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Summability and global solvability of the linearization problem for a singular vector field

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1. Introduction

Let $y = (y_1, \ldots , y_n) \in \mathbb{C}^n$, $n \geq 2$ be the variable in $\mathbb{C}^n$, and consider a holomorphic vector field in some domain of $\mathbb{C}^n$ containing the origin

$$\mathcal{X} = \sum_{j=1}^{n} a_j(y)\frac{\partial}{\partial y_j}. \tag{1}$$

We assume that the number of singular points of $\mathcal{X}$ is finite, hence the singular points are isolated. Moreover we suppose $a_j(0) = 0$ for $j = 1, \ldots , n$.

Assume that the change of coordinates preserving the origin

$$y = u(x), \ u = (u_1, \dots, u_n), \quad x = (x_1, \dots, x_n), \ n \geq 1$$

transforms $\mathcal{X}$ to its linear part. This is equivalent to

$$A(u(x)) \left(\frac{\partial u}{\partial x}\right)^{-1} = x\Lambda, \tag{2}$$

where $A(y) = (a_1(t), \dots, a_n(y))$ and $\frac{\partial u}{\partial x}$ is the Jacobian matrix, and $\Lambda = DA(0)$ is the linear part of $\mathcal{X}$ at the origin.

We define $v(x)$ by $u(x) = x + v(x)$, $v(x) = O(|x|^2)$ and we set

$$A(y) = y\Lambda + R(y), \ R(y) = (R_1(y), \dots, R_n(y)) = O(|y|^2).$$

Then we have $A(u) = (x+v)\Lambda + R(x+v)$ and $\frac{\partial u}{\partial x} = I + \frac{\partial v}{\partial x}$. Hence by (2) we have

$$A(u(x)) = (x+v)\Lambda + R(x+v) = x\Lambda\left(I + \frac{\partial v}{\partial x}\right).$$

It follows that our linearization condition can be written in

$$v\Lambda + R(x+v) = x\Lambda \frac{\partial v}{\partial x}. \tag{3}$$

This is a system of semilinear first order partial differential equations for v. Let λ_j, $j = 1, \dots, n$ be the eigenvalues of Λ. Poincaré's theorem asserts the existence of a local holomorphic solution provided the non resonance condition and the Poincaré condition, Re $\lambda_j > 0$ are verified. We note that the solution is not defined globally in general because the nonlinear term $R(x+v)$ may cause the singularity.

We note that similar relation like (3) holds at every isolated singular point of $\mathcal{X}$.

Instead of solving (3) globally we introduce a parameter η in the equation and we want to construct an approximate global transformation. Namely, we approximate our equation with the following

$$v\Lambda + R(x+v) = \eta^{-1} x \Lambda \frac{\partial v}{\partial x}. \tag{4}$$

Clearly if $\eta = 1$, then we have the linearization equation (3).

Motivations

O. Costin and R. Costin study the simultaneous normalization of vector field in the domain containing two isolated points by using transseries (= exponential log series expansion). They showed that the theorem like simultaneous Poincaré's theorem at two equilibrium points does not hold in general.

In this talk I will show the global solvability of (4) by virtue of the Borel sum with respect to η of some formal series solution. Then we show that the solution of (4) is naturally related to the solution of the original equation (3).

2. Formal solution

Assume that Λ is a diagonal matrix, $\Lambda = \mathrm{diag}\,(\lambda_1, \ldots, \lambda_n)$. Define

$$\mathcal{L} = \sum_{j=1}^{n} \lambda_j x_j \frac{\partial}{\partial x_j}. \tag{5}$$

Then (4) is written in

$$\eta^{-1}\mathcal{L}v_j = \lambda_j v_j + R_j(x + v(x)), \qquad j = 1, \ldots, n. \tag{6}$$

Definition 1 A singular perturbative solution (SP-solution in short) $v(x, \eta)$ of (6) is the formal power series in η^{-1} of the form

$$v(x, \eta) = \sum_{\nu=0}^{\infty} \eta^{-\nu} v_\nu(x) = v_0(x) + \eta^{-1} v_1(x) + \cdots, \tag{7}$$

where the coefficients $v_\nu(x)$ are holomorphic vector functions of x in some open set independent of ν.

We want to construct the SP-solution of (6) in the following form

$$v_j \equiv v_j(x, \eta) = \sum_{\nu=0}^{\infty} v_\nu^j(x)\eta^{-\nu}, \qquad v_\nu^j(x) = O(|x|^2),\; j = 1, \ldots, n. \tag{8}$$

By substituting the expansion (8) into (6), we obtain

$$\mathcal{L}v_j = \sum_{\nu=0}^{\infty} \mathcal{L}v_\nu^j(x)\eta^{-\nu}, \tag{9}$$

$$\begin{aligned} R_j(x+v) &= R_j(x+v_0+v_1\eta^{-1}+v_2\eta^{-2}+\cdots) \\ &= R_j(x+v_0)+\eta^{-1}\sum_{k=1}^{n}\left(\frac{\partial R_j}{\partial z_k}\right)(x+v_0)v_1^k+O(\eta^{-2}). \end{aligned} \tag{10}$$

By comparing the coefficients of η, $\eta^0 = 1$ we obtain

$$\lambda_j v_0^j(x) + R_j(x_1+v_0^1,\ldots,x_n+v_0^n) = 0. \tag{11}$$

$$\mathcal{L}v_0^j = \lambda_j v_1^j + \sum_{k=1}^{n}\left(\frac{\partial R_j}{\partial z_k}\right)(x+v_0)v_1^k. \tag{12}$$

In the following we assume that v_0 is determined as a holomorphic function in $\Omega(v_0)$ which contains the origin. In order to determine $v_\nu(x)$ $(\nu \geq 2)$ we compare the coefficients of $\eta^{-\nu}$ of (6). We differentiate (10) with respect to $\varepsilon = \eta^{-1}$, $\nu - 1$ times and we put $\varepsilon = 0$. We obtain

$$\begin{aligned} \mathcal{L}v_{\nu-1}^j &= \lambda_j v_\nu^j + \sum_{k=1}^{n}\left(\frac{\partial R_j}{\partial z_k}\right)(x+v_0)v_\nu^k \\ &+ \text{(terms consisting of } v_k^j,\ k \leq \nu-1 \text{ and } j = 1,\ldots,n). \end{aligned} \tag{13}$$

Define

$$\Sigma_0 := \{x \in \mathbb{C}^n;\ \det(\Lambda + \nabla R(x + v_0(x))) = 0\}. \tag{14}$$

In the rest of talk we assume

$$0 \notin \Sigma_0. \tag{15}$$

Note that (15) implies $\lambda_k \neq 0$ for every k. The next theorem gives the existence of the SP-solution.

Proposition 1 Assume (15). Then every coefficient of the SP-solution (8) is uniquely determined as a holomorphic function in a neighborhood of the origin $x = 0$ independent of ν.

Remark. Let $\widetilde{\mathbb{C}^n \setminus \Sigma_0}$ be the universal covering space of $\mathbb{C}^n \setminus \Sigma_0$. We can make analytic continuation of the formal SP-solution in Proposition 1 from the origin to $\widetilde{\mathbb{C}^n \setminus \Sigma_0} \cap \Omega(v_0)$, provided that $R(x)$ is an entire function on $x \in \mathbb{C}^n$.

3. Definition of Borel sum

The formal Borel transform of $v(x, \eta)$ is defined by

$$B(v)(x, \zeta) := \sum_{\nu=0}^{\infty} v_\nu(x) \frac{\zeta^\nu}{\Gamma(\nu + 1)}, \tag{16}$$

where $\Gamma(z)$ is the Gamma function.

For an opening $\theta > 0$ and the direction ξ we define the sector $S_{\theta,\xi}$ with the bisecting direction ξ and opening θ by

$$S_{\theta,\xi} = \left\{ z \in \mathbb{C};\ |\arg z - \xi| < \frac{\theta}{2} \right\}. \tag{17}$$

We say that $v(x,\eta)$ is Borel summable in the direction ξ with respect to η if $B(v)(x,\zeta)$ converges in some neighborhood of the origin of (x,ζ), and there exist a neighborhood U of the origin $x = 0$ and a $\theta > 0$ such that $B(v)(x,\zeta)$ can be analytically continued to $U \times S_{\theta,\xi}$ having exponential growth of order 1 with respect to ζ in $S_{\theta,\xi}$. The Borel sum $V(x,\eta)$ of $v(x,\eta)$ is, then, given by the Laplace transform

$$V(x,\eta) := \int_{L_\xi} \zeta^{-1} e^{-\zeta\eta} B(v)(x,\zeta) d\zeta \tag{18}$$

where the integral is taken on the ray L_ξ starting from the origin to the infinity in the direction ξ.

4. Convergence of the formal Borel transform

Theorem 2 Assume that $R(x)$ is an entire function on $x \in \mathbb{C}^n$. Let v be the SP-solution given by (8). Let K be the compact set in $\widetilde{\mathbb{C}^n \setminus \Sigma_0} \cap \Omega(v_0)$. Suppose that every $v_\nu(x)$ in v is analytic in some neighborhood of K independent of ν. Then there exist a neighborhood U of K and a neighborhood W of the origin $\zeta = 0$ in $\mathbb{C}$ such that the formal Borel transform $B(v)(x, \zeta)$ converges in $U \times W$.

Remark. If K is a neighborhood of the origin $x = 0$, then we only need to assume that $R(x)$ is analytic in some neighborhood of the origin $x \in \mathbb{C}^n$.

Proof. The compact set K can be covered by a finite number of open balls. Hence it is sufficient to show our theorem when K is a subset of an open small ball. One may also assume that the center of the ball is the origin. We use the notation $u \ll v$ when v is the majorant function of u. Let $\rho > 0$ and define

$$\phi_\rho(x) := \left(1 - \frac{x_1 + \cdots + x_n}{\rho}\right)^{-1}. \tag{19}$$

The set of holomorphic functions at the origin such that $u \ll \phi_\rho C$ for some $C \geq 0$ forms a Banach space with the norm $\|u\|$ given by the infimum of C satisfying $u \ll \phi_\rho C$.

First we will estimate the differentiation. For any integers $1 \leq j \leq n$ and $k \geq 1$ we have

$$\frac{\partial}{\partial x_j} \phi_\rho(x)^k = \frac{k}{\rho} \phi_\rho(x)^{k+1}. \tag{20}$$

On the other hand, because $(\Lambda + \nabla R)^{-1} x_j$ is analytic at the origin for $1 \leq j \leq n$ we have

$$(\Lambda + \nabla R)^{-1} x_j \ll K\phi_\rho \tag{21}$$

for some $K > 0$. We now estimate v_1. By virtue of (12) we have $v_1 = (\Lambda + \nabla R)^{-1} \mathcal{L} v_0$. We have $v_0 \ll \|v_0\| \phi_\rho$. Hence, by (20) and (21) we have $v_1 \ll \|v_0\| C_1 \phi_\rho^3$ for some $C_1 > 0$. We will show that there exist $C \geq 1$ independent of $\nu \geq 1$ such that

$$v_m \ll C^{2m-1} m! \phi_\rho^{4m-1}, \qquad m = 1, 2, \ldots \tag{22}$$

The rest proof is done by induction.

5. Summability at the origin

Define C_0 as the smallest convex closed cone with vertex at the origin containing λ_j $(j = 1, 2, \ldots, n)$. Then we have

Theorem 3 Suppose (15). Assume that $\nabla R(x+v_0)$ is a diagonal matrix. Assume that there exist a real ξ such that

$$|\arg \lambda_j - \xi| < \pi/4 \quad \text{for } j = 1, 2, \ldots, n. \tag{23}$$

Then there exists a neighborhood U of the origin of x such that $v(x, \eta)$ is Borel-summable in the direction η such that $\eta^{-1} \in (C_0)^c$ and $x \in U$, where $(C_0)^c$ is the complement of C_0 in the complex plane.

6. Convolution

We estimate the convolution. Let Ω be an open set containing the sector $S_{\pi,\theta}$ in (17) and the disk $\{|z|<r_0\}$ for small $r_0>0$ such that $z\in\Omega$ implies $z+t\in\Omega$ for every real number $t\le 0$. Let $c>0$ and let $\mathcal{H}(\Omega)$ be defined by

$$\mathcal{H}(\Omega) := \left\{ f\in H(\Omega) \mid \exists K \text{ such that } |f(z)|\le Ke^{-c\mathbf{Re}\,z}(1+|z|)^{-2}, \forall z\in\Omega \right\}, \tag{24}$$

where $H(\Omega)$ is the set of holomorphic functions in Ω. Obviously, $\mathcal{H}(\Omega)$ is the Banach space with the norm

$$\|f\|_\Omega := \sup_{z\in\Omega} |f(z)|(1+|z|)^2 e^{c\mathbf{Re}\,z}. \tag{25}$$

Let $f,g\in\mathcal{H}(\Omega)$ be given. The convolution $f*g$ of f and g is defined by

$$(f*g)(z) := \frac{d}{dz}\int_0^z f(z-t)g(t)dt = \frac{d}{dz}\int_0^z f(t)g(z-t)dt. \tag{26}$$

Let $f'(z)=(df/dz)(z)$. We will show the following

Proposition 4 For every $f,g\in\mathcal{H}(\Omega)$ such that $f(0)=g(0)=0$ and $f',g'\in\mathcal{H}(\Omega)$ we have $f*g\in\mathcal{H}(\Omega)$ with the estimate

$$\|f*g\|_\Omega \le 8\|f'\|_\Omega\|g\|_\Omega, \quad \|f*g\|_\Omega \le 8\|f\|_\Omega\|g'\|_\Omega. \tag{27}$$

Proof. **Because $f * g = g * f$ we will prove the first inequality of (27). We have**

$$(f * g)(z) = \frac{d}{dz}\int_0^z f(z-t)g(t)dt = f(0)g(z) + \int_0^z f'(z-t)g(t)dt = \int_0^z f'(z-t)g(t)dt.$$

By (25) and by taking the path of integration from 0 to z we have

$$\begin{aligned} \left|\int_0^z f'(z-t)g(t)dt\right| &\leq \|f'\|_\Omega \|g\|_\Omega e^{-c\mathbf{Re}\,z}\int_0^z (1+|z-t|)^{-2}(1+|t|)^{-2}|dt| \quad (28) \\ &\leq \|f'\|_\Omega \|g\|_\Omega e^{-c\mathbf{Re}\,z}\int_0^{|z|} (1+|z|-s)^{-2}(1+s)^{-2}ds. \end{aligned}$$

We divide the integral in the right-hand side into two parts, $s \leq \frac{|z|}{2}$ and $s > \frac{|z|}{2}$. If $s \leq \frac{|z|}{2}$, then we have $(1+|z|-s)^{-2} \leq 4(1+|z|)^{-2}$, while in case $s > \frac{|z|}{2}$ we have $(1+s)^{-2} \leq 4(1+|z|)^{-2}$. Hence we have

$$\int_0^{|z|/2} \frac{1}{(1+|z|-s)^2(1+s)^2}ds \leq \frac{4}{(1+|z|)^2}\int_0^{|z|/2}(1+s)^{-2}ds \leq \frac{4}{(1+|z|)^2}. \quad (29)$$

One can similarly estimate the other part like $\int_{|z|/2}^{|z|}(1+|z|-s)^{-2}(1+s)^{-2}ds \leq 4(1+|z|)^{-2}$. Therefore we see that the left-hand side term of (28) can be estimated by $8\|f'\|_\Omega\|g\|_\Omega e^{-c\mathbf{Re}\,z}(1+|z|)^{-2}$. This ends the proof.

7. Proof of Theorem 3

We will consider (6) or, equivalently,

$$\mathcal{L}v = \eta\Lambda v + \eta R(x+v).$$

Set $v = v_0 + u$, where $\Lambda v_0 + R(x+v_0) = 0$. In terms of the definition of $\mathcal{L}$ we obtain

$$\mathcal{L}u = -\mathcal{L}v_0 + \eta(\Lambda + \nabla R(x+v_0))u + \eta \sum_{|\beta|\geq 2} r_\beta(x+v_0)u^\beta. \tag{30}$$

Let $\hat{u}(y) := \mathcal{B}(u)$ be the Borel transform of u with respect to η, where y is the dual variable of η. By the Borel transform of (30) we obtain

$$\mathcal{L}\hat{u} = -\mathcal{L}v_0 + (\Lambda + \nabla R(x+v_0))\frac{\partial \hat{u}}{\partial y} + \frac{\partial}{\partial y}\sum_{|\beta|\geq 2} r_\beta(x+v_0)(\hat{u})_*^\beta, \tag{31}$$

where $(\hat{u})_*^\beta = (\hat{u}_1)_*^{\beta_1}\cdots(\hat{u}_n)_*^{\beta_n}$, $\beta = (\beta_1, \ldots, \beta_n)$, and $(\hat{u}_j)_*^{\beta_j}$ is the β_j-convolution product, $(\hat{u}_j)_*^{\beta_j} = \hat{u}_j * \cdots * \hat{u}_j$.

Let $(\nabla R)_j$ be the j-th diagonal component of the matrix ∇R. Consider the linear part of (31)

$$\mathcal{L}w_j - (\lambda_j + (\nabla R)_j(x+v_0))\frac{\partial w_j}{\partial y} = f, \qquad j = 1, 2, \ldots, n, \tag{32}$$

where $f \equiv f(x,y)$ is a holomorphic function of $x \in \mathbb{C}^n$ and $y \in \mathbb{C}$. Consider the characteristic equation corresponding to (32)

$$\frac{d\zeta}{\zeta} = \frac{dx_k}{\lambda_k} = -\frac{dy}{\lambda_j + (\nabla R)_j(x+v_0)}, \qquad k = 1,2,\ldots,n-1. \tag{33}$$

By integration we have

$$x_k = c_k \zeta^{\lambda_k} \ (k = 1,2,\ldots,n-1),\ \ y = y_0 - \Phi(\zeta, b), \tag{34}$$

where c_k's and y_0 are some constants. Fix a branch of v_0 and define

$$\Phi(\zeta, b) \equiv \Phi_j(\zeta, b) = \int_b^\zeta \frac{\lambda_j + (\nabla R)_j(x + v_0(x))}{s} ds, \tag{35}$$

where $x = (x_1, \ldots, x_n)$, $x_k = c_k s^{\lambda_k}$ $(k = 1,2,\ldots,n-1)$ and $b \in \mathbb{C}$. Note that the relations (34) give the (multi-valued) change of variable between (x_k, ζ, y) and (c_k, ζ, y_0).

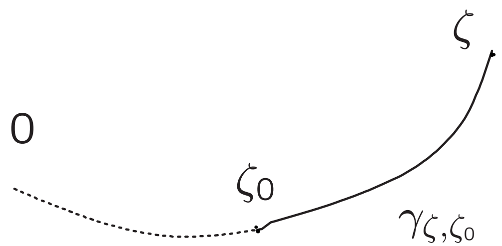
We will prove the existence of the SP-solution when x is in some neighborhood of the origin.

We will prove the solvability of (31) when x is in some open set and $y \in \Omega$.

(Proof. Let v be the formal SP-solution in Proposition 2 and define $\hat{u}(x,y) := B(v) - v_0$, where $B(v)$ is the formal Borel transform of v. We know that $\hat{u}(x,y)$ is analytic when y is in some neighborhood of the origin and x is in some open set in $\widetilde{\mathbb{C}^n \setminus \Sigma_0} \cap \Omega(v_0)$. Moreover, by definition $\hat{u}$ is the solution of (31) in some neighborhood of $y = 0$ such that $\hat{u}(x, 0) \equiv 0$ in x. We will show that every solution of (31) which is analytic at $y = 0$ and satisfies $\hat{u}(x, 0) \equiv 0$ is uniquely determined. Indeed, by the definition of convolution product of $y^i/i!$ and $y^j/j!$ is $y^{i+j}/(i+j)!$. Hence, if we expand $\hat{u}$ in the power series of y and insert (31), then every coefficient of the expansion can be uniquely determined from the recurrence relation because $\Lambda + \nabla R(x + v_0)$ is invertible. Therefore, if we can show the existence of the solution of (31) which is analytic in (x, y) with x in some open set and $y \in \Omega$ and of exponential growth with respect to y in Ω, then we have the analytic continuation of the formal Borel sum of v with exponential growth in $y \in \Omega$.)

Solvability of linear equation

The linear part of (31) is given by (32). In the following we omit the suffix j of w_j and write it w instead of w_j.



Let b be in some neighborhood of the origin of L. Then the solution of (32) such that $w(\eta) \to 0$ as $\zeta \to 0$ is given by

$$w \equiv P_0 f = \int_{\gamma_{\zeta_0,\zeta}} f(s^{\lambda_1} c_1, \cdots, s^{\lambda_{n-1}} c_{n-1}, s; y_0 - \Phi(s,b)) ds, \tag{36}$$

where the integral is taken along the path γ_{ζ,ζ_0} which is the Stokes curve of $\Phi(\cdot, b)$ emanating from the origin and passes ζ and ζ_0 in this order. Here we change the variables in (36) after integration in terms of (34).

Well-definedness of the integrand in (36).

First we show that

$$\Phi(s,b) = \lambda_j \log\left(\frac{s}{b}\right) + o(s,b) \quad \text{when } s, b \to 0. \tag{37}$$

Proof. We know that $(\nabla R)_j(x + v_0(x)) = O(|x|)$ as $x \to 0$. Because $\mathrm{Re}\,\lambda_k > 0$, the integral $\int_b^s t^{-1} (\nabla R)_j(x + v_0(x)) dt$ with $x_k = c_k t^{\lambda_k}$ has the limit when $s \to 0$ in some sector. Hence we have (37).

By (34) we have $y_0 - \Phi(s,b) = y - \Phi(s,b) + \Phi(\zeta, b) = y + \Phi(\zeta, s)$. By the definition of the Stokes curve we have that $\mathrm{Im}\Phi(\zeta, s) = 0$ if $s \in \gamma_{\zeta,\zeta_0}$. On the other hand one can easily show that $\mathrm{Re}\,\Phi(\zeta, s)$ is a monotone function of ζ on the Stokes curve. In view of (37) $\mathrm{Re}\,\Phi(\zeta, s)$ tends to $-\infty$ as $\zeta \to 0$. Hence $\mathrm{Re}\,\Phi(\zeta, s)$ is a monotone increasing function on the Stokes curve as $|\zeta|$ increases. We have $\mathrm{Re}\,\Phi(\zeta, s) \leq 0$ if $s \in \gamma_{\zeta,\zeta_0}$. In view of the assumption on Ω we have $y_0 - \text{Re}\,\Phi(s, b) = y + \text{Re}\,\Phi(\zeta, s) \in \Omega$ for every $y \in \Omega$ and $s \in \gamma_{\zeta,\zeta_0}$.

Next we take a neighborhood U_0 of the origin such that the formal SP-solution is holomorphic in U_0. Let γ_{ζ,ζ_0} be the Stokes curve as above. We want to substitute $x_k = s^{\lambda_k} c_k$ into the integrand of (36) for $s \in \gamma_{\zeta,\zeta_0}$. In order to show that this is possible uniformly when ζ tends to zero along the Stokes curve emanating from the origin it is sufficient to show that $|x_k|$ is sufficiently small. For this purpose we will consider

$$\log x_k = \log c_k + \lambda_k \log s = \log(c_k b^{\lambda_k}) + \frac{\lambda_k}{\lambda_j} \lambda_j \log\left(\frac{s}{b}\right). \tag{38}$$

By virtue of (37), $\lambda_j \log(s/b)$ is close to $\Phi(s, b)$ and hence Im $(\lambda_j \log(s/b))$ is close to Im $\Phi(s, b)$. Because Im $\Phi(s, b)$ is constant on every Stokes curve, we may consider Im $(\lambda_j \log(\zeta_0/b))$ instead of Im $(\lambda_j \log(s/b))$. It follows that there exists $K_0 > 0$ depending only on ζ_0/b such that

$$-K_0 < \text{Im}(\lambda_j \log(s/b)) < K_0.$$

On the other hand, Re $\Phi(s, b)$ is monotone increasing along the Stokes curve. It is bounded by Re $\Phi(\zeta_0, b)$. By taking the maximum on $|\zeta_0| = const$ there exists K_1 independent of $|\zeta_0| = const$ such that Re $(\lambda_j \log(s/b)) \le K_1$ for all $s \in \gamma_{\zeta,\zeta_0}$. On the other hand, by the strong Poincaré condition (23) we see that Re $(\lambda_k/\lambda_j) > 0$ for every j and k. Hence there exists ε_0 such that $(\lambda_k/\lambda_j)\lambda_j \log(s/b)$ is contained in the set $\{z; |arg\, z - \pi| < \pi/2 - \varepsilon\}$ for all $s \in \gamma_{\zeta,\zeta_0}$ except for a bounded set.

Because Re $(\log(c_k b^{\lambda_k}))$ tends to $-\infty$ when b tends to zero, we choose b sufficiently small, then choose ζ_0 so that $|\zeta_0|/|b|$ so small. We see that the right-hand side of (38) stays in the left-half plane such that the real part is arbitrarily small. Therefore we see that x_k lies in a sufficiently small neighborhood of the origin for $s \in \gamma_{\zeta,\zeta_0}$ uniformly when ζ moves to 0 along the Stokes curve. This proves that the substitution $x_k = s^{\lambda_k} c_k$ for $s \in \gamma_{\zeta,\zeta_0}$ into the integrand of (36) is well defined uniformly when $\zeta \to 0$ and ζ_0.

The integrability in (36) is clear for every given b because the integrand is continuous and the integral is taken on a compact smooth curve.

Definition of the function space.

Let D be the open connected set in some neighborhood of the origin of $x \in \mathbb{C}^n$. Then we define

$$\mathcal{H}(D,\Omega) := \left\{ f \in H(D,\Omega) \mid \exists K,\ \sup_{x\in D} |f(x,y)| \le K e^{-c\mathbf{Re}\, y}(1+|y|)^{-2}, \forall y \in \Omega \right\}, \tag{39}$$

where $H(D,\Omega)$ is the holomorphic function in $(x,y) \in D \times \Omega$. The space $\mathcal{H}(D,\Omega)$ is a Banach space with the norm $\|f\| = \sup K$ where K is given in (39).

Estimate of w and w_y in (36) for $f \in \mathcal{H}(D,\Omega)$

In the following we assume that there exists an $\varepsilon_0 > 0$ such that $|\zeta|/|\zeta_0| \geq \varepsilon_0$. We now estimate w in (36). We recall that $\Phi(s, b)$ is asymptotically equals to $\lambda_j \log(s/b)$ as $s \to 0$. Therefore one may assume that the integral is taken along the curve $\mathrm{Im}\,\lambda_j \log(s/b) = c$ for some c. Set $\lambda_j = \alpha + i\beta$ $(\alpha > 0)$ and $\log(s/b) = x + iy$. Then one can see that the curve $\mathrm{Im}\,\lambda_j \log(s/b) = c$ can be written in $\beta x + \alpha y = c$, and the integration is taken for some $x_1 \geq x \geq x_0$, where x_0 corresponds to ζ. In view of the relation $s = be^{x+iy}$, we have $ds = be^{x+iy}(dx + idy) = be^{x+iy}(1 - \beta i/\alpha)dx$. Because there appears a positive power of s in the integrand of (36) in view of the above argument, a positive power of e^x appears from the integrand. We next estimate the growth of $y_0 - \Phi(s, b)$. In terms of (34) we have

$$\begin{aligned} \exp\left(-c\mathbf{Re}\left(y_0 - \Phi(s, b)\right)\right) &= \exp\left(-c\mathbf{Re}\left(y + \Phi(\zeta, b) - \Phi(s, b)\right)\right) \\ &= \exp\left(-c\mathbf{Re}\left(y + \Phi(\zeta, s)\right)\right). \end{aligned} \tag{40}$$

Because $\mathrm{Re}\,\Phi(\zeta, s)$ is decreasing in ζ as ζ tends to zero along the Stokes curve, we have $\mathrm{Re}\,\Phi(\zeta, s) \leq 0$. Hence we need to estimate $e^{-c\mathbf{Re}\,\Phi(\zeta,s)}$. We have that $\Phi(\zeta, s)$ is asymptotically equal to $\lambda_j \log(\zeta/s)$. Set $\log(\zeta/s) = x + iy$ and $\lambda_j = \alpha + i\beta$ with $\alpha > 0$. Then we have $\mathrm{Re}\,(\lambda_j \log(\zeta/s)) = \alpha x - \beta y$. On the other hand the definition of the Stokes curves yields $\beta x + \alpha y = c$ for some c. Hence $\alpha x - \beta y = (\alpha + \beta^2\alpha^{-1})x - c\beta\alpha^{-1}$. Noting that $x = \log(|\zeta|/|s|) > \log(|\zeta|/|\zeta_0|) > \log \varepsilon_0$, we have

$$\begin{aligned} \exp(-c(\alpha x - \beta y)) &= \exp(-(\alpha + \beta^2\alpha^{-1})cx - c^2\beta\alpha^{-1}) \\ &\leq \exp\left((\alpha + \beta^2\alpha^{-1})c \log \varepsilon_0^{-1} - c^2\beta\alpha^{-1}\right) =: K_0. \end{aligned}$$

This proves that

$$\exp\left(-c\mathbf{Re}\left(y_0 - \Phi(s,b)\right)\right) \le K_0 \exp\left(-c\mathbf{Re}\, y\right). \tag{41}$$

Next we will estimate $|y_0 - \Phi(s,b)| = |y + \Phi(\zeta, s)|$ from the below. Because $\mathbf{Im}\,\Phi(\zeta,s) = 0$ on the Stokes curve and $\mathbf{Re}\,\Phi(\zeta,s) \le 0$, there exists $C_1 > 0$ independent of ζ and s such that

$$(1 + |y_0 - \Phi(s,b)|)^{-2} \le C_1 (1 + |y|)^{-2} \quad \text{for all } y \in \Omega. \tag{42}$$

By (41) and (42) there exists $C_3 > 0$ such that

$$\|w\| \le C_3 \|f\|.$$

Proof.

$$\begin{aligned} \|w\| \quad &\le \quad \sup\Big(|(1+|y|)^2 \exp\left(c\mathbf{Re}\, y\right) \times \\ &\int \|f\| \frac{\exp\left(-c\mathbf{Re}\left(y_0 - \Phi(s,b)\right)\right)}{(1+|y_0 - \Phi(s,b)|)^2} |ds|\Big) \le C_2 \|f\| \int |ds| \le C_3 \|f\|, \end{aligned} \tag{43}$$

for some $C_2 > 0$ and $C_3 > 0$.

We shall show

$$\|w_y\| \le C_4 \|f\| \tag{44}$$

for some $C_4 > 0$ independent of f.

Proof. Noting that $y_0 - \Phi(s,b) = y + \Phi(\zeta, s)$ we make the change of variable $\sigma = y + \Phi(\zeta, s)$ in (36) from s to σ. We have

$$d\sigma = -\frac{\lambda_j + \nabla R_j}{s} ds.$$

Note that the right-hand side is independent of y. We have $\sigma = y$ for $s = \zeta$ and $\sigma = y + \tilde{\zeta}_0$, where $\tilde{\zeta}_0 = \Phi(\zeta, \zeta_0)$. Clearly, $s \in \gamma_{\zeta_0, \zeta}$ is expressed as $\sigma \in y + \gamma_{\tilde{\zeta}_0, \zeta}$, where $\gamma_{\tilde{\zeta}_0, \zeta}$ is the straight line connecting 0 and $\tilde{\zeta}_0$. Then (36) is written in

$$w = -\int_{\gamma_{\tilde{\zeta}_0, \zeta}} f(s^{\lambda_1} c_1, \cdots, s^{\lambda_{n-1}} c_{n-1}, s; \sigma) \frac{d\sigma}{\partial_s \Phi(s, \zeta)}, \tag{45}$$

where $\sigma - y = \Phi(\zeta, s) \sim \lambda_j \log(\zeta/s)$ and s is independent of y. Hence we have

$$\begin{aligned} w_y &= -f(\zeta_0^{\lambda_1} c_1, \cdots, \zeta_0^{\lambda_{n-1}} c_{n-1}, \zeta_0; y + \tilde{\zeta}_0) \frac{1}{\partial_s \Phi(\zeta_0, \zeta)} \\ &+ f(\zeta^{\lambda_1} c_1, \cdots, \zeta^{\lambda_{n-1}} c_{n-1}, \zeta; y) \frac{1}{\partial_s \Phi(\zeta, \zeta)}. \end{aligned} \tag{46}$$

Using (46) we have (44) by the same argument as $\|w\|$ since $\partial_s \Phi(\zeta_0, \zeta)^{-1}$ and $\partial_s \Phi(\zeta, \zeta)^{-1}$ are bounded.

Approximate sequence.

We will solve (31) in $\mathcal{H}(D,\Omega)$. We define the approximate sequence $\hat{u}_n$ $(n = 0, 1, 2, \dots)$ by

$$\hat{u}_1 = -P_0 \mathcal{L} v_0, \quad \hat{u}_0 = 0 \tag{47}$$

$$\hat{u}_2 = P_0 \sum_{|\beta| \geq 2} r_\beta(x + v_0) \frac{\partial}{\partial y} (\hat{u}_1)_*^\beta - P_0 \mathcal{L} v_0, \tag{48}$$

$$\vdots$$

$$\hat{u}_{n+1} = P_0 \sum_{|\beta| \geq 2} r_\beta(x + v_0) \frac{\partial}{\partial y} (\hat{u}_n)_*^\beta - P_0 \mathcal{L} v_0, \tag{49}$$

where $n = 1, 2, \dots$

Apriori estimate.

By definition for any $\varepsilon > 0$ we can take the domain D sufficiently small that $\|\mathcal{L} v_0\| \leq \varepsilon$. By (43) we have

$$\|\hat{u}_1\| \leq \|P_0 \mathcal{L} v_0\| \leq C \|\mathcal{L} v_0\| \leq C\varepsilon. \tag{50}$$

Similarly by using (44) we have $\|(\hat{u}_1)_y\| \leq C\varepsilon$.

We have the apriori estimate

$$\|\hat{u}_n\| \leq C\varepsilon(1 + \varepsilon), \quad \|(\hat{u}_n)_y\| \leq C\varepsilon(1 + \varepsilon), \ n = 0, 1, 2, \dots \tag{51}$$

We will show the convergence of $\hat{u}_n$ by standard argument.

Revomal of singularities

-Hartogs type theorem for functions with exponential growth -

Let D and D' be domains such that $D \cap D' \neq \emptyset$ and let v_D and $v_{D'}$ be the corresponding Borel sum in D and D', respectively. Because the Borel sum with respect to η is unique for every x, we have that $v_D = v_{D'}$ on $D \cap D'$, from which we have an analytic continuation of v_D to $D \cup D'$. By choosing the sequence of open sets D we make an analytic continuation of v_D to the set $(\mathbb{C} \setminus 0)^n \cap B_0$, where B_0 is some open ball centered at the origin. By the uniqueness of the Borel sum the analytic continuation of $\hat{v}_D(x, y)$ with respect to x in the set $(\mathbb{C} \setminus 0)^n \cap B_0$, $y \in \Omega$ is single-valued. We also note that in view of the construction of $\hat{v}_D$ the growth estimate with respect to y of $\hat{v}_D(x, y)$ is uniform for $x \in (\mathbb{C} \setminus 0)^n \cap B_0$. Therefore we can define $\hat{v}(x, y) := \hat{v}_D(x, y)$ on $x \in (\mathbb{C} \setminus 0)^n \cap B_0$ and $y \in \Omega$ by taking $x \in D$.

The function $\hat{v}(x, y)$ may have singularity on $x \in (\mathbb{C}^n \setminus (\mathbb{C} \setminus 0)^n) \cap B_0$, $y \in \Omega$. We will prove that the singularity is removable. First consider the singularity with codimension 1. For simplicity, take $y_0 \in \Omega$, $x'_0 = (x^0_2, \ldots, x^0_n)$ with $x^0_j \neq 0$ and consider the expansion

$$\hat{v}(x, y) = \sum_{\nu \geq 0, j \geq 0} \hat{v}_{\nu,j}(x_1)(x' - x'_0)^\nu (y - y_0)^j. \tag{52}$$

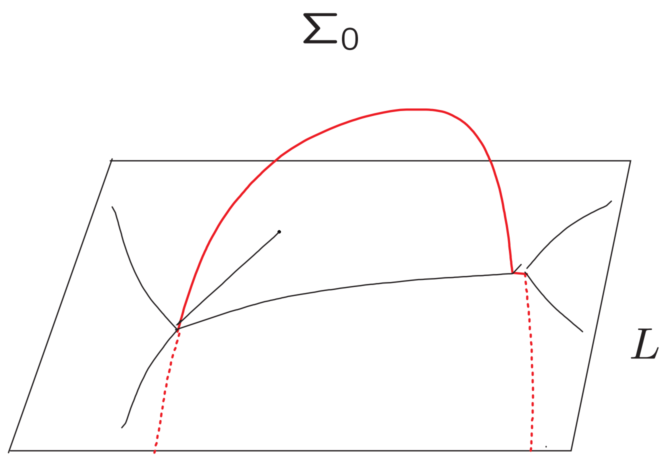
By what we have proved in the above, the right hand side is convergent if $x' - x'_0$ and $y - y_0$ are sufficiently small and $x_1 \neq 0$. Moreover, by the boundedness of $\widehat{v}(x, y)$ when $x_1 \to 0$ and the Cauchy's integral formula we have that $\widehat{v}_{\nu,j}(x_1)$ is holomorphic and single-valued and bounded in some neighborhood of the origin except for $x_1 = 0$. Hence the singularity of $\widehat{v}_{\nu,j}(x_1)$ is removable for every ν and j. In the same way one can show that the singularity of codimension one is removable.

Next we consider the singularity of codimension 2. For the sake of simplicity, consider the one $x_1 = x_2 = 0$, $x''_0 = (x_3^0, \ldots, x_n^0)$ with $x_j^0 \neq 0$. By considering in the same way as in the codimension one case we have the expansion similar to (52) where $x' - x'_0$ and $\widehat{v}_{\nu,j}(x_1)$ are replaced by $x'' - x''_0$ and $\widehat{v}_{\nu,j}(x_1, x_2)$, respectively. Because $\widehat{v}_{\nu,j}(x_1, x_2)$ is holomorphic and single-valued except for $x_1 = x_2 = 0$, we see that the singularity is removable by Hartogs theorem. Hence we see that the singularity of codimension 2 is removable. As for the singularity of higher codimension ≥ 3 we can argue in the same way by using Hartogs theorem. We see that $\widehat{v}(x, y)$ is holomorphic and single-valued on $x \in \mathbb{C}^n \cap B_0$, $y \in \Omega$.

The exponential growth of $\widehat{v}(x, y)$ when $y \to \infty$ in $y \in \Omega$ for $x \in \mathbb{C}^n \cap B_0$. Set some c_k to be equal to zero and make the same argument as for $\widehat{v}_D(x, y)$. By what we have proved in the above, we have the assertion. Hence we have proved the solvability of (31), and the summability of the our solution as desired.

8. Some geometry.

Let $v_0(x)$ and Σ_0 be given by (11) and (14), respectively. Because Σ_0 is a main analytic set, it has the pure codimension one. Hence, by the well known embedding theorem in several complex variables, for every point b of Σ_0 there exists a complex line L such that $b \in \Sigma_0 \cap L$ is isolated in L. In the following we assume that L is given by $x_j = 0$ $(1 \leq j \leq n-1)$ and $\Sigma_0 \cap L$ consists of isolated points in L. We denote the variable in L by ζ. We may also assume $\lambda_n = 1$ without loss of generality by dividing the equation with λ_n.



9. Preparations

In Theorem 3 we have proved Borel summability of the formal SP-solution $v(x, \eta)$ in a neighborhood of the origin $x = 0$. We will study Borel summability at other points $\xi \in (\mathbb{C}^n \setminus \Sigma_0) \cap \Omega_0$, $\xi \neq 0$.

Write $\xi = (\xi', \xi_n)$ and determine $(c_k)_k$ by the relations $\zeta = \xi_n$,(34) with $x' = \xi'$. Determine L with $x' = \xi'$. Define the set $T_0 \subset L$ by

$$T_0 := \Sigma_0 \cap \{(\xi', \zeta); \zeta \in L\}.$$

Let $a \in T_0$. With (c_k) given in the above, define $\Phi(s, a)$ by (35) and the curve S_a by the set of points s such that $\text{Im}\,\Phi(s, a) = 0$, respectively. Clearly, $\xi_n \notin T_0$ because $\xi \in (\mathbb{C}^n \setminus \Sigma_0) \cap \Omega_0$. Hence the following two cases occur:

(a) $\xi_n \notin S_a$ for any $a \in \Sigma_0$.

(b) $\xi_n \in S_a$ for some $a \in \Sigma_0$.

Because v_0 has a singularity on Σ_0 in general a branch cut emanating from Σ_0 may appear. We have

Theorem 5 Assume that $R(x)$ is an entire function on $x \in \mathbb{C}^n$ and that $\nabla R(x + v_0)$ is a diagonal matrix. Suppose $0 \notin \Sigma_0$. Moreover, assume that $\text{Re}\,\lambda_j > 0$ for $j = 1, 2, \ldots, n$. Let $\xi \in (\mathbb{C}^n \setminus \Sigma_0) \cap \Omega(v_0)$, $\xi \neq 0$. Then we have

The case (a). There exist a neighborhood D of ξ and an $\varepsilon > 0$ such that if $\|v_0\| < \varepsilon$, then $v(x, \eta)$ is Borel-summable in the direction η with $\eta^{-1} \in (C_0)^c$ for any $x \in D$, where $(C_0)^c$ is the complement of C_0 in the complex plane.

The case (b). Assume that $\xi_n \in S_a$ which is not a branch cut of $\Phi'(s, \cdot) = (d/ds)\Phi(s, \cdot)$. Then there exist a neighborhood D of ξ and an $\varepsilon > 0$ such that if $\|v_0\| < \varepsilon$, then $v(x, \eta)$ is Borel-summable in the direction η with $\eta^{-1} \in (C_0)^c$ for any $x \in D$.

10. Global summability

By using the results in the preceeding sections we can show the following facts. Given a domain K whose closure is compact. Then there exists an $\varepsilon > 0$ such that if $\|v_0\| < \varepsilon$ and v_0 is holomorphic in K, then the SP-solution is Borel summable in the direction η with $\eta^{-1} \in (C_0)^c$ for any $x \in K$. Indeed, one can make analytic continuation by covering K with a finite number of open sets. We note that the Borel sum gives the desired solution of our equation.

11. Connection problem across singular directions and Poincaré's theorem

Consider the connection problem with respect to η of the summed SP-solution $V(x, \eta)$ of (6) across a singular direction in C_0. Let E_0 be given by

$$E_0 := \left\{ \frac{\langle \lambda, \alpha \rangle}{\lambda_k}; k = 1, 2, \ldots, n, \alpha \in \mathbb{Z}^n_{\geq 0}, |\alpha| \geq 2 \right\}. \tag{53}$$

We can show:
E_0 is contained in the right half-plane $\mathrm{Re}\,\eta > 0$ and $\eta/|\eta|$ $(\eta \in E_0)$ are dense in some sector of the complex plane.

Note that E_0 gives the singular directions for the Borel sum $V(x, \eta)$ which is dense in C_0. A connection problem occurs at a singular direction in

C_0. **We shall study the analytic continuation of** $V(x,\eta)$ **from the negative real axis to the point** $\eta = 1$.

Theorem 6 **Assume there exists a real** ξ **such that** $|\arg \lambda_j - \xi| < \pi/4$ **for** $j = 1, 2, \ldots, n$ **and that** λ_j $(j = 1, 2, \ldots, n)$ **be linearly independent over** $\mathbb{Z}$. **Then there exists a neighborhood** W **of the origin of** $x \in \mathbb{C}^n$ **such that the connection coefficient across every singular direction in** C_0 **vanishes. Especially,** $V(x,\eta)$ **is a single-valued meromorphic function with respect to** η **with poles on** E_0 **and analytic in** x **when** $x \in W$.

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Thank you very much

for attending to the conference