = 0$. When $h_2(x,p)$ is generic, then there are just 2^n (formal and convergent) solutions.

generic: Siegel condition or Poincaré condition for linearlizability of a degenerate vector field at 0

$$H_{z+h(p,x)} = \sum_{j=1}^{n} \Big(\frac{\partial h}{\partial x_j} + p_j \Big) \frac{\partial}{\partial p_j} - \sum_{j=1}^{n} \frac{\partial h}{\partial p_j} \frac{\partial}{\partial x_j}$$

Proof. Conditions so that $H_{z+h(p,x)}$ is linearlizable

$\Rightarrow$ $\exists \mathcal{T} : z + h \mapsto z + h_2$ $\quad$ ($\Rightarrow$ easy to analyze).

$X = \sum_{\nu=1}^{N} a_\nu(t) \frac{\partial}{\partial t_\nu}$ $\quad (a_\nu(0) = 0)$, $\quad A := \Big(\frac{\partial a_i}{\partial t_j}(0) \Big) \in M(N, \mathbb{C})$

$\lambda_1, \ldots, \lambda_N$: eigenvalues of A.

Poincaré : $\exists \theta$ such that $\theta < \arg \lambda_j < \theta + \pi$ and (C)

(C) $\qquad \sum_\nu \alpha_\nu \lambda_\nu \neq \lambda_j \qquad (\alpha \in \mathbb{Z}^N_{\geq 0},\ |\alpha| \geq 2,\ \forall j)$

Siegel : $\exists K > 0$ such that A is diagonizable and (C) and

$$|\sum_\nu \alpha_\nu \lambda_\nu - \lambda_j| \geq |\alpha|^{-K} \quad (\alpha \in \mathbb{Z}^N_{\geq 0},\ |\alpha| \geq 2,\ \forall j)$$

$\Rightarrow$ is proved by the facts

1. $\exists$ formal contact transf. $\hat{\mathcal{T}} : z + h \mapsto z + h_2$

2. Any formal coordinate transf. (with some initial cond.) keeping H_{z+h_2} invariant is analytic. Apply this to $\hat{\mathcal{T}} \circ \mathcal{T}^{-1}$.

Remark. Generic (degenerate) case $\Rightarrow D(V) = \{0\}$.

$$h_2(p, x) = \langle p, Up\rangle + 2\langle p, Vx\rangle + \langle x, Wx\rangle \Rightarrow A = \begin{pmatrix} 2^tV + I_n & 2W \\ -2U & -2V \end{pmatrix}$$

A is non-singular $\Rightarrow D(V) = \{0\}$.

1. is proved by using a generating function Ω of a symplectic transf.

$$g(q, y) := \langle q, Wq\rangle + 2\langle q, ({}^tV + I_n)y\rangle + \langle y, Uy\rangle,$$

$$\begin{cases} \Omega = h(-\frac{\partial\Omega}{\partial x}, x) - g(y, \frac{\partial\Omega}{\partial y}), \\ \Omega = \langle x, y\rangle + \text{higher order terms.} \end{cases}$$

$$p_j = \frac{\partial\Omega}{\partial x_j},\ q_j = \frac{\partial\Omega}{\partial y_j} \ \Rightarrow\ (p, x) \mapsto (q, y),\ h \mapsto g\ (\mapsto h_2)$$

Under the condition (Poincaré) or (Siegel),

$$\{u(x) \mid u + h(\tfrac{\partial u}{\partial x}, x) = 0\} \underset{\text{contact}}{\overset{\sim}{\to}} \{v(x) \mid v + h_2(\tfrac{\partial v}{\partial x}, x) = 0\}$$

Here $u(x)$ and $v(x)$ vanish at 0 with their first derivatives.

The solution $v(x)$ is a homogeneous polynomial of degree 2.

Any formal solution $v(x)$ or $u(x)$ converges.

The number of the solutions may be 0 or infinite.

When the eigenvalues of A are mutually different, then

$$\#\{0 \in \text{Lagrangean} \subset \bar{V}\} = 2^n$$

and the number of solutions is at most 2^n and at least one.
($\Leftarrow$ The solution of $u = \sum_{j=0}^{n} C_j x_j \frac{\partial u}{\partial x_j}$ is trivial if C_j are generic).

The number of solutions is easily determined by A or h_2 by the condition of the submersion.

The problem (for $h_2(p,x) \mod$ contact transformations) is related to characterize

$$\{A \in M(n,\mathbb{C})\}/\sim$$

Here

$$A \sim A' \overset{\text{def}}{\Leftrightarrow} \exists G \in GL(n,\mathbb{C}) \text{ such that } {}^tGAG = A'$$

In our case, ${}^tA - A$ is non-singular. ($\Leftarrow$ Some elementary devisors)

Books by Gantmacher, Mal'cev

[Yaglom 1950] ($\mathbb{R}$ case)

Most degenerate case:

Theorem [O '74]. If $\dim D(V) \geq n$, then $D(V)$ is nonsingular and

$$\exists \mathcal{T} : V \mapsto \{p_1 = \cdots = p_z = z = 0\}$$

In this case we call V is maximally degenerate.

Some interesting cases:

$$u - \sum_{j=1}^{\ell} x_j \frac{\partial u}{\partial x_j} = 0$$

with the solution $u(x) = \sum_{j=1}^{\ell} \varphi_j(x_{\ell+1}, \ldots, x_n) x_j$.
Legendre transf: $z \mapsto z - \sum_{j=1}^{\ell} x_j p_j$, $p_j \mapsto -x_j$, $x_j \mapsto p_j$ $(1 \leq j \leq \ell)$

Future Problems.

Other cases

(Multi-)summability

References

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