Parametric Stokes phenomena of Gauss’s hypergeometric differential equation with a large parameter

Mika TANDA (Kinki Univ.)
Collaborator : Takashi AOKI (Kinki Univ.)

Global Study of Differential Equations in the Complex Domain
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1. Hypergeometric differential equations

2. Stokes graphs

3. Voros coefficients

4. Borel sums of Voros coefficients

5. Parametric Stokes phenomena
We consider the following differential equation with a large parameter $\eta$:

$$\left( - \frac{d^2}{dx^2} + \eta^2 Q \right) \psi = 0 \quad \cdots \quad (*)$$
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with $Q = Q_0 + \eta^{-2}Q_1$,

$$Q_0 = \frac{(\alpha - \beta)^2 x^2 + 2(2\alpha\beta - \alpha\gamma - \beta\gamma)x + \gamma^2}{4x^2(x - 1)^2}, \quad Q_1 = -\frac{x^2 - x + 1}{4x^2(x - 1)^2}.$$
Introduce

We consider the following differential equation with a large parameter $\eta$:

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The equation (*) comes from the Classical HGDE (Gauss’s hypergeometric differential equation):

$$x(1 - x)\frac{d^2w}{dx^2} + (c - (a + b + 1)x)\frac{dw}{dx} - abw = 0.$$
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Introduce a large parameter $\eta$ by setting

$$a = \frac{1}{2} + \eta\alpha, \quad b = \frac{1}{2} + \eta\beta, \quad c = 1 + \eta\gamma$$
We consider the following differential equation with a large parameter $\eta$:

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$$a = \frac{1}{2} + \eta\alpha, \quad b = \frac{1}{2} + \eta\beta, \quad c = 1 + \eta\gamma$$

and eliminate the first-order term by

$$\psi = x^{\frac{1}{2} + \frac{\eta\alpha}{2}} (1 - x)^{\frac{1}{2} + \frac{\eta(\alpha + \beta - \gamma)}{2}} w.$$
Our equation (*) has the following formal solutions

(WKB solutions):

\[ \psi_\pm = \frac{1}{\sqrt{S_{\text{odd}}}} \exp(\pm \int_a^x S_{\text{odd}} \, dx) \]

- \( a \) is a zero of \( \sqrt{Q_0} \, dx \). (\( a \) is a turning point.)
- a formal solution \( S = S_{\text{odd}} + S_{\text{even}} = \sum_{j=-1}^{\infty} \eta^{-j}S_j \)
  to Riccati equation

\[ \frac{dS}{dx} + S^2 = \eta^2 Q. \]

- \( S_{-1} = \sqrt{Q_0}, \ 2S_{-1}S_0 + \frac{dS_{-1}}{dx} = 0, \ldots \)
A **Stokes curve** is an integral curve of $\text{Im} \sqrt{Q_0} \, dx = 0$ emanating from a turning point.

A **Stokes graph** of our equation is a collection of all Stokes curves, turning points $a_k (k = 0, 1)$ and singular points $b_0 = 0, b_1 = 1, b_2 = \infty$. 

Assumption (i):

There are two distinct turning points $a_0, a_1$ and $a_0, a_1, 0; 1; 1$.

Assumptions (ii) and (iii):

There is no Stokes curves which connect turning point(s).
Stokes graphs

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We assume

(i) \( \alpha \beta \gamma (\alpha - \beta)(\alpha - \gamma)(\alpha + \beta - \gamma) \neq 0 \)

(ii) \( \text{Re} \, \alpha \, \text{Re} \, \beta \, \text{Re} \, (\gamma - \alpha) \, \text{Re} \, (\gamma - \beta) \neq 0 \)

(iii) \( \text{Re} \, (\alpha - \beta) \, \text{Re} \, (\alpha + \beta - \gamma) \, \text{Re} \, \gamma \neq 0 \)
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**Assumption (i)**

$\implies$ There are two distinct turning points $a_0, a_1$ and $a_0, a_1 \neq 0, 1, \infty$. 
A **Stokes curve** is an integral curve of \( \text{Im} \sqrt{Q_0} \, dx = 0 \) emanating from a turning point.

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Assumption (i)

\( \implies \) There are two distinct turning points \( a_0, a_1 \) and \( a_0, a_1 \neq 0, 1, \infty \).

Assumptions (ii) and (iii)

\( \implies \) There is no Stokes curves which connect turning point(s).
We assume that \((\alpha, \beta, \gamma)\) are contained in (i). Let \(n_0, n_1\) and \(n_2\) be numbers of Stokes curves that flow into 0, 1 and \(\infty\), respectively. \(\hat{n}\) will denote \((n_0, n_1, n_2)\).
We assume that \((\alpha, \beta, \gamma)\) are contained in (i). Let \(n_0, n_1\) and \(n_2\) be numbers of Stokes curves that flow into 0, 1 and \(\infty\), respectively. \(\hat{n}\) will denote \((n_0, n_1, n_2)\).

- \(\hat{n}\) characterizes topological configuration of Stokes graphs.
- \(\hat{n}\) is constant on a connected component of the set of all \((\alpha, \beta, \gamma)\) satisfying (ii) and (iii).

We defined

\[
\begin{align*}
\omega_1 &= \{(\alpha, \beta, \gamma) \in \mathbb{C}^3 | 0 < \text{Re}\alpha < \text{Re}\gamma < \text{Re}\beta\}, \\
\omega_2 &= \{(\alpha, \beta, \gamma) \in \mathbb{C}^3 | 0 < \text{Re}\alpha < \text{Re}\beta < \text{Re}\gamma < \text{Re}\alpha + \text{Re}\beta\}, \\
\omega_3 &= \{(\alpha, \beta, \gamma) \in \mathbb{C}^3 | 0 < \text{Re}\gamma < \text{Re}\alpha < \text{Re}\beta\}, \\
\omega_4 &= \{(\alpha, \beta, \gamma) \in \mathbb{C}^3 | 0 < \text{Re}\gamma < \text{Re}\alpha + \text{Re}\beta < \text{Re}\beta\}.
\end{align*}
\]

If \((\alpha, \beta, \gamma)\) are contained in \(\omega_h\) \((h = 1, 2, 3, 4)\) respectively, we give a characterization of the Stokes geometry of our equation.
\[ \text{Re } \gamma > 0 \text{ fixed} \]

**Re} \alpha-\text{Re} \beta \text{ plane}**

\[ \omega_4 \quad \omega_1 \quad \omega_3 \]

\[ 0 \]

\[ \text{Re } \alpha \]

\[ \text{Re } \beta \]
Examples of Stokes graphs for \((\alpha, \beta, \gamma)\) of values

\((0.1, 2, 1) \in \omega_1, \ (2, 3.9, 4) \in \omega_2, \ (1.1, 2, 1) \in \omega_3, \ (-0.1, 2, 1) \in \omega_4.\)

\[ \hat{n} = (2, 2, 2), \quad \hat{n} = (4, 1, 1), \quad \hat{n} = (1, 4, 1), \quad \hat{n} = (1, 1, 4). \]
Examples of Stokes graphs for \((\alpha, \beta, \gamma)\) of values

\[(0.1, 2, 1) \in \omega_1, \quad (2, 3.9, 4) \in \omega_2, \quad (1.1, 2, 1) \in \omega_3, \quad (-0.1, 2, 1) \in \omega_4.\]

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\]

We denote by \(\iota_j \ (j = 0, 1, 2)\) the following mappings.

We have to keep in mind that \(Q\) is invariant under these involutions.

\[
\begin{align*}
\iota_0 : \quad (\alpha, \beta, \gamma) & \mapsto (-\alpha, -\beta, -\gamma) \\
\iota_1 : \quad & \mapsto (\gamma - \alpha, \gamma - \beta, \gamma) \\
\iota_2 : \quad & \mapsto (\beta, \alpha, \gamma)
\end{align*}
\]

Each domain is covered by one of \(\omega_h \ (h = 1, 2, 3, 4)\) via involutions in the configuration.
Hypergeometric differential equations

Stokes graphs

Voros coefficients

Borel sums of Voros coefficients

Parametric Stokes phenomena

\[ G = \text{the group generated by } \nu_k \ (k = 0, 1, 2) \]

and set

\[ \Pi_j = \bigcup_{r \in G} r(\omega_j). \]
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1. If \((\alpha, \beta, \gamma) \in \Pi_1\), then \(\hat{n} = (2, 2, 2)\).
2. If \((\alpha, \beta, \gamma) \in \Pi_2\), then \(\hat{n} = (4, 1, 1)\).
3. If \((\alpha, \beta, \gamma) \in \Pi_3\), then \(\hat{n} = (1, 4, 1)\).
4. If \((\alpha, \beta, \gamma) \in \Pi_4\), then \(\hat{n} = (1, 1, 4)\).
Hypergeometric differential equations
Stokes graphs
Voros coefficients
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Parametric Stokes phenomena

\[ G = \text{the group generated by } \iota_k \ (k = 0, 1, 2) \]

and set

\[ \Pi_j = \bigcup_{r \in G} r(\omega_j). \]

Theorem 1 (Aoki T. and Tanda M. [2] 2013)

(1) If \((\alpha, \beta, \gamma) \in \Pi_1\), then \(\hat{n} = (2, 2, 2)\).
(2) If \((\alpha, \beta, \gamma) \in \Pi_2\), then \(\hat{n} = (4, 1, 1)\).
(3) If \((\alpha, \beta, \gamma) \in \Pi_3\), then \(\hat{n} = (1, 4, 1)\).
(4) If \((\alpha, \beta, \gamma) \in \Pi_4\), then \(\hat{n} = (1, 1, 4)\).

\begin{figure}
\centering
\includegraphics[width=\textwidth]{chart.png}
\caption{Re\(\beta\)-Re\(\alpha\)-Re\(\beta\) planes}
\end{figure}
Voros coefficients

\[ \sqrt{Q_0} \sim -\frac{\gamma}{2x} \quad \text{at } x = 0, \]
\[ \sqrt{Q_0} \sim \frac{\alpha + \beta - \gamma}{2(x-1)} \quad \text{at } x = 1, \]
\[ \sqrt{Q_0} \sim \frac{\beta - \alpha}{2x} \quad \text{at } x = \infty, \]

\( V_j \) for \((b_j, a)\) \((b_0 = 0, b_1 = 1, b_2 = \infty)\) has following form: the Voros coefficient

\[ V_j = V_j(\alpha, \beta, \gamma; \eta) := \int_{b_j}^{a} (S_{\text{odd}} - \eta S_{-1}) \, dx, \]

Since residues of \(S_{\text{odd}}\) and \(\eta S_{-1}\) as the singular points coincide, \(V_j\) are well defined and we have a formal power series \(V_j\) in \(\eta^{-1}\).

The Voros coefficient \(V_j(\alpha, \beta, \gamma)\) describes the discrepancy between WKB solutions normalized at \(a\) and those normalized at \(b_j\).
Voros coefficients

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The Voros coefficient \(V_j(\alpha, \beta, \gamma)\) describes the discrepancy between WKB solutions normalized at \(a\) and those normalized at \(b_j\).

\[ V_j(\alpha, \beta, \gamma; \eta) \text{ for } (j, a) \quad (j = 0, 1, 2) \text{ has following forms:} \]

\[ V_0 = -\frac{1}{2} \sum_{n=2}^{\infty} \frac{B_n \eta^{1-n}}{n(n-1)} \left\{ (1 - 2^{1-n}) \left( \frac{1}{\alpha^{n-1}} + \frac{1}{\beta^{n-1}} + \frac{1}{(\gamma - \alpha)^{n-1}} + \frac{1}{(\gamma - \beta)^{n-1}} \right) + \frac{2}{\gamma^{n-1}} \right\}, \]

\[ V_1 = \frac{1}{2} \sum_{n=2}^{\infty} \frac{B_n \eta^{1-n}}{n(n-1)} \left\{ (1 - 2^{1-n}) \left( \frac{1}{\alpha^{n-1}} + \frac{1}{\beta^{n-1}} - \frac{1}{(\gamma - \alpha)^{n-1}} - \frac{1}{(\gamma - \beta)^{n-1}} \right) \right. \]
\[ \left. + \frac{2}{(\alpha + \beta - \gamma)^{n-1}} \right\}, \]

\[ V_2 = \frac{1}{2} \sum_{n=2}^{\infty} \frac{B_n \eta^{1-n}}{n(n-1)} \left\{ (1 - 2^{1-n}) \left( \frac{1}{\alpha^{n-1}} - \frac{1}{\beta^{n-1}} - \frac{1}{(\gamma - \alpha)^{n-1}} + \frac{1}{(\gamma - \beta)^{n-1}} \right) \right. \]
\[ \left. - \frac{2}{(\beta - \alpha)^{n-1}} \right\}. \]

Here, \( B_n \) are Bernoulli numbers defined by

\[ \frac{te^t}{e^t - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} t^n. \]
Borel sums of Voros coefficients

Re $\gamma > 0$ fixed  Re$\alpha$-Re$\beta$ plane

$\omega_h (h = 1, 2, 3, 4)$:
Stokes regions
of Voros coefficients

$V_j$ are Borel summable in each $\omega_h (h = 1, 2, 3, 4)$.
$V_j^h$ : The Borel sums of $V_j$ in $\omega_h (h = 1, 2, 3, 4)$. 

(i) The Borel sums $V_j^1$ of $V_j$ in $\omega_1$ have following forms:

$$V_0^1 = \frac{1}{2} \log \frac{\Gamma\left(\frac{1}{2} + \alpha \eta\right) \Gamma\left(\frac{1}{2} + \beta \eta\right) \Gamma\left(\frac{1}{2} + (\gamma - \alpha) \eta\right) (\beta - \gamma)^{(\beta - \gamma) \eta} \gamma^{2 \gamma \eta - 1}}{\Gamma\left(\frac{1}{2} + (\beta - \gamma) \eta\right) \Gamma^2(\gamma \eta) \alpha^\alpha \beta^\beta \eta (\gamma - \alpha)^{(\gamma - \alpha) \eta} \eta},$$

$$V_1^1 = \frac{1}{2} \log \frac{\Gamma\left(\frac{1}{2} + (\gamma - \alpha) \eta\right) \Gamma^2((\alpha + \beta - \gamma) \eta) \alpha^\alpha \beta^\beta \eta (\beta - \gamma)^{(\beta - \gamma) \eta} \eta}{\Gamma\left(\frac{1}{2} + \alpha \eta\right) \Gamma\left(\frac{1}{2} + \beta \eta\right) \Gamma\left(\frac{1}{2} + (\beta - \gamma) \eta\right) (\gamma - \alpha)^{(\gamma - \alpha) \eta} (\alpha + \beta - \gamma)^{2(\alpha + \beta - \gamma) \eta - 1}},$$

$$V_2^1 = \frac{1}{2} \log \frac{\Gamma\left(\frac{1}{2} + \beta \eta\right) \Gamma\left(\frac{1}{2} + (\gamma - \alpha) \eta\right) \Gamma\left(\frac{1}{2} + (\beta - \gamma) \eta\right) \alpha^\alpha \beta^\beta \eta (\beta - \alpha)^{2(\beta - \alpha) \eta - 1}}{\Gamma\left(\frac{1}{2} + \alpha \eta\right) \Gamma^2((\beta - \alpha) \eta) \beta^\beta \eta (\gamma - \alpha)^{(\gamma - \alpha) \eta} (\beta - \gamma)^{(\beta - \gamma) \eta} \eta}.$$
(ii) The Borel sums $V^2_j$ of $V_j$ in $\omega_2$ have following forms:

$V^2_0 = \frac{1}{2} \log \frac{\Gamma\left(\frac{1}{2} + \alpha \eta\right) \Gamma\left(\frac{1}{2} + \beta \eta\right) \Gamma\left(\frac{1}{2} + (\gamma - \alpha) \eta\right) \Gamma\left(\frac{1}{2} + (\gamma - \beta) \eta\right) \gamma^{2 \eta \gamma^{-1}}}{\Gamma^2(\gamma \eta) \alpha^\alpha \beta^\beta \eta (\gamma - \alpha)^{(\gamma - \alpha) \eta} (\gamma - \beta)^{(\gamma - \beta) \eta} 2 \pi \eta}$.

$V^2_1 = \frac{1}{2} \log \frac{\Gamma\left(\frac{1}{2} + (\gamma - \alpha) \eta\right) \Gamma\left(\frac{1}{2} + (\gamma - \beta) \eta\right) \Gamma^2((\alpha + \beta - \gamma) \eta) \alpha^\alpha \beta^\beta \eta}{\Gamma\left(\frac{1}{2} + \alpha \eta\right) \Gamma\left(\frac{1}{2} + \beta \eta\right) (\gamma - \alpha)^{(\gamma - \alpha) \eta} (\gamma - \beta)^{(\gamma - \beta) \eta} (\alpha + \beta - \gamma)^{2(\alpha + \beta - \gamma) \eta^{-1}} 2 \pi}$.

$V^2_2 = \frac{1}{2} \log \frac{\Gamma\left(\frac{1}{2} + \beta \eta\right) \Gamma\left(\frac{1}{2} + (\gamma - \alpha) \eta\right) \alpha^\alpha \eta (\gamma - \beta)^{(\gamma - \beta) \eta} (\beta - \alpha)^{2(\beta - \alpha) \eta^{-1}} 2 \pi}{\Gamma\left(\frac{1}{2} + \alpha \eta\right) \Gamma\left(\frac{1}{2} + (\gamma - \beta) \eta\right) \Gamma^2((\beta - \alpha) \eta) \beta^\beta \eta (\gamma - \alpha)^{(\gamma - \alpha) \eta} \eta}$. 
(iii) The Borel sums $V^3_j$ of $V_j$ in $\omega_3$ have following forms:

$$V^3_0 = \frac{1}{2} \log \frac{\Gamma(\frac{1}{2} + \alpha \eta) \Gamma(\frac{1}{2} + \beta \eta)(\alpha - \gamma)^{\eta}(\beta - \gamma)^{\eta} \gamma^{2\eta - 1} 2\pi}{\Gamma(\frac{1}{2} + (\alpha - \gamma) \eta) \Gamma(\frac{1}{2} + (\beta - \gamma) \eta) \Gamma^2(\gamma \eta) \alpha^{\alpha \eta} \beta^{\beta \eta} \eta}.$$

$$V^3_1 = \frac{1}{2} \log \frac{2\pi \Gamma^2((\alpha + \beta - \gamma) \eta) \alpha^{\alpha \eta} \beta^{\beta \eta} \gamma^{2(\alpha + \beta - \gamma) \eta - 1}}{\Gamma(\frac{1}{2} + \alpha \eta) \Gamma(\frac{1}{2} + \beta \eta) \Gamma(\frac{1}{2} + (\alpha - \gamma) \eta) \Gamma(\frac{1}{2} + (\beta - \gamma) \eta) (\alpha + \beta - \gamma)^{2(\alpha + \beta - \gamma) \eta - 1}},$$

$$V^3_2 = \frac{1}{2} \log \frac{2\pi \Gamma(\frac{1}{2} + \beta \eta) \Gamma(\frac{1}{2} + (\beta - \gamma) \eta) \alpha^{\alpha \eta} \beta^{\beta \eta} \gamma^{2(\beta - \alpha) \eta - 1}}{\Gamma(\frac{1}{2} + \alpha \eta) \Gamma(\frac{1}{2} + (\alpha - \gamma) \eta) \Gamma^2((\beta - \alpha) \eta) \beta^{\beta \eta} \gamma^{(\beta - \gamma) \eta} \eta}}.$$
(iv) The Borel sums $V^4_j$ of $V_j$ in $\omega_4$ have following forms:

\[
V^4_0 = \frac{1}{2} \log \frac{\Gamma\left(\frac{1}{2} + \beta \eta\right) \Gamma\left(\frac{1}{2} + (\gamma - \alpha) \eta\right)(-\alpha)^{-\alpha \eta}(\beta - \gamma)^{(\beta - \gamma)\eta}\gamma^{2\gamma \eta - 1}2\pi}{\Gamma\left(\frac{1}{2} - \alpha \eta\right) \Gamma\left(\frac{1}{2} + (\beta - \gamma) \eta\right) \Gamma^2(\gamma \eta)\beta^{\beta \eta}(\gamma - \alpha)^{(\gamma - \alpha)\eta}}.
\]

\[
V^4_1 = \frac{1}{2} \log \frac{\Gamma\left(\frac{1}{2} - \alpha \eta\right) \Gamma\left(\frac{1}{2} + (\gamma - \alpha) \eta\right) \Gamma^2((\alpha + \beta - \gamma) \eta)\beta^{\beta \eta}(\beta - \gamma)^{(\beta - \gamma)\eta}\eta}{\Gamma\left(\frac{1}{2} + \beta \eta\right) \Gamma\left(\frac{1}{2} + (\beta - \gamma) \eta\right)(-\alpha)^{-\alpha \eta}(\gamma - \alpha)^{(\gamma - \alpha)\eta}(\alpha + \beta - \gamma)^{2(\alpha + \beta - \gamma)\eta - 1}2\pi}.
\]

\[
V^4_2 = \frac{1}{2} \log \frac{\Gamma\left(\frac{1}{2} - \alpha \eta\right) \Gamma\left(\frac{1}{2} + \beta \eta\right) \Gamma\left(\frac{1}{2} + (\gamma - \alpha) \eta\right) \Gamma\left(\frac{1}{2} + (\beta - \gamma) \eta\right) \beta^{\beta \eta}(\beta - \alpha)^{2(\beta - \alpha)\eta - 1}}{\Gamma^2((\beta - \alpha) \eta)(-\alpha)^{-\alpha \eta} \beta^{\beta \eta}(\gamma - \alpha)^{(\gamma - \alpha)\eta}(\beta - \gamma)^{(\beta - \gamma)\eta}2\pi \eta}.
\]
Proof

We consider the proof of the Borel sum $V^1_0$ and $V^2_0$ of Voros coefficient $V_0$ in $\omega_1$ and $\omega_2$, respectively.

The Borel sums $V^1_0$ and $V^2_0$ of $V_0$ in $\omega_1$ and $\omega_2$ have following forms:

$$V^1_0 = \frac{1}{2} \log \frac{\Gamma\left(\frac{1}{2} + \alpha \eta\right)\Gamma\left(\frac{1}{2} + \beta \eta\right)\Gamma\left(\frac{1}{2} + (\gamma - \alpha)\eta\right)(\beta - \gamma)^{(\beta - \gamma)\eta}\gamma^{2\gamma\eta - 1}}{\Gamma\left(\frac{1}{2} + (\beta - \gamma)\eta\right)\Gamma^2(\gamma \eta)\alpha^\eta \beta^\eta (\gamma - \alpha)^{(\gamma - \alpha)\eta} \eta},$$

$$V^2_0 = \frac{1}{2} \log \frac{\Gamma\left(\frac{1}{2} + \alpha \eta\right)\Gamma\left(\frac{1}{2} + \beta \eta\right)\Gamma\left(\frac{1}{2} + (\gamma - \alpha)\eta\right)\Gamma\left(\frac{1}{2} + (\gamma - \beta)\eta\right)\gamma^{2\eta\gamma - 1}}{\Gamma^2(\gamma \eta)\alpha^\eta \beta^\eta (\gamma - \alpha)^{(\gamma - \alpha)\eta} (\gamma - \beta)^{(\gamma - \beta)\eta} 2\pi \eta}.$$

The idea of proof of the Theorem 3 is to use the method developed by Takei [10] (2008). To find the Borel sums $V^1_0$ and $V^2_0$, we first take the Borel transform $V_{0,B}(\alpha, \beta, \gamma; y)$ of $V_0$. 
By the definition, we have the Borel transform $V_{0,B}(\alpha, \beta, \gamma; y)$ as follows.

$$V_{0,B} = \frac{1}{2} \sum_{n=2}^{\infty} \frac{B_n y^{n-2}}{n!} \left\{ (1 - 2^{1-n}) \left( \frac{1}{\alpha^{n-1}} - \frac{1}{\beta^{n-1}} - \frac{1}{(\gamma - \alpha)^{n-1}} - \frac{1}{(\gamma - \beta)^{n-1}} \right) - \frac{2}{\gamma^{n-1}} \right\}.$$ 

### Bernoulli number

$$\tilde{g}(t) = \sum_{n=2}^{\infty} \frac{B_n}{n!} t^n = \frac{t e^t}{e^t - 1} - 1 - \frac{t}{2}.$$ 

$$\frac{1}{e^{2x} - 1} = \frac{1}{2} \left( \frac{1}{e^x - 1} - \frac{1}{e^x + 1} \right)$$

$$g_0(t) = \tilde{g}(t) \frac{t}{y^2} = \frac{1}{y} \left( \frac{1}{\exp \frac{y}{t} - 1} + \frac{1}{2} - \frac{t}{y} \right),$$

$$g_1(t) = \frac{1}{\exp \frac{y}{2t} - 1} + \frac{1}{\exp \frac{y}{2t} + 1} - \frac{2t}{y}.$$
Hence the Borel transform $V_{0,B}(\alpha, \beta, \gamma; y)$ is written in the form

$$V_{0,B}(\alpha, \beta, \gamma; y) = \frac{1}{4y} \left\{ g_1(\gamma - \alpha) + g_1(\gamma - \beta) + g_1(\alpha) + g_1(\beta) \right\} - g_0(\gamma).$$

We introduce the following auxiliary infinite series:

$$\tilde{V}_0 = V_0 + \mu(\gamma - \alpha) + \mu(\gamma - \beta) + \mu(\alpha) + \mu(\beta),$$

where

$$\mu(t) = \frac{1}{4} - \frac{t\eta}{2 \log(1 + \frac{1}{2t\eta})} = -\frac{1}{4} \sum_{n=0}^{\infty} \frac{1}{n+2} \frac{1}{(-2t\eta)^{(n+1)}}.$$

The Borel transform $\mu_B(t; y)$ of $\mu(t)$ is

$$\mu_B(t; y) = \frac{1}{8t} \sum_{n=0}^{\infty} \frac{1}{(n+2)n!} \left( -\frac{y}{2t} \right)^n = \frac{t}{4y^2} \left\{ \left( -\frac{y}{t} - 2 \right) \exp\left( -\frac{y}{2t} \right) + 2 \right\}.$$

The Borel transform $\tilde{V}_{0,B}$ of $\tilde{V}_0$ is related to $V_{0,B}$ by

$$g(t) = \frac{1}{2y} e^{-\frac{1}{2}y} \left( \frac{1}{e^{\frac{1}{2}y} - 1} + \frac{1}{2} - \frac{t}{y} \right).$$

Then we have

$$\tilde{V}_{0,B} = g(\gamma - \alpha) + g(\gamma - \beta) + g(\alpha) + g(\beta) - g_0(\gamma).$$
Next we consider the Borel sums $V^1_0$ and $V^2_0$ of $V_0$. We use the following integral formula representation of the logarithm of the $\Gamma$-function ([5] BMP, 1997).

$$L(\theta) = \int_0^\infty \left( \frac{1}{e^s - 1} + \frac{1}{2} - \frac{1}{s} \right) \frac{e^{-\theta s}}{s} \, ds = \log \frac{\Gamma(\theta)}{\sqrt{2\pi}} - (\theta - \frac{1}{2}) \log \theta + \theta,$$

where $\text{Re} \, \theta$ is positive.

We can compute the inverse Laplace transform

$$\int_0^\infty g(\alpha) e^{-\eta y} \, dy$$

of $g(\alpha)$ by using $\tilde{V}_{0,B}$ if $\text{Re} \, \alpha$ is positive. If $\text{Re} \, \alpha$ is negative, we make use of the relation

$$g(t) = -g(-t).$$
If \((\alpha, \beta, \gamma)\) are contained in \(\omega_1\), \(\Re \alpha, \Re \beta, \Re \gamma - \alpha\) and \(\Re \gamma\) are positive and \(\Re \gamma - \beta\) is negative.
If \((\alpha, \beta, \gamma)\) are contained in \(\omega_2\), \(\Re \alpha, \Re \beta, \Re \gamma - \alpha, \Re \gamma - \beta\) and \(\Re \gamma\) are positive.

We use the Borel transforms \(V^1_{0,B}\) and \(V^2_{0,B}\) of Voros coefficients \(V_0\):
\[
V^1_{0,B} = g(\alpha) + g(\beta) + g(\gamma - \alpha) - g(\beta - \gamma) - g_0(\gamma),
\]
\[
V^2_{0,B} = g(\alpha) + g(\beta) + g(\gamma - \alpha) + g(\gamma - \beta) - g_0(\gamma).
\]

Similarly we can compute the inverse Laplace transforms and we have:
\[
\tilde{V}^1_0 = L\left(\frac{1}{2} + (\gamma - \alpha)\eta\right) - L\left(\frac{1}{2} + (\beta - \gamma)\eta\right) + L\left(\frac{1}{2} + \alpha\eta\right) + L\left(\frac{1}{2} + \beta\eta\right) - L(\gamma\eta),
\]
\[
\tilde{V}^2_0 = L\left(\frac{1}{2} + (\gamma - \alpha)\eta\right) + L\left(\frac{1}{2} + (\gamma - \beta)\eta\right) + L\left(\frac{1}{2} + \alpha\eta\right) + L\left(\frac{1}{2} + \beta\eta\right) - L(\gamma\eta).
\]
Since $\mu$ is a convergent power series of $\eta^{-1}$, we have

$$V^1_0 = \tilde{V}^1_0 + \mu(\gamma - \alpha) - \mu(\beta - \gamma) + \mu(\alpha) + \mu(\beta),$$

$$V^2_0 = \tilde{V}^2_0 + \mu(\gamma - \alpha) + \mu(\gamma - \beta) + \mu(\alpha) + \mu(\beta).$$

Then we obtain

The Borel sums $V^1_0$ and $V^2_0$ of $V_0$ in $\omega_1$ and $\omega_2$ have following forms:

$$V^1_0 = \frac{1}{2} \log \frac{\Gamma(\frac{1}{2} + \alpha \eta)\Gamma(\frac{1}{2} + \beta \eta)\Gamma(\frac{1}{2} + (\gamma - \alpha)\eta)(\beta - \gamma)^{(\beta - \gamma)\eta}\gamma^{2\gamma \eta - 1}}{\Gamma(\frac{1}{2} + (\beta - \gamma)\eta)\Gamma^2(\gamma \eta)\alpha^\alpha \eta^\beta^\beta \eta(\gamma - \alpha)^{(\gamma - \alpha)\eta}}.$$

$$V^2_0 = \frac{1}{2} \log \frac{\Gamma(\frac{1}{2} + \alpha \eta)\Gamma(\frac{1}{2} + \beta \eta)\Gamma(\frac{1}{2} + (\gamma - \alpha)\eta)\Gamma(\frac{1}{2} + (\gamma - \beta)\eta)\gamma^{2\gamma \eta - 1}}{\Gamma^2(\gamma \eta)\alpha^\alpha \eta^\beta^\beta \eta(\gamma - \alpha)^{(\gamma - \alpha)\eta}(\gamma - \beta)^{(\gamma - \beta)\eta} 2\pi \eta}}.$$
Relation between Borel sums of $V_j$ in two regions

Re $\gamma > 0$ fixed
Re$\alpha$-Re$\beta$ plane

We compare $V_j^2$ ($j = 0, 1, 2$) with the analytic continuation $V_j^1$ to $\omega_2$. 
Theorem 4

The Borel sums $V^1_j$ of Voros coefficients $V_j$ can be analytically continued over $\omega_2$ ($j = 0, 1, 2$). The analytic continuations of the Borel sums $V^1_j$ to $\omega_2$ are related to $V^2_j$ as follows:

$$V^1_2 = V^2_2 - \frac{1}{2} \log(\exp 2(\gamma - \beta)\eta\pi i + 1),$$

$$V^1_j = V^2_j + \frac{1}{2} \log(\exp 2(\gamma - \beta)\eta\pi i + 1) \quad (j = 0, 1).$$

Here $\beta - \gamma = (\gamma - \beta)e^{-\pi i}$. 

Proof

We compare $V^2_0$ with the analytic continuation $V^1_0$ to $\omega_2$.

Let us recall the Borel sums $V^1_0$ and $V^2_0$ of $V_0$ in $\omega_1$ and $\omega_2$ have the following forms, respectively:

$$V^2_0 = \frac{1}{2} \log \frac{\Gamma\left(\frac{1}{2} + \alpha \eta\right)\Gamma\left(\frac{1}{2} + \beta \eta\right)\Gamma\left(\frac{1}{2} + (\gamma - \alpha)\eta\right)\Gamma\left(\frac{1}{2} + (\gamma - \beta)\eta\right)\gamma^{2\eta\gamma - 1}}{\Gamma^2(\gamma \eta)\alpha^{\alpha \eta} \beta^{\beta \eta} (\gamma - \alpha)^{(\gamma - \alpha) \eta} (\gamma - \beta)^{(\gamma - \beta) \eta} 2\pi \eta}.$$  

$$V^1_0 = \frac{1}{2} \log \frac{\Gamma\left(\frac{1}{2} + \alpha \eta\right)\Gamma\left(\frac{1}{2} + \beta \eta\right)\Gamma\left(\frac{1}{2} + (\gamma - \alpha)\eta\right)(\beta - \gamma)^{(\beta - \gamma)\eta} \gamma^{2\gamma \eta - 1}}{\Gamma\left(\frac{1}{2} + (\beta - \gamma) \eta\right)\Gamma^2(\gamma \eta)\alpha^{\alpha \eta} \beta^{\beta \eta} (\gamma - \alpha)^{(\gamma - \alpha) \eta} \eta},$$  

$$= \frac{1}{2} \log \frac{\Gamma\left(\frac{1}{2} + \alpha \eta\right)\Gamma\left(\frac{1}{2} + \beta \eta\right)\Gamma\left(\frac{1}{2} + (\gamma - \alpha)\eta\right)(\gamma - \beta)^{(\beta - \gamma)\eta} \gamma^{2\gamma \eta - 1}}{\Gamma\left(\frac{1}{2} + (\beta - \gamma) \eta\right)\Gamma^2(\gamma \eta)\alpha^{\alpha \eta} \beta^{\beta \eta} (\gamma - \alpha)^{(\gamma - \alpha) \eta} \eta} - \frac{(\beta - \gamma) \eta \pi i}{2},$$  

Subtracting $V^2_0$ from the analytic continuation $V^1_0$ to $\omega_2$, we have

$$V^1_0 - V^2_0 = \frac{1}{2} \log \frac{2\pi}{\Gamma\left(\frac{1}{2} + (\gamma - \beta) \eta\right)\Gamma\left(\frac{1}{2} + (\beta - \gamma) \eta\right)} - \frac{(\gamma - \beta) \eta \pi i}{2}.$$  

Proof

We compare $V_0^2$ with the analytic continuation $V_0^1$ to $\omega_2$.

Let us recall the Borel sums $V_0^1$ and $V_0^2$ of $V_0$ in $\omega_1$ and $\omega_2$ have the following forms, respectively:

$$V_0^2 = \frac{1}{2} \log \frac{\Gamma\left(\frac{1}{2} + \alpha \eta \right) \Gamma\left(\frac{1}{2} + \beta \eta \right) \Gamma\left(\frac{1}{2} + (\gamma - \alpha) \eta \right) \Gamma\left(\frac{1}{2} + (\gamma - \beta) \eta \right) \gamma^{2\eta \gamma - 1}}{\Gamma^2(\gamma \eta) \alpha^{\alpha \eta} \beta^{\beta \eta} (\gamma - \alpha)^{(\gamma - \alpha) \eta} (\gamma - \beta)^{(\gamma - \beta) \eta} 2\pi \eta}.$$

$$V_0^1 = \frac{1}{2} \log \frac{\Gamma\left(\frac{1}{2} + \alpha \eta \right) \Gamma\left(\frac{1}{2} + \beta \eta \right) \Gamma\left(\frac{1}{2} + (\gamma - \alpha) \eta \right) (\beta - \gamma)^{(\beta - \gamma) \eta} \gamma^{2\gamma \eta - 1}}{\Gamma\left(\frac{1}{2} + (\beta - \gamma) \eta \right) \Gamma^2(\gamma \eta) \alpha^{\alpha \eta} \beta^{\beta \eta} (\gamma - \alpha)^{(\gamma - \alpha) \eta} \eta}$$

$$= \frac{1}{2} \log \frac{\Gamma\left(\frac{1}{2} + \alpha \eta \right) \Gamma\left(\frac{1}{2} + \beta \eta \right) \Gamma\left(\frac{1}{2} + (\gamma - \alpha) \eta \right) (\gamma - \beta)^{(\beta - \gamma) \eta} \gamma^{2\gamma \eta - 1}}{\Gamma\left(\frac{1}{2} + (\beta - \gamma) \eta \right) \Gamma^2(\gamma \eta) \alpha^{\alpha \eta} \beta^{\beta \eta} (\gamma - \alpha)^{(\gamma - \alpha) \eta} \eta} - \frac{(\beta - \gamma) \eta \pi i}{2},$$

Subtracting $V_0^2$ from the analytic continuation $V_0^1$ to $\omega_2$, we have

$$V_0^1 - V_0^2 = \frac{1}{2} \log \frac{2\pi}{\Gamma\left(\frac{1}{2} + (\gamma - \beta) \eta \right) \Gamma\left(\frac{1}{2} + (\beta - \gamma) \eta \right)} - \frac{(\gamma - \beta) \eta \pi i}{2}.$$
Proof

We compare $V^2_0$ with the analytic continuation $V^1_0$ to $\omega_2$.
Let us recall the Borel sums $V^1_0$ and $V^2_0$ of $V_0$ in $\omega_1$ and $\omega_2$ have the following forms, respectively:

$$V^2_0 = \frac{1}{2} \log \frac{\Gamma(\frac{1}{2} + \alpha \eta)\Gamma(\frac{1}{2} + \beta \eta)\Gamma(\frac{1}{2} + (\gamma - \alpha)\eta)\Gamma(\frac{1}{2} + (\gamma - \beta)\eta)\gamma^{2\eta\gamma - 1}}{\Gamma^2(\gamma \eta)\alpha^{\alpha\eta}\beta^{\beta\eta}(\gamma - \alpha)^{(\gamma - \alpha)\eta}(\gamma - \beta)^{(\gamma - \beta)\eta}2\pi\eta}.$$ 

$$V^1_0 = \frac{1}{2} \log \frac{\Gamma(\frac{1}{2} + \alpha \eta)\Gamma(\frac{1}{2} + \beta \eta)\Gamma(\frac{1}{2} + (\gamma - \alpha)\eta)(\beta - \gamma)^{(\beta - \gamma)\eta}\gamma^{2\gamma\eta - 1}}{\Gamma(\frac{1}{2} + (\beta - \gamma)\eta)\Gamma^2(\gamma \eta)\alpha^{\alpha\eta}\beta^{\beta\eta}(\gamma - \alpha)^{(\gamma - \alpha)\eta}\eta},$$

$$= \frac{1}{2} \log \frac{\Gamma(\frac{1}{2} + \alpha \eta)\Gamma(\frac{1}{2} + \beta \eta)\Gamma(\frac{1}{2} + (\gamma - \alpha)\eta)(\gamma - \beta)^{(\beta - \gamma)\eta}\gamma^{2\gamma\eta - 1}}{\Gamma(\frac{1}{2} + (\beta - \gamma)\eta)\Gamma^2(\gamma \eta)\alpha^{\alpha\eta}\beta^{\beta\eta}(\gamma - \alpha)^{(\gamma - \alpha)\eta}\eta} - \frac{(\beta - \gamma)\eta\pi i}{2},$$

Subtracting $V^2_0$ from the analytic continuation $V^1_0$ to $\omega_2$, we have

$$V^1_0 - V^2_0 = \frac{1}{2} \log \frac{2\pi}{\Gamma(\frac{1}{2} + (\gamma - \beta)\eta)\Gamma(\frac{1}{2} + (\beta - \gamma)\eta)} - \frac{(\gamma - \beta)\eta\pi i}{2}.$$
Proof

We compare $V_0^2$ with the analytic continuation $V_0^1$ to $\omega_2$.

Let us recall the Borel sums $V_0^1$ and $V_0^2$ of $V_0$ in $\omega_1$ and $\omega_2$ have the following forms, respectively:

$$V_0^2 = \frac{1}{2} \log \frac{\Gamma\left(\frac{1}{2} + \alpha \eta\right)\Gamma\left(\frac{1}{2} + \beta \eta\right)\Gamma\left(\frac{1}{2} + (\gamma - \alpha)\eta\right)\Gamma\left(\frac{1}{2} + (\gamma - \beta)\eta\right)\gamma^{2n\gamma - 1}}{\Gamma^2(\gamma \eta)\alpha^\alpha \beta^\beta n(\gamma - \alpha)^{(\gamma - \alpha)\eta} (\gamma - \beta)^{(\gamma - \beta)\eta} 2\pi \eta}.$$  

$$V_0^1 = \frac{1}{2} \log \frac{\Gamma\left(\frac{1}{2} + \alpha \eta\right)\Gamma\left(\frac{1}{2} + \beta \eta\right)\Gamma\left(\frac{1}{2} + (\gamma - \alpha)\eta\right)(\beta - \gamma)^{(\beta - \gamma)\eta} \gamma^{2\gamma \eta - 1}}{\Gamma\left(\frac{1}{2} + (\beta - \gamma)\eta\right)\Gamma^2(\gamma \eta)\alpha^\alpha \beta^\beta n(\gamma - \alpha)^{(\gamma - \alpha)\eta} \eta},$$

$$= \frac{1}{2} \log \frac{\Gamma\left(\frac{1}{2} + \alpha \eta\right)\Gamma\left(\frac{1}{2} + \beta \eta\right)\Gamma\left(\frac{1}{2} + (\gamma - \alpha)\eta\right)(\gamma - \beta)^{(\beta - \gamma)\eta} \gamma^{2\gamma \eta - 1} - (\beta - \gamma)\eta \pi i}{\Gamma\left(\frac{1}{2} + (\beta - \gamma)\eta\right)\Gamma^2(\gamma \eta)\alpha^\alpha \beta^\beta n(\gamma - \alpha)^{(\gamma - \alpha)\eta} \eta} - \frac{(\beta - \gamma)\eta \pi i}{2},$$

Subtracting $V_0^2$ from the analytic continuation $V_0^1$ to $\omega_2$, we have

$$V_0^1 - V_0^2 = \frac{1}{2} \log \frac{2\pi}{\Gamma\left(\frac{1}{2} + (\gamma - \beta)\eta\right)\Gamma\left(\frac{1}{2} + (\beta - \gamma)\eta\right)} - \frac{(\gamma - \beta)\eta \pi i}{2}.$$
Hypergeometric differential equations
Stokes graphs
Voros coefficients
Borel sums of Voros coefficients
Parametric Stokes phenomena

Well-known

\[\Gamma\left(\frac{1}{2} + t\right)\Gamma\left(\frac{1}{2} - t\right) = \frac{\pi}{\cos \pi t},\]

\[\cos t = \frac{e^{it} + e^{-it}}{2}.\]

Hence we obtain

\[V^1_0 - V^2_0 = \frac{1}{2} \log(\exp 2(\gamma - \beta)\eta \pi i + 1).\]

In the same way, we can compute the relations between \(V^1_1\) and \(V^2_1\) and \(V^1_2\) and \(V^2_2\). Hence we have

\[V^1_1 - V^2_1 = \frac{1}{2} \log(\exp 2(\gamma - \beta)\eta \pi i + 1),\]

\[V^1_2 - V^2_2 = -\frac{1}{2} \log(\exp 2(\gamma - \beta)\eta \pi i + 1).\]
Parametric Stokes phenomena for the WKB solution

Re $\gamma > 0$ fixed
Re $\alpha$-Re $\beta$ plane

The WKB solutions are Borel summable in $R_1$ and $R_2$ (Koike and R. Schäfke, [7] to appear.)
**Parametric Stokes phenomena for the WKB solution**

**Re $\gamma > 0$ fixed**

**Re$\alpha$-Re$\beta$ plane**

$R_1$ and $R_2$:

*The Stokes regions.*

$0 < R_1 < a_1$ and $0 < R_2 < a_1$

$Re(\beta - \gamma)$

$0 < a_0 < 1$

$0 < a_1 < 1$

$(\alpha, \beta, \gamma) = (0.5, 1.05, 1)$ in $\omega_1$

$(0.5, 1 + 0.01i, 1)$

$(0.5, 0.95, 1)$ in $\omega_2$

The WKB solutions are Borel summable in $R_1$ and $R_2$ (Koike and R. Schäfke, [7] to appear.)
Let $\psi^1$ and $\psi^2$ denote the Borel sum of the WKB solution:

$$
\psi = \frac{1}{\sqrt{S_{\text{odd}}}} \exp(\int_{a_0}^{x} S_{\text{odd}} \, dx)
$$

in the Stokes region $R^1$ and $R^2$, respectively.

**Theorem 4**

We obtain the following relation between $\psi^1$ and $\psi^2$.

$$
\psi^1 = (1 + \exp(2\pi i(\gamma - \beta)\eta))^{-\frac{1}{2}} \psi^2.
$$
Let $\psi^1$ and $\psi^2$ denote the Borel sum of the WKB solution:

$$\psi = \frac{1}{\sqrt{S_{\text{odd}}}} \exp\left(\int_{a_0}^{x} S_{\text{odd}} dx\right)$$

in the Stokes region $R^1$ and $R^2$, respectively.

**Theorem 4**

We obtain the following relation between $\psi^1$ and $\psi^2$.

$$\psi^1 = (1 + \exp(2\pi i(\gamma - \beta)\eta))^{-\frac{1}{2}} \psi^2.$$
Moreover we use the following result by Koike and Schäfke ([7] to appear):

\[ \psi^{(0),1} = \psi^{(0),2} \]

and the following relation by Theorem 4:

\[ V_0^1 = V_0^2 + \frac{1}{2} \log(\exp 2(\gamma - \beta)\eta \pi i + 1). \]

Here \( \beta - \gamma = (\gamma - \beta)e^{-\pi i} \). Hence we have

\[ \psi^1 = (\exp(-V_0^1))\psi^{(0),1} \]
\[ = (1 + \exp(2\pi i(\gamma - \beta)\eta))^{-\frac{1}{2}} (\exp(-V_0^2))\psi^{(0),2} \]
\[ = (1 + \exp(2\pi i(\gamma - \beta)\eta))^{-\frac{1}{2}} \psi^2. \]

In the similar manner, we can compute other relations.
$Q$ : invariant under involutions $\iota_m$ $(m = 0, 1, \cdots, 6)$

$\iota_0$ : $(\alpha, \beta, \gamma) \mapsto (-\alpha, -\beta, -\gamma)$
$\iota_1$ : $\mapsto (\gamma - \beta, \gamma - \alpha, \gamma)$
$\iota_2$ : $\mapsto (\beta, \alpha, \gamma)$
$\iota_3 = \iota_1 \iota_2$ : $\mapsto (\gamma - \alpha, \gamma - \beta, \gamma)$
$\iota_4 = \iota_0 \iota_2$ : $\mapsto (-\beta, -\alpha, -\gamma)$
$\iota_5 = \iota_0 \iota_1$ : $\mapsto (\beta - \gamma, \alpha - \gamma, -\gamma)$
$\iota_6 = \iota_0 \iota_1 \iota_2$ : $\mapsto (\alpha - \gamma, \beta - \gamma, -\gamma)$

$\omega_{hm} = \iota_m(\omega_h)$ : Images in $\omega_h$ by $\iota_m$.
Here, $h = 1, 2, 3, 4$, $m = 0, 1, \cdots, 6$. 
**Reα-Reβ planes**

Reγ > 0

Reγ < 0
Re$\alpha$-Re$\beta$ planes

Re$\gamma > 0$

Re$\gamma < 0$
Hypergeometric differential equations
Stokes graphs
Voros coefficients
Borel sums of Voros coefficients
Parametric Stokes phenomena

\[ (\alpha, \beta, \gamma) = (0.5, 1.05, 1) \text{ in } \omega_1 \quad (0.5, 1 + 0.01i, 1) \quad \text{ and } \quad (0.5, 0.95, 1) \text{ in } \omega_2 \]

\[ \iota_1 : (\alpha, \beta, \gamma) \mapsto (\gamma - \beta, \gamma - \alpha, \gamma) \]
The WKB solutions are Borel summable in $R_{11}$ and $R_{21}$. The top and bottom Stokes graphs are same graphs, respectively.
Let $\psi^{11}$ and $\psi^{21}$ denote the Borel sum of the WKB solution:

$$\psi = \frac{1}{\sqrt{S_{\text{odd}}}} \exp\left( \int_{a_0}^{x} S_{\text{odd}} dx \right)$$

in the Stokes region $\mathcal{R}^{11}$ and $\mathcal{R}^{21}$, respectively.

**Theorem 4**

We obtain the following relation between $\psi^1$ and $\psi^2$.

$$\psi^1 = (1 + \exp(2\pi i (\gamma - \beta)\eta))^{-\frac{1}{2}} \psi^2.$$ 

$$\iota_1 : (\alpha, \beta, \gamma) \mapsto (\gamma - \beta, \gamma - \alpha, \gamma)$$

**Theorem 5**

We obtain the following relation between $\psi^{11}$ and $\psi^{21}$.

$$\psi^{11} = (1 + \exp(2\pi i \alpha \eta))^{-\frac{1}{2}} \psi^{21}.$$
References


Thank you for your attention.