Parametric Stokes phenomena of Gauss's hypergeometric differential equation with a large parameter

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Global Study of Differential Equations in the Complex Domain
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Introduce

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The equation $(*)$ comes from the Classical HGDE (Gauss's hypergeometric differential equation):

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and eliminate the first-order term by

$$\psi = x^{\frac{1}{2}+\frac{\eta\gamma}{2}}(1-x)^{\frac{1}{2}+\frac{\eta(\alpha+\beta-\gamma)}{2}}w.$$

Our equation ($*$) has the following formal solutions (**WKB solutions**) :

$$\psi_{\pm} = \frac{1}{\sqrt{S_{\mathrm{odd}}}} \exp\left(\pm \int_a^x S_{\mathrm{odd}}\, dx\right)$$

- a is a zero of $\sqrt{Q_0}dx$. (a is a turning point.)
- a formal solution $S = S_{\mathrm{odd}} + S_{\mathrm{even}} = \sum_{j=-1}^{\infty} \eta^{-j} S_j$ to Riccati equation
$$\frac{dS}{dx} + S^2 = \eta^2 Q.$$
- $S_{-1} = \sqrt{Q_0},\ 2S_{-1}S_0 + \frac{dS_{-1}}{dx} = 0, \cdots$

Stokes graphs

- **A Stokes curve is an integral curve of $\mathrm{Im}\sqrt{Q_0}\,dx = 0$ emanating from a turning point.**
- **A Stokes graph of our equation is a collection of all Stokes curves, turning points $a_k (k = 0, 1)$ and singular points $b_0 = 0, b_1 = 1, b_2 = \infty$.**

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We assume

(i) $\alpha\beta\gamma(\alpha-\beta)(\alpha-\gamma)(\alpha+\beta-\gamma) \neq 0$
(ii) $\mathrm{Re}\,\alpha\,\mathrm{Re}\,\beta\,\mathrm{Re}\,(\gamma-\alpha)\,\mathrm{Re}\,(\gamma-\beta) \neq 0$
(iii) $\mathrm{Re}\,(\alpha-\beta)\,\mathrm{Re}\,(\alpha+\beta-\gamma)\,\mathrm{Re}\,\gamma \neq 0$

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Assumption (i)
$\Longrightarrow$ There are two distinct turning points a_0, a_1 and $a_0, a_1 \neq 0, 1, \infty$.

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Assumption (i)
$\Longrightarrow$ There are two distinct turning points a_0, a_1 and $a_0, a_1 \neq 0, 1, \infty$.
Assumptions (ii) and (iii)
$\Longrightarrow$ There is no Stokes curves which connect turning point(s).

We assume that (α, β, γ) are contained in (i). Let n_0, n_1 and n_2 be numbers of Stokes curves that flow into 0, 1 and ∞, respectively. $\hat{n}$ will denote (n_0, n_1, n_2).

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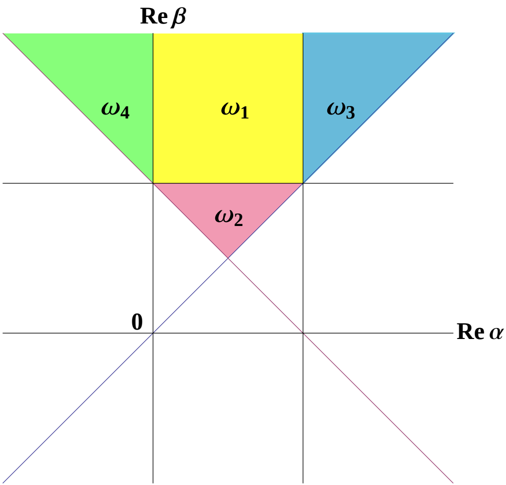
- **$\hat{n}$ characterizes topological configration of Stokes graphs.**
- **$\hat{n}$ is constant on a connected component of the set of all (α, β, γ) satisfying (ii) and (iii).**

We defined

$$\begin{aligned}
\omega_1 &= \{(\alpha,\beta,\gamma) \in \mathbb{C}^3 \mid 0 < \mathrm{Re}\alpha < \mathrm{Re}\gamma < \mathrm{Re}\beta\}, \\
\omega_2 &= \{(\alpha,\beta,\gamma) \in \mathbb{C}^3 \mid 0 < \mathrm{Re}\alpha < \mathrm{Re}\beta < \mathrm{Re}\gamma < \mathrm{Re}\alpha + \mathrm{Re}\beta\}, \\
\omega_3 &= \{(\alpha,\beta,\gamma) \in \mathbb{C}^3 \mid 0 < \mathrm{Re}\gamma < \mathrm{Re}\alpha < \mathrm{Re}\beta\}, \\
\omega_4 &= \{(\alpha,\beta,\gamma) \in \mathbb{C}^3 \mid 0 < \mathrm{Re}\gamma < \mathrm{Re}\alpha + \mathrm{Re}\beta < \mathrm{Re}\beta\}.
\end{aligned}$$

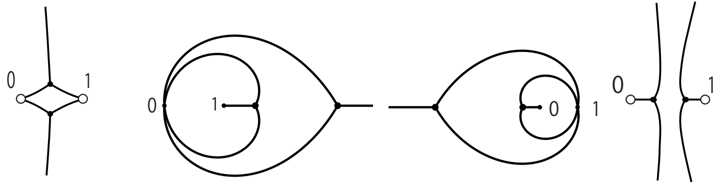
If (α, β, γ) are contained in ω_h $(h = 1, 2, 3, 4)$ respectively, we give a characterization of the Stokes geometry of our equation.

Re $\gamma > 0$ fixed
Reα-Reβ plane



Examples of Stokes graphs for (α, β, γ) of values

$$(0.1, 2, 1) \in \omega_1, \quad (2, 3.9, 4) \in \omega_2, \quad (1.1, 2, 1) \in \omega_3, \quad (-0.1, 2, 1) \in \omega_4.$$



$$\hat{n} = (2, 2, 2), \qquad \hat{n} = (4, 1, 1), \qquad \hat{n} = (1, 4, 1), \qquad \hat{n} = (1, 1, 4).$$

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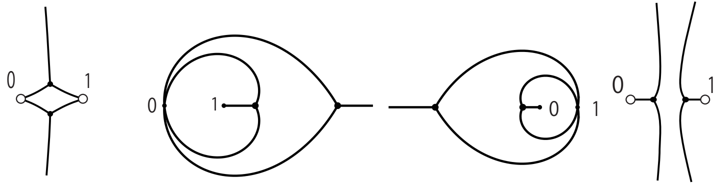

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We denote by ι_j ($j = 0, 1, 2$) the following mappings.

We have to keep in mind that Q is invariant under these involutions.

$$\begin{aligned} \iota_0 : \quad & (\alpha, \beta, \gamma) \mapsto (-\alpha, -\beta, -\gamma) \\ \iota_1 : \quad & \mapsto (\gamma - \alpha, \gamma - \beta, \gamma) \\ \iota_2 : \quad & \mapsto (\beta, \alpha, \gamma) \end{aligned}$$

Each domain is covered by one of ω_h ($h = 1, 2, 3, 4$) via involutions in the configuration.

$$G = \text{the group generated by } \iota_k \ (k = 0, 1, 2)$$

and set

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Theorem 1 (Aoki T. and Tanda M. [2] 2013)

(1) If $(\alpha, \beta, \gamma) \in \Pi_1$, then $\hat{n} = (2, 2, 2)$. (2) If $(\alpha, \beta, \gamma) \in \Pi_2$, then $\hat{n} = (4, 1, 1)$.
(3) If $(\alpha, \beta, \gamma) \in \Pi_3$, then $\hat{n} = (1, 4, 1)$. (4) If $(\alpha, \beta, \gamma) \in \Pi_4$, then $\hat{n} = (1, 1, 4)$.

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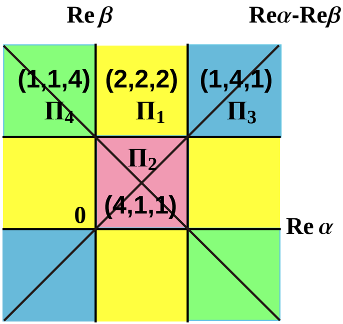
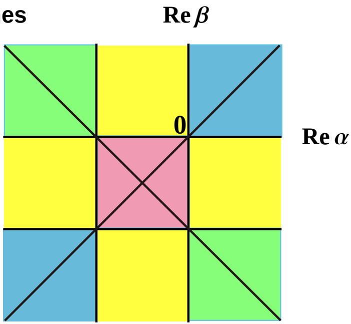
Reα-Reβ planes



Re $\gamma > 0$



Re $\gamma < 0$

Voros coefficients

$$\sqrt{Q_0} \sim -\frac{\gamma}{2x} \qquad \text{at } x = 0,$$
$$\sqrt{Q_0} \sim \frac{\alpha+\beta-\gamma}{2(x-1)} \qquad \text{at } x = 1,$$
$$\sqrt{Q_0} \sim \frac{\beta-\alpha}{2x} \qquad \text{at } x = \infty,$$

V_j for (b_j, a) $(b_0 = 0, b_1 = 1, b_2 = \infty)$ has following form: the Voros coefficient

$$V_j = V_j(\alpha, \beta, \gamma; \eta) := \int_{b_j}^{a} (S_{\mathrm{odd}} - \eta S_{-1})dx,$$

Since residues of S_{odd} and ηS_{-1} as the singular points coincide, V_j are well defined and we have a formal power series V_j in η^{-1}.
The Voros coefficient $V_j(\alpha, \beta, \gamma)$ describes the discrepancy between WKB solutions normalized at a and those normalized at b_j.

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Theorem 2 (Aoki T. and Tanda M. [2] 2013)

$V_j(\alpha,\beta,\gamma;\eta)$ for (j,a) $(j=0,1,2)$ has following forms:

$$V_0 = -\frac{1}{2}\sum_{n=2}^{\infty}\frac{B_n\eta^{1-n}}{n(n-1)}\left\{(1-2^{1-n})\left(\frac{1}{\alpha^{n-1}}+\frac{1}{\beta^{n-1}}+\frac{1}{(\gamma-\alpha)^{n-1}}+\frac{1}{(\gamma-\beta)^{n-1}}\right)+\frac{2}{\gamma^{n-1}}\right\},$$

$$V_1 = \frac{1}{2}\sum_{n=2}^{\infty}\frac{B_n\eta^{1-n}}{n(n-1)}\left\{(1-2^{1-n})\left(\frac{1}{\alpha^{n-1}}+\frac{1}{\beta^{n-1}}-\frac{1}{(\gamma-\alpha)^{n-1}}-\frac{1}{(\gamma-\beta)^{n-1}}\right)+\frac{2}{(\alpha+\beta-\gamma)^{n-1}}\right\},$$

$$V_2 = \frac{1}{2}\sum_{n=2}^{\infty}\frac{B_n\eta^{1-n}}{n(n-1)}\left\{(1-2^{1-n})\left(\frac{1}{\alpha^{n-1}}-\frac{1}{\beta^{n-1}}-\frac{1}{(\gamma-\alpha)^{n-1}}+\frac{1}{(\gamma-\beta)^{n-1}}\right)-\frac{2}{(\beta-\alpha)^{n-1}}\right\}.$$

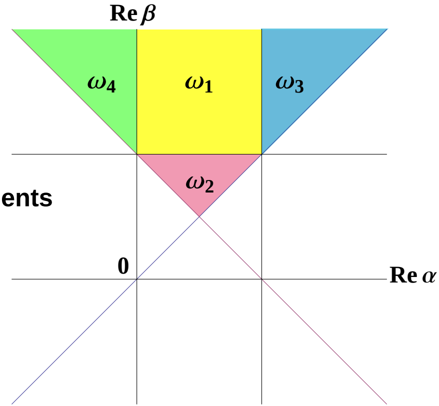
Here, B_n are Bernoulli numbers defined by

$$\frac{te^t}{e^t-1}=\sum_{n=0}^{\infty}\frac{B_n}{n!}t^n.$$

Borel sums of Voros coefficients

$\mathrm{Re}\,\gamma > 0$ **fixed** $\mathrm{Re}\alpha$-$\mathrm{Re}\beta$ **plane**

$\omega_h(h = 1, 2, 3, 4)$:
Stokes regions
of Voros coefficients



V_j **are Borel summable in each** $\omega_h(h = 1, 2, 3, 4)$.
V_j^h **: The Borel sums of** V_j **in** $\omega_h(h = 1, 2, 3, 4)$.

Theorem 3 (Aoki T. and Tanda M. [4] to appear)

(i) The Borel sums V_j^1 of V_j in ω_1 have following forms:

$$V_0^1 = \frac{1}{2}\log\frac{\Gamma(\frac{1}{2}+\alpha\eta)\Gamma(\frac{1}{2}+\beta\eta)\Gamma(\frac{1}{2}+(\gamma-\alpha)\eta)(\beta-\gamma)^{(\beta-\gamma)\eta}\gamma^{2\gamma\eta-1}}{\Gamma(\frac{1}{2}+(\beta-\gamma)\eta)\Gamma^2(\gamma\eta)\alpha^{\alpha\eta}\beta^{\beta\eta}(\gamma-\alpha)^{(\gamma-\alpha)\eta}\eta},$$

$$V_1^1 = \frac{1}{2}\log\frac{\Gamma(\frac{1}{2}+(\gamma-\alpha)\eta)\Gamma^2((\alpha+\beta-\gamma)\eta)\alpha^{\alpha\eta}\beta^{\beta\eta}(\beta-\gamma)^{(\beta-\gamma)\eta}\eta}{\Gamma(\frac{1}{2}+\alpha\eta)\Gamma(\frac{1}{2}+\beta\eta)\Gamma(\frac{1}{2}+(\beta-\gamma)\eta)(\gamma-\alpha)^{(\gamma-\alpha)\eta}(\alpha+\beta-\gamma)^{2(\alpha+\beta-\gamma)\eta-1}},$$

$$V_2^1 = \frac{1}{2}\log\frac{\Gamma(\frac{1}{2}+\beta\eta)\Gamma(\frac{1}{2}+(\gamma-\alpha)\eta)\Gamma(\frac{1}{2}+(\beta-\gamma)\eta)\alpha^{\alpha\eta}(\beta-\alpha)^{2(\beta-\alpha)\eta-1}}{\Gamma(\frac{1}{2}+\alpha\eta)\Gamma^2((\beta-\alpha)\eta)\beta^{\beta\eta}(\gamma-\alpha)^{(\gamma-\alpha)\eta}(\beta-\gamma)^{(\beta-\gamma)\eta}\eta}.$$

(ii) The Borel sums V_j^2 of V_j in ω_2 have following forms:

$$V_0^2 = \frac{1}{2}\log\frac{\Gamma(\frac{1}{2}+\alpha\eta)\Gamma(\frac{1}{2}+\beta\eta)\Gamma(\frac{1}{2}+(\gamma-\alpha)\eta)\Gamma(\frac{1}{2}+(\gamma-\beta)\eta)\gamma^{2\eta\gamma-1}}{\Gamma^2(\gamma\eta)\alpha^{\alpha\eta}\beta^{\beta\eta}(\gamma-\alpha)^{(\gamma-\alpha)\eta}(\gamma-\beta)^{(\gamma-\beta)\eta}2\pi\eta},$$

$$V_1^2 = \frac{1}{2}\log\frac{\Gamma(\frac{1}{2}+(\gamma-\alpha)\eta)\Gamma(\frac{1}{2}+(\gamma-\beta)\eta)\Gamma^2((\alpha+\beta-\gamma)\eta)\alpha^{\alpha\eta}\beta^{\beta\eta}\eta}{\Gamma(\frac{1}{2}+\alpha\eta)\Gamma(\frac{1}{2}+\beta\eta)(\gamma-\alpha)^{(\gamma-\alpha)\eta}(\gamma-\beta)^{(\gamma-\beta)\eta}(\alpha+\beta-\gamma)^{2(\alpha+\beta-\gamma)\eta-1}2\pi},$$

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(iii)The Borel sums V_j^3 of V_j in ω_3 have following forms:

$$V_0^3 = \frac{1}{2}\log\frac{\Gamma(\frac{1}{2}+\alpha\eta)\Gamma(\frac{1}{2}+\beta\eta)(\alpha-\gamma)^{(\alpha-\gamma)\eta}(\beta-\gamma)^{(\beta-\gamma)\eta}\gamma^{2\gamma\eta-1}2\pi}{\Gamma(\frac{1}{2}+(\alpha-\gamma)\eta)\Gamma(\frac{1}{2}+(\beta-\gamma)\eta)\Gamma^2(\gamma\eta)\alpha^{\alpha\eta}\beta^{\beta\eta}\eta}.$$

$$V_1^3 = \frac{1}{2}\log\frac{2\pi\Gamma^2((\alpha+\beta-\gamma)\eta)\alpha^{\alpha\eta}\beta^{\beta\eta}(\alpha-\gamma)^{(\alpha-\gamma)\eta}(\beta-\gamma)^{(\beta-\gamma)\eta}\eta}{\Gamma(\frac{1}{2}+\alpha\eta)\Gamma(\frac{1}{2}+\beta\eta)\Gamma(\frac{1}{2}+(\alpha-\gamma)\eta)\Gamma(\frac{1}{2}+(\beta-\gamma)\eta)(\alpha+\beta-\gamma)^{2(\alpha+\beta-\gamma)\eta-1}},$$

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(iv) The Borel sums V_j^4 of V_j in ω_4 have following forms:

$$V_0^4 = \frac{1}{2}\log\frac{\Gamma(\frac{1}{2}+\beta\eta)\Gamma(\frac{1}{2}+(\gamma-\alpha)\eta)(-\alpha)^{-\alpha\eta}(\beta-\gamma)^{(\beta-\gamma)\eta}\gamma^{2\gamma\eta-1}2\pi}{\Gamma(\frac{1}{2}-\alpha\eta)\Gamma(\frac{1}{2}+(\beta-\gamma)\eta)\Gamma^2(\gamma\eta)\beta^{\beta\eta}(\gamma-\alpha)^{(\gamma-\alpha)\eta}\eta},$$

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Proof

We consider the poof of the Borel sum V_0^1 and V_0^2 of Voros coefficient V_0 in ω_1 and ω_2, respectively.

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$$V_0^1 = \frac{1}{2}\log\frac{\Gamma(\frac{1}{2}+\alpha\eta)\Gamma(\frac{1}{2}+\beta\eta)\Gamma(\frac{1}{2}+(\gamma-\alpha)\eta)(\beta-\gamma)^{(\beta-\gamma)\eta}\gamma^{2\gamma\eta-1}}{\Gamma(\frac{1}{2}+(\beta-\gamma)\eta)\Gamma^2(\gamma\eta)\alpha^{\alpha\eta}\beta^{\beta\eta}(\gamma-\alpha)^{(\gamma-\alpha)\eta}\eta},$$

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The idea of proof of the Theorem 3 is to use the method developed by Takei [10] (2008). To find the Borel sums V_0^1 and V_0^2, we first take the Borel transform $V_{0,B}(\alpha,\beta,\gamma;y)$ of V_0.

By the definition, we have the Borel transform $V_{0,B}(\alpha,\beta,\gamma;y)$ as follows.

$$V_{0,B} = \frac{1}{2}\sum_{n=2}^{\infty}\frac{B_n y^{n-2}}{n!}\left\{(1-2^{1-n})\left(-\frac{1}{\alpha^{n-1}}-\frac{1}{\beta^{n-1}}-\frac{1}{(\gamma-\alpha)^{n-1}}-\frac{1}{(\gamma-\beta)^{n-1}}\right)-\frac{2}{\gamma^{n-1}}\right\}.$$

Bernoulli number

$$\tilde{g}(t) = \sum_{n=2}^{\infty}\frac{B_n}{n!}t^n = \frac{te^t}{e^t-1}-1-\frac{t}{2}.$$

$$\frac{1}{e^{2x}-1} = \frac{1}{2}\left(\frac{1}{e^x-1}-\frac{1}{e^x+1}\right)$$

$$g_0(t) = \tilde{g}(t)\frac{t}{y^2} = \frac{1}{y}\left(\frac{1}{\exp\frac{y}{t}-1}+\frac{1}{2}-\frac{t}{y}\right),$$

$$g_1(t) = \frac{1}{\exp\frac{y}{2t}-1}+\frac{1}{\exp\frac{y}{2t}+1}-\frac{2t}{y}.$$

Hence the Borel transform $V_{0,B}(\alpha,\beta,\gamma;y)$ is written in the form

$$V_{0,B}(\alpha,\beta,\gamma;y) = \frac{1}{4y}\{g_1(\gamma-\alpha)+g_1(\gamma-\beta)+g_1(\alpha)+g_1(\beta)\} - g_0(\gamma).$$

We introduce the following auxiliary infinite series:

$$\tilde{V}_0 = V_0 + \mu(\gamma-\alpha)+\mu(\gamma-\beta)+\mu(\alpha)+\mu(\beta),$$

where

$$\mu(t) = \frac{1}{4} - \frac{t\eta}{2}\log(1+\frac{1}{2t\eta}) = -\frac{1}{4}\sum_{n=0}^{\infty}\frac{1}{n+2}(-2t\eta)^{-(n+1)}.$$

The Borel transform $\mu_B(t;y)$ of $\mu(t)$ is

$$\mu_B(t;y) = \frac{1}{8t}\sum_{n=0}^{\infty}\frac{1}{(n+2)n!}\left(-\frac{y}{2t}\right)^n = \frac{t}{4y^2}\left\{\left(-\frac{y}{t}-2\right)\exp\left(-\frac{y}{2t}\right)+2\right\}.$$

The Borel transform $\tilde{V}_{0,B}$ of $\tilde{V}_0$ is related to $V_{0,B}$ by

$$g(t) = \frac{1}{2y}e^{-\frac{1}{2t}y}\left(\frac{1}{e^{\frac{1}{t}y}-1}+\frac{1}{2}-\frac{t}{y}\right).$$

Then we have

$$\tilde{V}_{0,B} = g(\gamma-\alpha)+g(\gamma-\beta)+g(\alpha)+g(\beta)-g_0(\gamma).$$

Next we consider the Borel sums V_0^1 and V_0^2 of V_0.
We use the following integral formula representation of the logarithm of the Γ-function ([5] BMP, 1997).

$$L(\theta) = \int_0^\infty \left(\frac{1}{e^s - 1} + \frac{1}{2} - \frac{1}{s} \right) \frac{e^{-\theta s}}{s} \, ds = \log \frac{\Gamma(\theta)}{\sqrt{2\pi}} - (\theta - \frac{1}{2}) \log \theta + \theta,$$

where $\mathrm{Re}\,\theta$ is positive.
We can compute the inverse Laplace transform

$$\int_0^\infty g(\alpha) e^{-y\eta} \, dy$$

of $g(\alpha)$ by using $\tilde{V}_{0,B}$ if Re α is positive. If Re α is negative, we make use of the relation

$$g(t) = -g(-t).$$

If (α, β, γ) are contained in ω_1, Re α, Re β, Re $\gamma - \alpha$ and Re γ are positive and Re $\gamma - \beta$ is negative.
If (α, β, γ) are contained in ω_2, Re α, Re β, Re $\gamma - \alpha$, Re $\gamma - \beta$ and Re γ are positive.

We use the Borel transforms $V^1_{0,B}$ and $V^2_{0,B}$ of Voros coefficients V_0 :

$$V^1_{0,B} = g(\alpha) + g(\beta) + g(\gamma - \alpha) - g(\beta - \gamma) - g_0(\gamma),$$

$$V^2_{0,B} = g(\alpha) + g(\beta) + g(\gamma - \alpha) + g(\gamma - \beta) - g_0(\gamma).$$

Similarly we can compute the inverse Laplace transforms and we have:

$$\tilde{V}^1_0 = L(\frac{1}{2} + (\gamma - \alpha)\eta) - L(\frac{1}{2} + (\beta - \gamma)\eta) + L(\frac{1}{2} + \alpha\eta) + L(\frac{1}{2} + \beta\eta) - L(\gamma\eta),$$

$$\tilde{V}^2_0 = L(\frac{1}{2} + (\gamma - \alpha)\eta) + L(\frac{1}{2} + (\gamma - \beta)\eta) + L(\frac{1}{2} + \alpha\eta) + L(\frac{1}{2} + \beta\eta) - L(\gamma\eta).$$

Since μ is a convergent power series of η^{-1}, we have

$$\begin{aligned} V_0^1 &= \tilde{V}_0^1 + \mu(\gamma-\alpha) - \mu(\beta-\gamma) + \mu(\alpha) + \mu(\beta), \\ V_0^2 &= \tilde{V}_0^2 + \mu(\gamma-\alpha) + \mu(\gamma-\beta) + \mu(\alpha) + \mu(\beta). \end{aligned}$$

Then we obtain

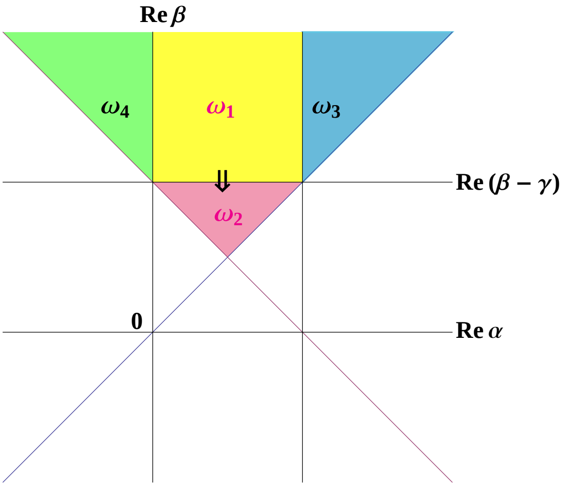
The Borel sums V_0^1 and V_0^2 of V_0 in ω_1 and ω_2 have following forms:

$$V_0^1 = \frac{1}{2}\log\frac{\Gamma(\frac{1}{2}+\alpha\eta)\Gamma(\frac{1}{2}+\beta\eta)\Gamma(\frac{1}{2}+(\gamma-\alpha)\eta)(\beta-\gamma)^{(\beta-\gamma)\eta}\gamma^{2\gamma\eta-1}}{\Gamma(\frac{1}{2}+(\beta-\gamma)\eta)\Gamma^2(\gamma\eta)\alpha^{\alpha\eta}\beta^{\beta\eta}(\gamma-\alpha)^{(\gamma-\alpha)\eta}\eta}.$$

$$V_0^2 = \frac{1}{2}\log\frac{\Gamma(\frac{1}{2}+\alpha\eta)\Gamma(\frac{1}{2}+\beta\eta)\Gamma(\frac{1}{2}+(\gamma-\alpha)\eta)\Gamma(\frac{1}{2}+(\gamma-\beta)\eta)\gamma^{2\eta\gamma-1}}{\Gamma^2(\gamma\eta)\alpha^{\alpha\eta}\beta^{\beta\eta}(\gamma-\alpha)^{(\gamma-\alpha)\eta}(\gamma-\beta)^{(\gamma-\beta)\eta}2\pi\eta},$$

Relation between Borel sums of V_j in two regions

$\mathrm{Re}\,\gamma > 0$ fixed
$\mathrm{Re}\alpha$-$\mathrm{Re}\beta$ plane



We compare V_j^2 ($j = 0, 1, 2$) with the analytic continuation V_j^1 to ω_2.

Theorem 4

The Borel sums V_j^1 of Voros coefficients V_j can be analytically continued over ω_2 $(j = 0, 1, 2)$. The analytic continuations of the Borel sums V_j^1 to ω_2 are related to V_j^2 as follows:

$$\begin{aligned} V_2^1 &= V_2^2 - \frac{1}{2}\log(\exp 2(\gamma - \beta)\eta\pi i + 1), \\ V_j^1 &= V_j^2 + \frac{1}{2}\log(\exp 2(\gamma - \beta)\eta\pi i + 1) \quad (j = 0, 1). \end{aligned}$$

Here $\beta - \gamma = (\gamma - \beta)e^{-\pi i}$.

Proof

We compare V_0^2 with the analytic continuation V_0^1 to ω_2.
Let us recall the Borel sums V_0^1 and V_0^2 of V_0 in ω_1 and ω_2 have the following forms, respectively:

$$V_0^2 = \frac{1}{2}\log\frac{\Gamma(\frac{1}{2}+\alpha\eta)\Gamma(\frac{1}{2}+\beta\eta)\Gamma(\frac{1}{2}+(\gamma-\alpha)\eta)\Gamma(\frac{1}{2}+(\gamma-\beta)\eta)\gamma^{2\eta\gamma-1}}{\Gamma^2(\gamma\eta)\alpha^{\alpha\eta}\beta^{\beta\eta}(\gamma-\alpha)^{(\gamma-\alpha)\eta}(\gamma-\beta)^{(\gamma-\beta)\eta}2\pi\eta}.$$

$$\begin{aligned} V_0^1 &= \frac{1}{2}\log\frac{\Gamma(\frac{1}{2}+\alpha\eta)\Gamma(\frac{1}{2}+\beta\eta)\Gamma(\frac{1}{2}+(\gamma-\alpha)\eta)(\beta-\gamma)^{(\beta-\gamma)\eta}\gamma^{2\gamma\eta-1}}{\Gamma(\frac{1}{2}+(\beta-\gamma)\eta)\Gamma^2(\gamma\eta)\alpha^{\alpha\eta}\beta^{\beta\eta}(\gamma-\alpha)^{(\gamma-\alpha)\eta}\eta}, \\ &= \frac{1}{2}\log\frac{\Gamma(\frac{1}{2}+\alpha\eta)\Gamma(\frac{1}{2}+\beta\eta)\Gamma(\frac{1}{2}+(\gamma-\alpha)\eta)(\gamma-\beta)^{(\beta-\gamma)\eta}\gamma^{2\gamma\eta-1}}{\Gamma(\frac{1}{2}+(\beta-\gamma)\eta)\Gamma^2(\gamma\eta)\alpha^{\alpha\eta}\beta^{\beta\eta}(\gamma-\alpha)^{(\gamma-\alpha)\eta}\eta} - \frac{(\beta-\gamma)\eta\pi i}{2}, \end{aligned}$$

Subtracting V_0^2 from the analytic continuation V_0^1 to ω_2, we have

$$V_0^1 - V_0^2 = \frac{1}{2}\log\frac{2\pi}{\Gamma(\frac{1}{2}+(\gamma-\beta)\eta)\Gamma(\frac{1}{2}+(\beta-\gamma)\eta)} - \frac{(\gamma-\beta)\eta\pi i}{2}.$$

Proof

We compare V_0^2 with the analytic continuation V_0^1 to ω_2.
Let us recall the Borel sums V_0^1 and V_0^2 of V_0 in ω_1 and ω_2 have the

following forms, respectively:

$$V_0^2 = \frac{1}{2}\log\frac{\Gamma(\frac{1}{2}+\alpha\eta)\Gamma(\frac{1}{2}+\beta\eta)\Gamma(\frac{1}{2}+(\gamma-\alpha)\eta)\Gamma(\frac{1}{2}+(\gamma-\beta)\eta)\gamma^{2\eta\gamma-1}}{\Gamma^2(\gamma\eta)\alpha^{\alpha\eta}\beta^{\beta\eta}(\gamma-\alpha)^{(\gamma-\alpha)\eta}(\gamma-\beta)^{(\gamma-\beta)\eta}2\pi\eta}.$$

$$\begin{aligned} V_0^1 &= \frac{1}{2}\log\frac{\Gamma(\frac{1}{2}+\alpha\eta)\Gamma(\frac{1}{2}+\beta\eta)\Gamma(\frac{1}{2}+(\gamma-\alpha)\eta)(\beta-\gamma)^{(\beta-\gamma)\eta}\gamma^{2\gamma\eta-1}}{\Gamma(\frac{1}{2}+(\beta-\gamma)\eta)\Gamma^2(\gamma\eta)\alpha^{\alpha\eta}\beta^{\beta\eta}(\gamma-\alpha)^{(\gamma-\alpha)\eta}\eta}, \\ &= \frac{1}{2}\log\frac{\Gamma(\frac{1}{2}+\alpha\eta)\Gamma(\frac{1}{2}+\beta\eta)\Gamma(\frac{1}{2}+(\gamma-\alpha)\eta)(\gamma-\beta)^{(\beta-\gamma)\eta}\gamma^{2\gamma\eta-1}}{\Gamma(\frac{1}{2}+(\beta-\gamma)\eta)\Gamma^2(\gamma\eta)\alpha^{\alpha\eta}\beta^{\beta\eta}(\gamma-\alpha)^{(\gamma-\alpha)\eta}\eta} - \frac{(\beta-\gamma)\eta\pi i}{2}, \end{aligned}$$

Subtracting V_0^2 from the analytic continuation V_0^1 to ω_2, we have

$$V_0^1 - V_0^2 = \frac{1}{2}\log\frac{2\pi}{\Gamma(\frac{1}{2}+(\gamma-\beta)\eta)\Gamma(\frac{1}{2}+(\beta-\gamma)\eta)} - \frac{(\gamma-\beta)\eta\pi i}{2}.$$

Proof

We compare V_0^2 with the analytic continuation V_0^1 to ω_2.
Let us recall the Borel sums V_0^1 and V_0^2 of V_0 in ω_1 and ω_2 have the following forms, respectively:

$$V_0^2 = \frac{1}{2}\log\frac{\Gamma(\frac{1}{2}+\alpha\eta)\Gamma(\frac{1}{2}+\beta\eta)\Gamma(\frac{1}{2}+(\gamma-\alpha)\eta)\Gamma(\frac{1}{2}+(\gamma-\beta)\eta)\gamma^{2\eta\gamma-1}}{\Gamma^2(\gamma\eta)\alpha^{\alpha\eta}\beta^{\beta\eta}(\gamma-\alpha)^{(\gamma-\alpha)\eta}(\gamma-\beta)^{(\gamma-\beta)\eta}2\pi\eta}.$$

$$\begin{aligned} V_0^1 &= \frac{1}{2}\log\frac{\Gamma(\frac{1}{2}+\alpha\eta)\Gamma(\frac{1}{2}+\beta\eta)\Gamma(\frac{1}{2}+(\gamma-\alpha)\eta)(\beta-\gamma)^{(\beta-\gamma)\eta}\gamma^{2\gamma\eta-1}}{\Gamma(\frac{1}{2}+(\beta-\gamma)\eta)\Gamma^2(\gamma\eta)\alpha^{\alpha\eta}\beta^{\beta\eta}(\gamma-\alpha)^{(\gamma-\alpha)\eta}\eta}, \\ &= \frac{1}{2}\log\frac{\Gamma(\frac{1}{2}+\alpha\eta)\Gamma(\frac{1}{2}+\beta\eta)\Gamma(\frac{1}{2}+(\gamma-\alpha)\eta)(\gamma-\beta)^{(\beta-\gamma)\eta}\gamma^{2\gamma\eta-1}}{\Gamma(\frac{1}{2}+(\beta-\gamma)\eta)\Gamma^2(\gamma\eta)\alpha^{\alpha\eta}\beta^{\beta\eta}(\gamma-\alpha)^{(\gamma-\alpha)\eta}\eta} - \frac{(\beta-\gamma)\eta\pi i}{2}, \end{aligned}$$

Subtracting V_0^2 from the analytic continuation V_0^1 to ω_2, we have

$$V_0^1 - V_0^2 = \frac{1}{2}\log\frac{2\pi}{\Gamma(\frac{1}{2}+(\gamma-\beta)\eta)\Gamma(\frac{1}{2}+(\beta-\gamma)\eta)} - \frac{(\gamma-\beta)\eta\pi i}{2}.$$

Proof

We compare V_0^2 with the analytic continuation V_0^1 to ω_2.
Let us recall the Borel sums V_0^1 and V_0^2 of V_0 in ω_1 and ω_2 have the following forms, respectively:

$$V_0^2 = \frac{1}{2}\log\frac{\Gamma(\frac{1}{2}+\alpha\eta)\Gamma(\frac{1}{2}+\beta\eta)\Gamma(\frac{1}{2}+(\gamma-\alpha)\eta)\Gamma(\frac{1}{2}+(\gamma-\beta)\eta)\gamma^{2\eta\gamma-1}}{\Gamma^2(\gamma\eta)\alpha^{\alpha\eta}\beta^{\beta\eta}(\gamma-\alpha)^{(\gamma-\alpha)\eta}(\gamma-\beta)^{(\gamma-\beta)\eta}2\pi\eta}.$$

$$\begin{aligned} V_0^1 &= \frac{1}{2}\log\frac{\Gamma(\frac{1}{2}+\alpha\eta)\Gamma(\frac{1}{2}+\beta\eta)\Gamma(\frac{1}{2}+(\gamma-\alpha)\eta)(\beta-\gamma)^{(\beta-\gamma)\eta}\gamma^{2\gamma\eta-1}}{\Gamma(\frac{1}{2}+(\beta-\gamma)\eta)\Gamma^2(\gamma\eta)\alpha^{\alpha\eta}\beta^{\beta\eta}(\gamma-\alpha)^{(\gamma-\alpha)\eta}\eta}, \\ &= \frac{1}{2}\log\frac{\Gamma(\frac{1}{2}+\alpha\eta)\Gamma(\frac{1}{2}+\beta\eta)\Gamma(\frac{1}{2}+(\gamma-\alpha)\eta)(\gamma-\beta)^{(\beta-\gamma)\eta}\gamma^{2\gamma\eta-1}}{\Gamma(\frac{1}{2}+(\beta-\gamma)\eta)\Gamma^2(\gamma\eta)\alpha^{\alpha\eta}\beta^{\beta\eta}(\gamma-\alpha)^{(\gamma-\alpha)\eta}\eta} - \frac{(\beta-\gamma)\eta\pi i}{2}, \end{aligned}$$

Subtracting V_0^2 from the analytic continuation V_0^1 to ω_2, we have

$$V_0^1 - V_0^2 = \frac{1}{2}\log\frac{2\pi}{\Gamma(\frac{1}{2}+(\gamma-\beta)\eta)\Gamma(\frac{1}{2}+(\beta-\gamma)\eta)} - \frac{(\gamma-\beta)\eta\pi i}{2}.$$

Well-known

$$\Gamma(\frac{1}{2}+t)\Gamma(\frac{1}{2}-t)=\frac{\pi}{\cos\pi t},$$
$$\cos t=\frac{e^{it}+e^{-it}}{2}.$$

Hence we obtain

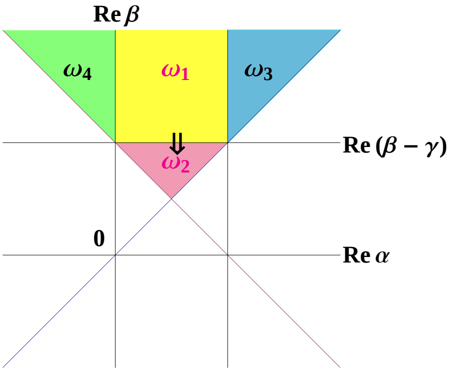
$$V_0^1-V_0^2=\frac{1}{2}\log(\exp 2(\gamma-\beta)\eta\pi i+1).$$

In the same way, we can compute the relations between V_1^1 and V_1^2 and V_2^1 and V_2^2. Hence we have

$$V_1^1-V_1^2=\frac{1}{2}\log(\exp 2(\gamma-\beta)\eta\pi i+1),$$
$$V_2^1-V_2^2=-\frac{1}{2}\log(\exp 2(\gamma-\beta)\eta\pi i+1).$$

Parametric Stokes phenomena for the WKB solution

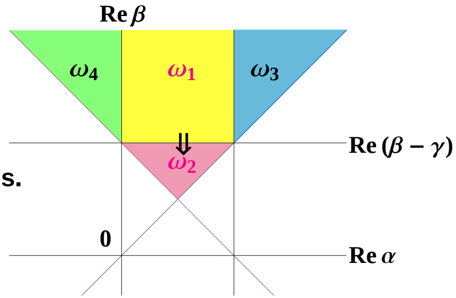
$\mathrm{Re}\,\gamma > 0$ fixed
$\mathrm{Re}\alpha$-$\mathrm{Re}\beta$ plane

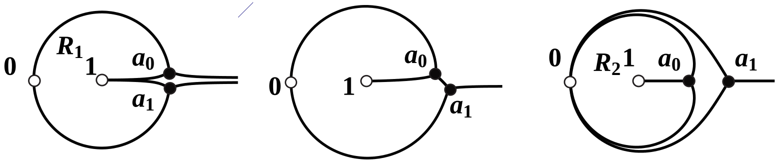


Parametric Stokes phenomena for the WKB solution

$\mathrm{Re}\,\gamma > 0$ fixed
$\mathrm{Re}\alpha$-$\mathrm{Re}\beta$ plane

R_1 and R_2 :
The Stokes regions.





$(\alpha,\beta,\gamma) = (0.5, 1.05, 1)$ in ω_1 $(0.5, 1 + 0.01i, 1)$ $(0.5, 0.95, 1)$ in ω_2

The WKB solutions are Borel summable in R_1 and R_2(Koike and R. Schäfke, [7] to appear.)

Let ψ^1 and ψ^2 denote the Borel sum of the WKB solution:

$$\psi = \frac{1}{\sqrt{S_{\mathrm{odd}}}} \exp\left(\int_{a_0}^{x} S_{\mathrm{odd}}\,dx\right)$$

in the Stokes region $\mathcal{R}^1$ and $\mathcal{R}^2$, respectively.

Theorem 4

We obtain the following relation between ψ^1 and ψ^2.

$$\psi^1 = (1 + \exp(2\pi i(\gamma - \beta)\eta))^{-\frac{1}{2}}\psi^2.$$

Let ψ^1 and ψ^2 denote the Borel sum of the WKB solution:

$$\psi = \frac{1}{\sqrt{S_{\mathrm{odd}}}} \exp\left(\int_{a_0}^{x} S_{\mathrm{odd}} dx\right)$$

in the Stokes region $\mathcal{R}^1$ and $\mathcal{R}^2$, respectively.

Theorem 4

We obtain the following relation between ψ^1 and ψ^2.

$$\psi^1 = (1 + \exp(2\pi i(\gamma - \beta)\eta))^{-\frac{1}{2}} \psi^2.$$

Out of Proof

We use the following relation:

$$\psi = \exp(-V_0)\psi^{(0)},$$

where

$$\psi^{(0)} = \frac{1}{\sqrt{S_{\mathrm{odd}}}} \exp\left(\int_0^x (S_{\mathrm{odd}} - \eta S_{-1}) dx \pm \eta \int_{a_0}^x S_{-1} dx\right).$$

Moreover we use the following result by Koike and Schäfke ([7] to appear):

$$\psi^{(0),1} = \psi^{(0),2}$$

and the following relation by Theorem 4:

$$V_0^1 = V_0^2 + \frac{1}{2}\log(\exp 2(\gamma - \beta)\eta\pi i + 1).$$

Here $\beta - \gamma = (\gamma - \beta)e^{-\pi i}$. Hence we have

$$\begin{aligned} \psi^1 &= (\exp(-V_0^1))\psi^{(0),1} \\ &= (1 + \exp(2\pi i(\gamma - \beta)\eta))^{-\frac{1}{2}}(\exp(-V_0^2))\psi^{(0),2} \\ &= (1 + \exp(2\pi i(\gamma - \beta)\eta))^{-\frac{1}{2}}\psi^2. \end{aligned}$$

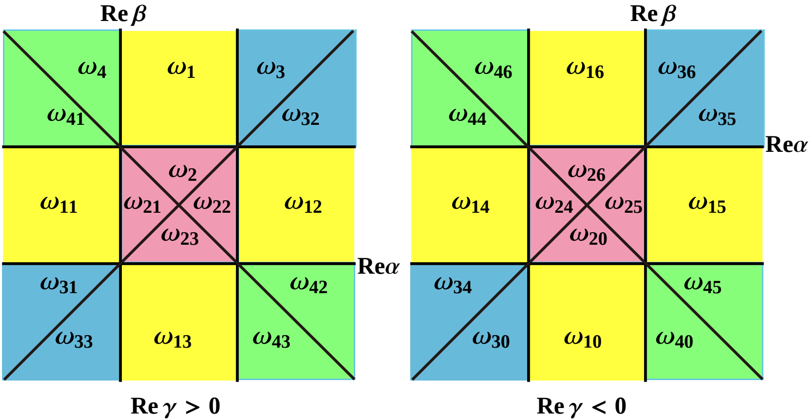
In the similar manner, we can compute other relations.

Q : :invariant under involutions ι_m ($m = 0, 1, \cdots, 6$)

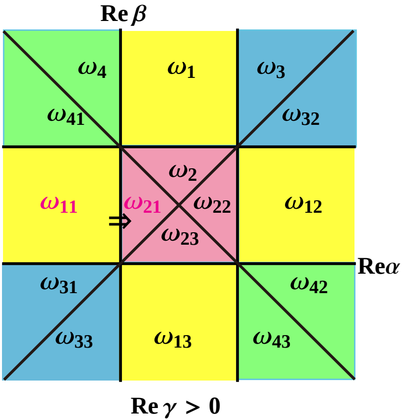
$$
\begin{array}{lll}
\iota_0 & : (\alpha, \beta, \gamma) & \mapsto (-\alpha, -\beta, -\gamma) \\
\iota_1 & : & \mapsto (\gamma - \beta, \gamma - \alpha, \gamma) \\
\iota_2 & : & \mapsto (\beta, \alpha, \gamma) \\
\iota_3 = \iota_1\iota_2 & : & \mapsto (\gamma - \alpha, \gamma - \beta, \gamma) \\
\iota_4 = \iota_0\iota_2 & : & \mapsto (-\beta, -\alpha, -\gamma) \\
\iota_5 = \iota_0\iota_1 & : & \mapsto (\beta - \gamma, \alpha - \gamma, -\gamma) \\
\iota_6 = \iota_0\iota_1\iota_2 & : & \mapsto (\alpha - \gamma, \beta - \gamma, -\gamma)
\end{array}
$$

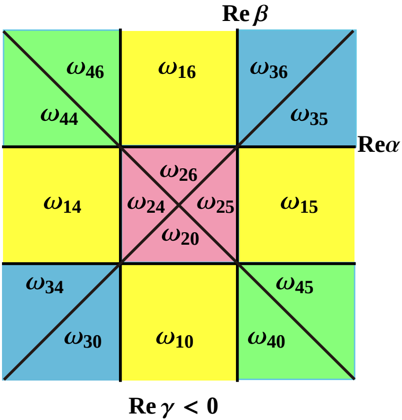
$\omega_{hm} = \iota_m(\omega_h)$: Images in ω_h by ι_m.
Here, $h = 1, 2, 3, 4$, $m = 0, 1, \cdots, 6$.

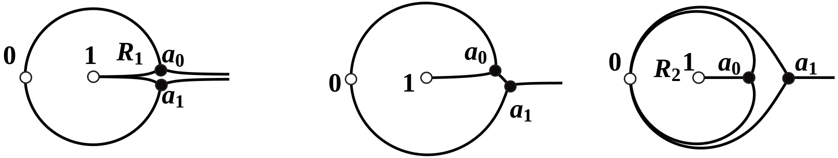
Reα-Reβ planes



Reα-Reβ planes

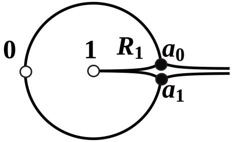


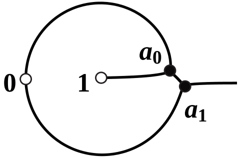


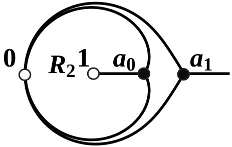


$(\alpha, \beta, \gamma) = (0.5, 1.05, 1)$ in ω_1 $(0.5, 1 + 0.01i, 1)$ $(0.5, 0.95, 1)$ in ω_2

$$\iota_1 : \quad (\alpha, \beta, \gamma) \mapsto (\gamma - \beta, \gamma - \alpha, \gamma)$$

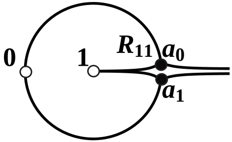


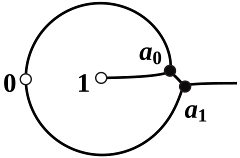


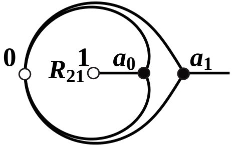


$(\alpha, \beta, \gamma) = (0.5, 1.05, 1)$ in ω_1 $(0.5, 1 + 0.01i, 1)$ $(0.5, 0.95, 1)$ in ω_2

$$\iota_1 : \quad (\alpha, \beta, \gamma) \mapsto (\gamma - \beta, \gamma - \alpha, \gamma)$$







$(\alpha, \beta, \gamma) = (-0.05, 0.5, 1)$ in ω_{11} $(-0.01i, 0.5, 1)$ $(0.05, 0.5, 1)$ in ω_{21}

The WKB solutions are Borel summable in R_{11} and R_{21}. The top and bottom Stokes graphs are same graphs, respectively.

Let ψ^{11} and ψ^{21} denote the Borel sum of the WKB solution:

$$\psi = \frac{1}{\sqrt{S_{\mathrm{odd}}}} \exp\left(\int_{a_0}^{x} S_{\mathrm{odd}} dx\right)$$

in the Stokes region $\mathcal{R}^{11}$ and $\mathcal{R}^{21}$, respectively.

Theorem 4

We obtain the following relation between ψ^1 and ψ^2.

$$\psi^1 = (1 + \exp(2\pi i(\gamma - \beta)\eta))^{-\frac{1}{2}} \psi^2.$$

$$\iota_1 : \quad (\alpha, \beta, \gamma) \mapsto (\gamma - \beta, \gamma - \alpha, \gamma)$$

Theorem 5

We obtain the following relation between ψ^{11} and ψ^{21}.

$$\psi^{11} = (1 + \exp(2\pi i \alpha \eta))^{-\frac{1}{2}} \psi^{21}.$$

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END

Thank you for your attention.