Post-Newtonian Approximation

Piotr Jaranowski

Faculty of Physcis, University of Białystok, Poland

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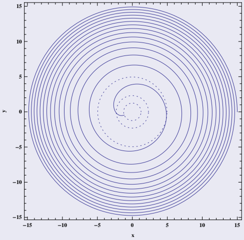
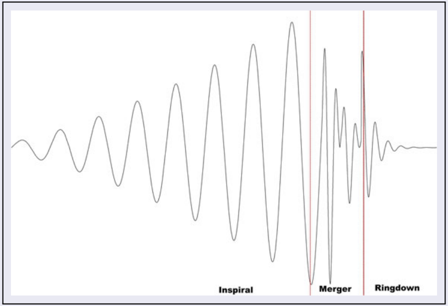
- We will concentrate in this lecture to the most important application of post-Newtonian approximation to general relativity:
  relativistic two-body problem,
  i.e. the problem of finding motion and gravitational radiation of selfgravitating relativistic systems which consist of two extended bodies.
- Relativistic binary systems exist in nature,
  they comprise compact objects:
  neutron stars or black holes.
  These systems emit gravitational waves,
  which experimenters try to detect
  within the LIGO/VIRGO/GEO600 projects.

Black-hole binary: Inspiral—merger—ringdown

- As the result of emission of gravitational waves, the size of the binary orbit decreases and the binary components move faster, consequently leading to emission of gravitational waves with increasing amplitude and frequency —producing a gravitational-wave chirp signal.





- To detect in noise of a detector weak gravitational waves (coming e.g. from a coalescing compact binary system) by means of matched filtering method,
  one has to construct a discrete bank of templates (waveforms) parametrized by possible values of the GW signal's parameters (e.g. masses and spins of binary components).
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- For construction of templates (waveforms) one has to know time evolution of the source of gravitational waves (i.e. compact binary system in the rest of this lecture).

Different approaches for solving relativistic two-body problem:

- numerical relativity (breakthrough in 2005);
- approximate 'analytical' methods:
  - post-Newtonian expansion,
  - perturbation-based self-force approach.

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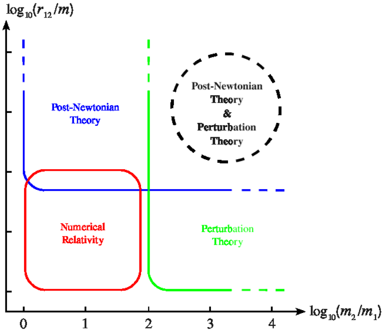
(Blanchet et al., Phys. Rev. D **81**, 064004 (2010))

- PN expansion:
  0th order—Newtonian gravity;
  nPN order—corrections of order

$$\left(\frac{v}{c}\right)^{2n} \sim \left(\frac{Gm}{rc^2}\right)^{n}$$

  to the Newtonian gravity.
- Perturbation approach: $\dfrac{m_1}{m_2} \ll 1$.



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to construct bank of templates based purely on numerical results.

- An accurate equal-mass, non-spinning 8 orbit BBH simulation takes $\sim$200 000 CPU hours.
- The computational cost of a BBH simulation scales with the mass ratio: an accurate 1:10 mass ratio, non-spinning 8 orbit simulation would thus take $\sim$2 000 000 CPU hours.

(I.Hinder, Class. Quantum Grav. **27**, 114004 (2010))

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- Gravitational-wave induced evolution of BBH computed numerically and by means of the PN approximations agree very well in the region, where the coalescing objects are sufficiently far away.
- Detection of GW signals (and extraction their parameters) from coalescing BBH by advanced LIGO/VIRGO detectors —the most promising waveforms are hybrid waveforms: PN (early inspiral) + numerical (late inspiral, merger, ringdown) or waveforms based on effective one-body formalism.

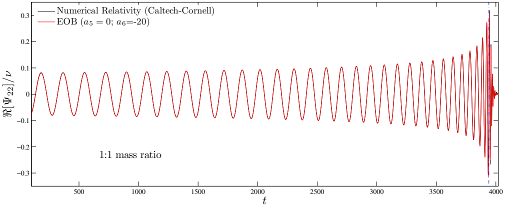
- Effective one-body (EOB) formalism is a formalism, based on approximate PN and BH perturbation theory results, which allows to model 'analytically' the whole coalescence of a BBH system, from its adiabatic inspiral up to vibrations of the 'resultant' Kerr BH.
  It is being developed by Damour and his collaborators since 1999.
- Main idea of EOB approach: To extend the domain of validity of PN and BH perturbation theories so as to approximately cover non-perturbative features of BBH evolution.

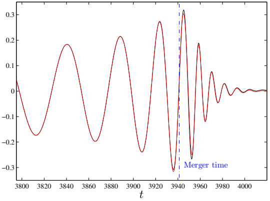
Utility (partially expected) of EOB formalism

- It provides accurate templates needed for detection of GW signals (and estimation of their parameters) originating from coalescences of BBHs,
- it gives a physical understanding of dynamics and radiation of BBHs,
- it can be extended to BH-NS or NS-NS systems (after incorporating tidal interactions).

Black-hole binary: EOB vs numerical relativity

T.Damour, A.Nagar, Phys.Rev.D **79**, 081503(R) (2009)





- In the coordinate system (t, x_s^i) centered on a gravitational-wave source the wave polarization functions can be computed from the equations

$$h_+(t, x_s^i) = \frac{G}{c^4\,R}\left(\frac{\mathrm{d}^2\mathcal{J}_{xx}}{\mathrm{d}t^2}\left(t-\frac{R}{c}\right) - \frac{\mathrm{d}^2\mathcal{J}_{yy}}{\mathrm{d}t^2}\left(t-\frac{R}{c}\right)\right), \tag{1a}$$

$$h_\times(t, x_s^i) = \frac{2G}{c^4\,R}\frac{\mathrm{d}^2\mathcal{J}_{xy}}{\mathrm{d}t^2}\left(t-\frac{R}{c}\right), \tag{1b}$$

for the wave propagating in the $+z$ direction of the coordinate system.

- Let us introduce coordinate system (t, x^i) about which we assume that some solar-system related observer measures the gravitational-wave field $h_{\mu\nu}^{\rm TT}$ in the neighborhood of its spatial origin.
- In the discussion of the detection of gravitational waves by solar-system-based detectors, the origin of these coordinates will be located at the solar system barycenter (SSB).
- Let us denote by $\mathbf{x}^*$ the constant 3-vector joining the origin of the coordinates (x^i) (i.e. the SSB) with the origin of the $(x_{\rm s}^i)$ coordinates (i.e. the 'center' of the source).
- We also assume that the spatial axes in both coordinate systems are parallel to each other, i.e. the coordinates x^i and $x_{\rm s}^i$ differ just by constant shifts determined by the components of the 3-vector $\mathbf{x}^*$,

$$x^i = x^{*i} + x_{\rm s}^i. \tag{2}$$

It means that in both coordinate systems the components $h_{\mu\nu}^{\rm TT}$ of the gravitational-wave field are numerically the same.

- Equations (1) are valid only when the z axis of the TT coordinate system is parallel to the 3-vector $\mathbf{x}^* - \mathbf{x}$ joining the observer located at $\mathbf{x}$ ($\mathbf{x}$ is the 3-vector joining the SSB with the observer's location) and the gravitational-wave source at the position $\mathbf{x}^*$.

  If one changes the location $\mathbf{x}$ of the observer, one has to rotate the spatial axes to ensure that Eqs. (1) are still valid.

- Let us now fix, in the whole region of interest, the direction of the $+z$ axis of both coordinate systems considered here, by choosing it to be antiparallel to the 3-vector $\mathbf{x}^*$ (so for the observer located at the SSB gravitational wave propagates along the $+z$ direction).

- To a very good accuracy one can assume that the size of the region where the observer can be located is very small compared to the distance from the SSB to the gravitational-wave source, which is equal to

$$r^* := |\mathbf{x}^*|.$$

  Our assumption thus means that $r \ll r^*$, where $r := |\mathbf{x}|$.

- Then $\mathbf{x}^* = (0, 0, -r^*)$ and the Taylor expansion of $R = |\mathbf{x} - \mathbf{x}^*|$ and $\mathbf{n} := (\mathbf{x} - \mathbf{x}^*)/R$ around $\mathbf{x}^*$ reads

$$|\mathbf{x} - \mathbf{x}^*| = r^* \left(1 + \frac{z}{r^*} + \frac{x^2 + y^2}{2(r^*)^2} + \mathcal{O}\big((x^l/r^*)^3\big)\right), \tag{3a}$$

$$\mathbf{n} = \left(-\frac{x}{r^*} + \frac{xz}{2(r^*)^2}, -\frac{y}{r^*} + \frac{yz}{2(r^*)^2}, 1 - \frac{x^2 + y^2}{2(r^*)^2}\right) + \mathcal{O}\big((x^l/r^*)^3\big). \tag{3b}$$

- One can compute the TT projection of the reduced mass quadrupole moment at the point $\mathbf{x}$ in the direction of the unit vector $\mathbf{n}$ (which in general is not parallel to the $+z$ axis). One gets

$$\mathcal{J}_{xx}^{\mathrm{TT}} = -\mathcal{J}_{yy}^{\mathrm{TT}} = \frac{1}{2}(\mathcal{J}_{xx} - \mathcal{J}_{yy}) + \mathcal{O}\big(x^l/r^*\big), \tag{4a}$$

$$\mathcal{J}_{xy}^{\mathrm{TT}} = \mathcal{J}_{yx}^{\mathrm{TT}} = \mathcal{J}_{xy} + \mathcal{O}\big(x^l/r^*\big), \tag{4b}$$

$$\mathcal{J}_{zi}^{\mathrm{TT}} = \mathcal{J}_{iz}^{\mathrm{TT}} = 0 + \mathcal{O}\big(x^l/r^*\big) \quad \text{for} \quad i = x, y, z. \tag{4c}$$

- To obtain the gravitational-wave field $h_{\mu\nu}^{\mathrm{TT}}$ in the coordinate system (t, x^i) one should plug Eqs. (4) into the quadrupole formula.
- If one neglects in Eqs. (4) the terms of the order of x^l/r^*, then in the whole region of interest covered by the single TT coordinate system, the gravitational-wave field $h_{\mu\nu}^{\mathrm{TT}}$ can be written in the form

$$\boxed{h_{\mu\nu}^{\mathrm{TT}}(t,\mathbf{x}) = h_+(t,\mathbf{x})\, e_{\mu\nu}^{+} + h_\times(t,\mathbf{x})\, e_{\mu\nu}^{\times} + \mathcal{O}\big(x^l/r^*\big),} \tag{5}$$

where e^+ and $e^\times$ are the polarization tensors and the functions h_+ and $h_\times$ are of the form given in Eqs. (1).

- Dependence of the $1/R$ factor in the amplitude of the wave polarizations (1) on the observer's position $\mathbf{x}$ (with respect to the SSB) is usually negligible, so $1/R$ in the amplitudes can be replaced by $1/r^*$ [this is consistent with the neglection of the $\mathcal{O}(x^l/r^*)$ terms we have just made in Eqs. (4)].
- This is not the case for the 2nd time derivative of $\mathcal{J}_{ij}$ in (1), which determines the time evolution of the wave polarizations phases and which is evaluated at the retarded time $t - R/c$. Here it is usually enough to take into account the first correction to r^* [given by Eq. (3)].

- After taking all this into account the wave polarizations (1) take the form

$$h_+(t,x^i) = \frac{G}{c^4\,r^*}\left(\frac{\mathrm{d}^2\mathcal{J}_{xx}}{\mathrm{d}t^2}\left(t-\frac{z+r^*}{c}\right) - \frac{\mathrm{d}^2\mathcal{J}_{yy}}{\mathrm{d}t^2}\left(t-\frac{z+r^*}{c}\right)\right), \tag{6a}$$

$$h_\times(t,x^i) = \frac{2G}{c^4\,r^*}\frac{\mathrm{d}^2\mathcal{J}_{xy}}{\mathrm{d}t^2}\left(t-\frac{z+r^*}{c}\right). \tag{6b}$$

The wave polarization functions (6) represent a plane gravitational wave propagating in the $+z$ direction.

- We consider a binary system made of two bodies with masses m_1 and m_2 (we will always assume $m_1 \geq m_2$). We introduce

$$M := m_1 + m_2, \quad \mu := \frac{m_1 m_2}{M}, \tag{7}$$

so M is the total mass of the system and μ is its reduced mass.

- It is also useful to introduce the dimensionless symmetric mass ratio

$$\nu := \frac{m_1 m_2}{M^2} = \frac{\mu}{M}. \tag{8}$$

The quantity ν satisfies $0 \leq \nu \leq 1/4$, the case $\nu = 0$ corresponds to the test-mass limit and $\nu = 1/4$ describes equal-mass binary.

- We start from deriving the wave polarization functions h_+ and $h_\times$ for waves emitted by a binary system in the case when the dynamics of the binary can reasonably be described within the Newtonian theory of gravitation.

- Let $\mathbf{r}_1$ and $\mathbf{r}_2$ denote the position vectors of the bodies, i.e. the 3-vectors connecting the origin of some reference frame with the bodies.
  We introduce the relative position vector,

$$\mathbf{r}_{12} := \mathbf{r}_1 - \mathbf{r}_2. \tag{9}$$

- The center-of-mass reference frame is defined by the requirement that

$$m_1\,\mathbf{r}_1 + m_2\,\mathbf{r}_2 = \mathbf{0}. \tag{10}$$

- Solving Eqs. (9) and (10) with respect to $\mathbf{r}_1$ and $\mathbf{r}_2$ one gets

$$\mathbf{r}_1 = \frac{m_2}{M}\mathbf{r}_{12}, \quad \mathbf{r}_2 = -\frac{m_1}{M}\mathbf{r}_{12}. \tag{11}$$

- In the center-of-mass reference frame we introduce the spatial coordinates $(x_\mathrm{c}, y_\mathrm{c}, z_\mathrm{c})$ such that the total orbital angular momentum vector $\mathbf{J}$ of the binary is directed along the $+z_\mathrm{c}$ axis.
- Then the trajectories of both bodies lie in the $(x_\mathrm{c}, y_\mathrm{c})$ plane, so the position vector $\mathbf{r}_a$ of the ath body ($a = 1, 2$) has components

$$\mathbf{r}_a = (x_{\mathrm{c}a}, y_{\mathrm{c}a}, 0),$$

and the relative position vector components are

$$\mathbf{r}_{12} = (x_{\mathrm{c}12}, y_{\mathrm{c}12}, 0),$$

where $x_{\mathrm{c}12} := x_{\mathrm{c}1} - x_{\mathrm{c}2}$ and $y_{\mathrm{c}12} := y_{\mathrm{c}1} - y_{\mathrm{c}2}$.
- It is convenient to introduce in the coordinate $(x_{\mathrm{c}12}, y_{\mathrm{c}12})$ plane the usual polar coordinates (r, ϕ):

$$x_{\mathrm{c}12} = r\cos\phi, \quad y_{\mathrm{c}12} = r\sin\phi. \tag{12}$$

- Within the Newtonian gravity the orbit of the relative motion is an ellipse (we consider here only gravitationally bound binaries).
  We place the focus of the ellipse at the origin of the (x_{c12}, y_{c12}) coordinates. In polar coordinates the ellipse is described by the equation

$$r(\phi) = \frac{a(1-e^2)}{1 + e\cos(\phi - \phi_0)}, \tag{13}$$

  where a is the semimajor axis, e is the eccentricity, and ϕ_0 is the angle of the orbital periapsis.
- The time dependence of the relative motion is determined by the Kepler's equation,

$$\dot{\phi} = \frac{2\pi}{P}(1-e^2)^{-3/2}\big(1 + e\cos(\phi - \phi_0)\big)^2, \tag{14}$$

  where P is the orbital period of the binary,

$$P = 2\pi\sqrt{\frac{a^3}{GM}}. \tag{15}$$

- The binary's binding energy E and the modulus J of its total angular orbital momentum are related to the parameters of the relative orbit through the equations

$$E = -\frac{GM\mu}{2a}, \quad J = \mu\sqrt{GMa(1-e^2)}. \tag{16}$$

- Let us now introduce the TT 'wave' coordinates $(x_\mathrm{w}, y_\mathrm{w}, z_\mathrm{w})$ in which the gravitational wave is traveling in the $+z_\mathrm{w}$ direction and with the origin at the SSB.
- The line along which the plane tangent to the celestial sphere at the location of the binary's center-of-mass [this plane is parallel to the $(x_\mathrm{w}, y_\mathrm{w})$ plane] intersects the orbital $(x_\mathrm{c}, y_\mathrm{c})$ plane is called the line of nodes.
- Let us adjust the center-of-mass and the wave coordinates in such a way, that the x axes of both coordinate systems are parallel to each other and to the line of nodes.

- Then the relation between these coordinates is determined by the rotation matrix S,

$$\begin{pmatrix} x_{\rm w} \\ y_{\rm w} \\ z_{\rm w} \end{pmatrix} = \begin{pmatrix} x^*_{\rm w} \\ y^*_{\rm w} \\ z^*_{\rm w} \end{pmatrix} + \mathsf{S} \begin{pmatrix} x_{\rm c} \\ y_{\rm c} \\ z_{\rm c} \end{pmatrix}, \quad \mathsf{S} := \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\iota & \sin\iota \\ 0 & -\sin\iota & \cos\iota \end{pmatrix}, \tag{17}$$

  where $(x^*_{\rm w}, y^*_{\rm w}, z^*_{\rm w})$ are the components of the vector $\mathbf{x}^*$ joining the SSB and the binary's center-of-mass, and ι ($0 \leq \iota \leq \pi$) is the angle between the orbital angular momentum vector $\mathbf{J}$ of the binary and the line of sight (i.e. the $+z_{\rm w}$ axis).
- We assume that the center of mass of the binary is at rest with respect to the SSB.

- In the center-of-mass reference frame the binary's moment of inertia tensor has changing in time components which are equal

$$\mathcal{I}_{\rm c}^{ij}(t) = m_1\, x_{\rm c1}^i(t)\, x_{\rm c1}^j(t) + m_2\, x_{\rm c2}^i(t)\, x_{\rm c2}^j(t). \tag{18}$$

- Making use of Eqs. (11) one finds the matrix $\mathcal{I}_{\rm c}$ built from the $\mathcal{I}_{\rm c}^{ij}$ components of the inertia tensor,

$$\mathcal{I}_{\rm c}(t) = \mu \begin{pmatrix} (x_{\rm c12}(t))^2 & x_{\rm c12}(t)\, y_{\rm c12}(t) & 0 \\ x_{\rm c12}(t)\, y_{\rm c12}(t) & (y_{\rm c12}(t))^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{19}$$

- The inertia tensor components in wave coordinates are related with those in center-of-mass coordinates through the relation

$$\mathcal{I}^{ij}_{\mathrm{w}}(t) = \sum_{k=1}^{3}\sum_{\ell=1}^{3} \frac{\partial x^i_{\mathrm{w}}}{\partial x^k_{\mathrm{c}}} \frac{\partial x^j_{\mathrm{w}}}{\partial x^\ell_{\mathrm{c}}} \mathcal{I}^{k\ell}_{\mathrm{c}}(t), \tag{20}$$

what in matrix notation reads

$$\mathcal{I}_{\mathrm{w}}(t) = \mathsf{S}\cdot\mathcal{I}_{\mathrm{c}}(t)\cdot\mathsf{S}^{\mathsf{T}}. \tag{21}$$

- To obtain the wave polarization functions h_+ and $h_\times$ we plug the components of the binary's inertia tensor into general equations [note that in these equations the components $\mathcal{J}_{ij}$ of the reduced quadrupole moment can be replaced by the components $\mathcal{I}_{ij}$ of the inertia tensor].
- We get [here $R = |\mathbf{x}^*|$ is the distance to the binary's center-of-mass and $t_r = t - (z_w + R)/c$ is the retarded time]

$$\begin{aligned} h_+(t,\mathbf{x}) = \frac{G\mu}{c^4 R}\Big(& \sin^2\iota\big(\dot r(t_r)^2 + r(t_r)\ddot r(t_r)\big) \\ & + (1+\cos^2\iota)\big(\dot r(t_r)^2 + r(t_r)\ddot r(t_r) - 2r(t_r)^2\dot\phi(t_r)^2\big)\cos 2\phi(t_r) \\ & - (1+\cos^2\iota)\big(4r(t_r)\dot r(t_r)\dot\phi(t_r) + r(t_r)^2\ddot\phi(t_r)\big)\sin 2\phi(t_r)\Big), \end{aligned} \tag{22a}$$

$$\begin{aligned} h_\times(t,\mathbf{x}) = \frac{2G\mu}{c^4 R}\cos\iota\Big(& \big(4r(t_r)\dot r(t_r)\dot\phi(t_r) + r(t_r)^2\ddot\phi(t_r)\big)\cos 2\phi(t_r) \\ & + \big(\dot r(t_r)^2 + r(t_r)\ddot r(t_r) - 2r(t_r)^2\dot\phi(t_r)^2\big)\sin 2\phi(t_r)\Big). \end{aligned} \tag{22b}$$

- Gravitational waves emitted by the binary diminish the binary's binding energy and total orbital angular momentum.
  This makes the orbital parameters changing in time.
- Let us first assume that these changes are so slow that they can be neglected during the time interval in which the observations are performed.
- Making use of Eqs. (14) and (13) one can then eliminate from the formulae (22) the first and the second time derivatives of r and ϕ, treating the parameters of the orbit, a and e, as constants.

$$r(\phi) = \frac{a(1-e^2)}{1+e\cos(\phi-\phi_0)} \tag{13}$$

$$\dot{\phi} = \frac{2\pi}{P}(1-e^2)^{-3/2}\big(1+e\cos(\phi-\phi_0)\big)^2 \tag{14}$$

- The result is the following:

$$h_+(t,\mathbf{x}) = \frac{4G^2M\mu}{c^4Ra(1-e^2)}\Big(A_0(t_r) + A_1(t_r)\,e + A_2(t_r)\,e^2\Big), \qquad (23a)$$

$$h_\times(t,\mathbf{x}) = \frac{4G^2M\mu}{c^4Ra(1-e^2)}\cos\iota\Big(B_0(t_r) + B_1(t_r)\,e + B_2(t_r)\,e^2\Big), \qquad (23b)$$

where the functions A_i and B_i, $i = 0, 1, 2$, are defined as follows

$$A_0(t) = -\frac{1}{2}(1+\cos^2\iota)\cos 2\phi(t),$$

$$A_1(t) = \frac{1}{4}\sin^2\iota\cos(\phi(t)-\phi_0) - \frac{1}{8}(1+\cos^2\iota)\Big(5\cos(\phi(t)+\phi_0) + \cos(3\phi(t)-\phi_0)\Big),$$

$$A_2(t) = \frac{1}{4}\sin^2\iota - \frac{1}{4}(1+\cos^2\iota)\cos 2\phi_0,$$

$$B_0(t) = -\sin 2\phi(t),$$

$$B_1(t) = -\frac{1}{4}\Big(\sin(3\phi(t)-\phi_0) + 5\sin(\phi(t)+\phi_0)\Big),$$

$$B_2(t) = -\frac{1}{2}\sin 2\phi_0.$$

- In the case of circular orbits the eccentricity e vanishes and the wave polarization functions (23) simplify to

$$h_+(t, \mathbf{x}) = -\frac{2G^2 M\mu}{c^4 R a}(1 + \cos^2 \iota) \cos 2\phi(t_r), \tag{25a}$$

$$h_\times(t, \mathbf{x}) = -\frac{4G^2 M\mu}{c^4 R a} \cos \iota \sin 2\phi(t_r), \tag{25b}$$

where a is the radius of the relative circular orbit and

$$\phi(t) = \phi_0 + \frac{2\pi}{P}(t - t_0).$$

- Let us now consider observational intervals short enough so it is necessary to take into account effects of radiation reaction changing the parameters of the bodies' orbits.
- In the leading order the rates of emission of energy and angular momentum carried by gravitational waves are given by the formulae

$$\mathcal{L}_E^{\mathrm{gw}} = \frac{G}{5c^5}\sum_{i=1}^{3}\sum_{j=1}^{3}\left\langle\left(\frac{\mathrm{d}^3\mathcal{J}_{ij}}{\mathrm{d}t^3}\right)^2\right\rangle, \quad \mathcal{L}_{J_i}^{\mathrm{gw}} = \frac{2G}{5c^5}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{\ell=1}^{3}\epsilon_{ijk}\left\langle\frac{\mathrm{d}^2\mathcal{J}_{j\ell}}{\mathrm{d}t^2}\frac{\mathrm{d}^3\mathcal{J}_{k\ell}}{\mathrm{d}t^3}\right\rangle.$$

- We plug into them the Newtonian trajectories of the bodies and perform time averaging over one orbital period of the motion. We get

$$\mathcal{L}_E^{\mathrm{gw}} = \frac{32}{5}\frac{G^4\mu^2M^3}{c^5a^5(1-e^2)^{7/2}}\left(1+\frac{73}{24}e^2+\frac{37}{96}e^4\right), \tag{26a}$$

$$\mathcal{L}_J^{\mathrm{gw}} = \frac{32}{5}\frac{G^{7/2}\mu^2M^{5/2}}{c^5a^{7/2}(1-e^2)^2}\left(1+\frac{7}{8}e^2\right). \tag{26b}$$

- It is easy to obtain equations describing the leading-order time evolution of the relative orbit parameters a and e.
- We first differentiate both sides of equations

$$E = -\frac{GM\mu}{2a}, \quad J = \mu\sqrt{GMa(1-e^2)},$$

with respect to time treating a and e as functions of time.
Then the rates of changing the binary's binding energy and angular momentum, $\mathrm{d}E/\mathrm{d}t$ and $\mathrm{d}J/\mathrm{d}t$, one replaces by minus the rates of emission of respectively energy $\mathcal{L}_E^{\mathrm{gw}}$ and angular momentum $\mathcal{L}_J^{\mathrm{gw}}$ carried by gravitational waves, given in Eqs. (26).

- The two equations such obtained can be solved with respect to the derivatives $\mathrm{d}a/\mathrm{d}t$ and $\mathrm{d}e/\mathrm{d}t$:

$$\frac{\mathrm{d}a}{\mathrm{d}t} = -\frac{64}{5}\frac{G^3\mu M^2}{c^5 a^3(1-e^2)^{7/2}}\left(1+\frac{73}{24}e^2+\frac{37}{96}e^4\right), \tag{27a}$$

$$\frac{\mathrm{d}e}{\mathrm{d}t} = -\frac{304}{15}\frac{G^3\mu^2 M^2}{c^5 a^4(1-e^2)^{5/2}}e\left(1+\frac{121}{304}e^2\right). \tag{27b}$$

Let us note two interesting features of the evolution described by the above equations:

- Eq. (27b) implies that initially circular orbits (for which $e=0$) remain circular during the evolution;
- the eccentricity e of the relative orbit decreases with time much more rapidly than the semimajor axis a, so gravitational-wave reaction induces the circularization of the binary orbits (roughly, when the semimajor axis is halved, the eccentricity goes down by a factor of three).

- Evolution of the orbital parameters caused by the radiation reaction influences gravitational waveforms. To see this influence we restrict our considerations to binaries along circular orbits.
- The decay of the radius a of the relative orbit is given by the equation [this is Eq. (27a) with $e = 0$]

$$\frac{\mathrm{d}a}{\mathrm{d}t} = -\frac{64}{5}\frac{G^3\mu M^2}{c^5 a^3}. \tag{28}$$

- Integration of this equation leads to

$$a(t) = a_0\left(\frac{t_c - t}{t_c - t_0}\right)^{1/4}, \tag{29}$$

where $a_0 := a(t = t_0)$ is the initial value of the orbital radius and $t_c - t_0$ is the formal 'lifetime' or 'coalescence time' of the binary, i.e. the time after which the distance between the bodies is zero.

- It means that t_c is the moment of the coalescence, $a(t_c) = 0$; it is equal to

$$t_c = t_0 + \frac{5}{256} \frac{c^5 a_0^4}{G^3 \mu M^2}. \tag{30}$$

- If one neglects radiation reaction effects, then along the Newtonian circular orbit the time derivative of the orbital phase, i.e. the orbital angular frequency ω, is constant and given by [see Eqs. (14) and (15)]

$$\omega = \dot{\phi} = \frac{2\pi}{P} = \sqrt{\frac{GM}{a^3}}. \tag{31}$$

- We now assume that the inspiral of the binary system caused by the radiation reaction is adiabatic, i.e. it can be thought as a sequence of circular orbits with radii $a(t)$ which slowly diminish in time according to Eq. (29).
- Adiabacity thus means that Eq. (31) is valid during the inspiral, but now angular frequency ω is a function of time, because of time dependence of the orbital radius a.

- Making use of Eqs. (29)–(31) we get the following equation for the time evolution of the instantaneous orbital angular frequency during the adiabatic inspiral:

$$\omega(t) = \omega_0 \left(1 - \frac{256(G\mathcal{M})^{5/3}}{5c^5}\omega_0^{8/3}(t - t_0)\right)^{-3/8}, \tag{32}$$

where $\omega_0 := \omega(t = t_0)$ is the initial value of the orbital angular frequency and where we have introduced the new parameter $\mathcal{M}$ (with dimension of a mass) called a chirp mass of the system:

$$\mathcal{M} := \mu^{3/5} M^{2/5}. \tag{33}$$

- To get the polarization waveforms h_+ and $h_\times$ which describes gravitational radiation emitted during the adiabatic inspiral of the binary system, we use the general formulae (22).
- In these formulae we neglect all the terms proportional to $\dot{r} \equiv \dot{a}$ or $\ddot{r} \equiv \ddot{a}$ (the radial coordinate r is identical with the radius a of the relative circular orbit, $r \equiv a$).
- By virtue of Eq. (31) $\ddot{\phi} \propto \dot{a}$, therefore we also neglect terms $\propto \ddot{\phi}$.
- All these simplifications lead to the following formulae:

$$h_+(t, \mathbf{x}) = -\frac{4(G\mathcal{M})^{5/3}}{c^4 R} \frac{1 + \cos^2 \iota}{2} \omega(t_\mathrm{r})^{2/3} \cos 2\phi(t_\mathrm{r}), \tag{34a}$$

$$h_\times(t, \mathbf{x}) = -\frac{4(G\mathcal{M})^{5/3}}{c^4 R} \cos \iota \, \omega(t_\mathrm{r})^{2/3} \sin 2\phi(t_\mathrm{r}). \tag{34b}$$

- Making use of Eq. (31) one can express the time dependence of the waveforms (34) in terms of the single function $a(t)$:

$$h_+(t,\mathbf{x}) = -\frac{4G^2\mu M}{c^4 R}\frac{1+\cos^2\iota}{2}\frac{1}{a(t_r)}\cos 2\left(\sqrt{GM}\int_{t_0}^{t_r} a(t')^{-3/2}\mathrm{d}t' + \phi_0\right), \tag{35a}$$

$$h_\times(t,\mathbf{x}) = -\frac{4G^2\mu M}{c^4 R}\cos\iota\,\frac{1}{a(t_r)}\sin 2\left(\sqrt{GM}\int_{t_0}^{t_r} a(t')^{-3/2}\mathrm{d}t' + \phi_0\right), \tag{35b}$$

where $\phi_0 := \phi(t = t_0)$ is the initial value of the orbital phase of the binary.

- It is sometimes useful to analyze the chirp waveforms h_+ and $h_\times$ in terms of the instantaneous frequency f_{gw} of the gravitational waves emitted by a coalescing binary system.
- The frequency f_{gw} is defined as

$$f_{\mathrm{gw}}(t) := 2 \times \frac{1}{2\pi} \frac{\mathrm{d}\phi(t)}{\mathrm{d}t}, \qquad (36)$$

where the extra factor 2 is due to the fact that [see Eqs. (34)] the instantaneous phase of the gravitational waveforms h_+ and $h_\times$ is 2 times the instantaneous phase $\phi(t)$ of the orbital motion (so the gravitational-wave frequency f_{gw} is twice the orbital frequency).

- Making use of Eqs. (29)–(31) one gets that for the binary with circular orbits the gravitational-wave frequency $f_{\rm gw}$ reads

$$f_{\rm gw}(t) = \frac{5^{3/8}c^{15/8}}{8\pi G^{5/8}}\frac{1}{\mathcal{M}^{5/8}}(t_{\rm c}-t)^{-3/8}. \tag{37}$$

- The chirp waveforms given in Eqs. (35) can be rewritten in terms of the gravitational-wave frequency $f_{\rm gw}$ as follows

$$h_+(t,\mathbf{x}) = -h_0(t_{\rm r})\frac{1+\cos^2\iota}{2}\cos\left(2\pi\int_{t_0}^{t_{\rm r}} f_{\rm gw}(t')\mathrm{d}t' + 2\phi_0\right), \tag{38a}$$

$$h_\times(t,\mathbf{x}) = -h_0(t_{\rm r})\cos\iota\,\sin\left(2\pi\int_{t_0}^{t_{\rm r}} f_{\rm gw}(t')\mathrm{d}t' + 2\phi_0\right), \tag{38b}$$

where $h_0(t)$ is the changing in time amplitude of the waveforms,

$$h_0(t) := \frac{4\pi^{2/3}G^{5/3}}{c^4}\frac{\mathcal{M}^{5/3}}{R}f_{\rm gw}(t)^{2/3}. \tag{39}$$

There are two problems, usually analyzed separately:

- problem of finding equations of motion (EOM),
- problem of computing gravitational-wave luminosities (of energy, angular momentum, and linear momentum).

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PN corrections for EOM/luminosities without spin-dependent effects

EOM	N	1PN	2PN	2.5PN	3PN	3.5PN	4PN	4.5PN	5PN	5.5PN	6PN	6.5PN
Energy/ angular momentum luminosity	—	—	—	N	—	1PN	1.5PN	2PN	2.5PN	3PN	3.5PN	4PN

The response function of the laser-interferometric detector to gravitational waves from coalescing compact binary
(made of nonspinning bodies) in circular orbits:

$$h(t) = \frac{C}{D}\big(\dot\phi(t)\big)^{2/3} \sin\big(2\phi(t) + \alpha\big),$$

D is the distance of the binary to the Earth,
C and α are some constants,
and $\phi(t)$ is the orbital phase of the binary
(so $\dot\phi(t) \equiv \mathrm{d}\phi(t)/\mathrm{d}t$ is the angular frequency).

Dimensionless post-Newtonian parameter for circular orbits:

$$x \equiv \frac{1}{c^2}(GM\dot\phi)^{2/3}.$$

The orbital phase $\phi(t)$ evolution is computed from the balance equation

$$\boxed{\frac{\mathrm{d}E}{\mathrm{d}t} = -\mathcal{L},}$$

which has the following PN expansion:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(E_{\mathrm{N}} + \frac{1}{c^2}E_{1\mathrm{PN}} + \frac{1}{c^4}E_{2\mathrm{PN}} + \frac{1}{c^6}E_{3\mathrm{PN}} + \frac{1}{c^8}E_{4\mathrm{PN}} + \mathcal{O}\big((v/c)^9\big)\right)$$

$$= -\Big(\mathcal{L}_{\mathrm{N}} + \frac{1}{c^2}\mathcal{L}_{1\mathrm{PN}} + \frac{1}{c^3}\mathcal{L}_{1.5\mathrm{PN}} + \frac{1}{c^4}\mathcal{L}_{2\mathrm{PN}} + \frac{1}{c^5}\mathcal{L}_{2.5\mathrm{PN}}$$

$$+ \frac{1}{c^6}\mathcal{L}_{3\mathrm{PN}} + \frac{1}{c^7}\mathcal{L}_{3.5\mathrm{PN}} + \frac{1}{c^8}\mathcal{L}_{4\mathrm{PN}} + \mathcal{O}\big((v/c)^9\big)\Big).$$

Bounding energy in the center-of-mass frame for circular orbits up to the 4PN order

$$E(x;\nu) = -\frac{\mu c^2 x}{2}\bigg(1 + e_{1\mathrm{PN}}(\nu)\,x + e_{2\mathrm{PN}}(\nu)\,x^2 + e_{3\mathrm{PN}}(\nu)\,x^3 + \Big(e_{4\mathrm{PN}}(\nu) + \frac{448}{15}\nu\ln x\Big)\,x^4 + \mathcal{O}\big((v/c)^{10}\big)\bigg),$$

$$e_{1\mathrm{PN}}(\nu) = -\frac{3}{4} - \frac{1}{12}\nu, \qquad e_{2\mathrm{PN}}(\nu) = -\frac{27}{8} + \frac{19}{8}\nu - \frac{1}{24}\nu^2,$$

$$e_{3\mathrm{PN}}(\nu) = -\frac{675}{64} + \left(\frac{34445}{576} - \frac{205}{96}\pi^2\right)\nu - \frac{155}{96}\nu^2 - \frac{35}{5184}\nu^3,$$

$$e_{4\mathrm{PN}}(\nu) = -\frac{3969}{128} + c_1\nu + \boxed{\left(-\frac{498449}{3456} + \frac{3157}{576}\pi^2\right)\nu^2 + \frac{301}{1728}\nu^3 + \frac{77}{31104}\nu^4}$$

(Jaranowski & Schäfer 2012–2013, Foffa & Sturani 2013),

$$\boxed{c_1 = -\frac{123671}{5760} + \frac{9037}{1536}\pi^2 + \frac{896}{15}(2\ln 2 + \gamma))}$$ (γ is the Euler's constant)

(Le Tiec, Blanchet, & Whiting 2012, Bini & Damour 2013).

Gravitational-wave luminosity for circular orbits up to the 3.5PN order

$$\mathcal{L}(x;\nu) = \frac{32c^5}{5G}\nu^2 x^5 \Bigg(1 + \ell_{1\mathrm{PN}}(\nu)\,x + 4\pi\,x^{3/2} + \ell_{2\mathrm{PN}}(\nu)\,x^2 + \ell_{2.5\mathrm{PN}}(\nu)\,x^{5/2} + \left(\ell_{3\mathrm{PN}}(\nu) - \frac{856}{105}\ln(16x)\right)x^3 + \ell_{3.5\mathrm{PN}}(\nu)\,x^{7/2} + \mathcal{O}\big((v/c)^8\big)\Bigg),$$

$$\ell_{1\mathrm{PN}}(\nu) = -\frac{1247}{336} - \frac{35}{12}\nu, \qquad \ell_{2\mathrm{PN}}(\nu) = -\frac{44711}{9072} + \frac{9271}{504}\nu + \frac{65}{18}\nu^2,$$

$$\ell_{2.5\mathrm{PN}}(\nu) = \left(-\frac{8191}{672} - \frac{535}{24}\nu\right)\pi,$$

$$\ell_{3\mathrm{PN}}(\nu) = \frac{6643739519}{69854400} + \frac{16}{3}\pi^2 - \frac{1712}{105}\gamma + \left(-\frac{134543}{7776} + \frac{41}{48}\pi^2\right)\nu - \frac{94403}{3024}\nu^2 - \frac{775}{324}\nu^3,$$

$$\ell_{3.5\mathrm{PN}}(\nu) = \left(-\frac{16285}{504} + \frac{214745}{1728}\nu + \frac{193385}{3024}\nu^2\right)\pi.$$

Post-Newtonian results

- Gravitational-wave luminosities for energy and angular momentum in the case of quasi-elliptical orbits were computed up to the 3PN order beyond the leading-order formula.
- PN effects modify not only the orbital phase evolution of the binary, but also the amplitudes of the wave polarizations h_+ and $h_\times$:
  —for quasi-circular orbits the 3PN-accurate corrections were computed;
  —for quasi-elliptical orbits the 2PN-accurate corrections were computed.
- It is not easy to obtain the wave polarizations h_+ and $h_\times$ as explicit functions of time taking into account all known higher-order PN corrections. In the case of quasi-elliptical orbits one has to deal with three different time scales: orbital period, periastron precession, and radiation reaction time scale.

- The spin of a rotating body is of the order

$$S \sim m a v_{\rm spin},$$

where m and a denote the mass and typical size of the body, respectively, and $v_{\rm spin}$ represents the velocity of the body's surface.

- We are interested in compact bodies, so

$$a \sim \frac{Gm}{c^2}, \quad \text{and then} \quad S \sim Gm^2 \frac{v_{\rm spin}}{c^2}.$$

Nomenclature on PN spin-dependent effects

- Maximally rotating bodies: $v_{\text{spin}} \sim c \Longrightarrow S \sim \dfrac{Gm^2}{c}$.

  Spin-orbit (i.e. linear in S) effects in EOM:
  $$1.5\text{PN} + 2.5\text{PN} + \cdots;$$
  spin-spin effects in EOM:
  $$2\text{PN} + 3\text{PN} + \cdots.$$

- Slowly rotating bodies: $v_{\text{spin}} \ll c \Longrightarrow S \sim \dfrac{Gm^2 v_{\text{spin}}}{c^2}$.

  Spin-orbit (i.e. linear in S) effects in EOM:
  $$2\text{PN} + 3\text{PN} + \cdots;$$
  spin-spin effects in EOM:
  $$3\text{PN} + 3\text{PN} + \cdots.$$

More nomenclature on PN spin-dependent effects

- Just words, no $1/c$ counting:
  leading-order (LO) + next-to-leading-order (NLO) + next-to-next-to-leading-order (NNLO) + $\cdots$.
- Formal counting:
  PN orders are counted in terms of $1/c$ originally present in the Einstein equations, i.e. the spin variables do not contribute to counting of $1/c$.
  Then, e.g., spin-orbit effects in EOM start as follows:
  $$1\text{PN} + 2\text{PN} + \cdots.$$

Structure of EOB formalism

1. Computation of inspiral + plunge waveform $h^{\text{insplunge}}(t)$:
   - PN conservative dynamics $\longrightarrow$ EOB-improved Hamiltonian;
   - PN radiation losses $\longrightarrow$ EOB radiation-reaction force;
   - PN waveform.
2. Computation of ringdown waveform $h^{\text{ringdown}}(t)$:
   - BH perturbation theory.

EOB-waveform: $h^{\text{EOB}}(t) = \theta(t_m - t)\, h^{\text{insplunge}}(t) + \theta(t - t_m)\, h^{\text{ringdown}}(t)$,

where $\theta(t)$ denotes Heaviside's step function and t_m is the time at which the two waveforms $h^{\text{insplunge}}$, h^{ringdown} are matched.

- Within the ADM approach the 3PN-accurate 2-point-mass conservative Hamiltonian $H\big(\mathbf{x}_1, \mathbf{x}_2, \mathbf{p}_1, \mathbf{p}_2\big)$ was derived. We need its reduction to the center-of-mass frame.
- Relative motion in the center-of-mass frame $(\mathbf{p}_1 + \mathbf{p}_2 = \mathbf{0})$:

$$\big(\mathbf{x}_1, \mathbf{x}_2, \mathbf{p}_1, \mathbf{p}_2\big) \to (\mathbf{q}, \mathbf{p});$$

reduced variables in the center-of-mass frame:

$$\mathbf{q} \equiv \frac{\mathbf{x}_1 - \mathbf{x}_2}{GM}, \qquad \mathbf{p} \equiv \frac{\mathbf{p}_1}{\mu} = -\frac{\mathbf{p}_2}{\mu}, \qquad \widehat{t} \equiv \frac{t}{GM}.$$

- 'Non-relativistic' Hamiltonian of the 2-point-mass system:

$$\widehat{H}^{\mathrm{NR}} \equiv \frac{H - (m_1 + m_2)c^2}{\mu}.$$

- Conservative 3PN-accurate Hamiltonian describing relative motion

$$\widehat{H}^{\mathsf{NR}}(\mathbf{q},\mathbf{p}) = \widehat{H}_{\mathrm{N}}(\mathbf{q},\mathbf{p}) + \frac{1}{c^2}\widehat{H}_{\mathrm{1PN}}(\mathbf{q},\mathbf{p}) + \frac{1}{c^4}\widehat{H}_{\mathrm{2PN}}(\mathbf{q},\mathbf{p}) + \frac{1}{c^6}\widehat{H}_{\mathrm{3PN}}(\mathbf{q},\mathbf{p}),$$

$$\widehat{H}_{\mathrm{N}}(\mathbf{q},\mathbf{p}) = \frac{1}{2}\mathbf{p}^2 - \frac{1}{q}, \quad \ldots$$

(the functional form of $\widehat{H}^{\mathsf{NR}}(\mathbf{q},\mathbf{p})$ is gauge dependent).

Units: $c = G = 1.$

Real two-body problem
(two masses m_1, m_2 orbiting around each other)

$$\updownarrow$$

Effective one-body problem
(one test particle of mass m_0
moving in some background metric $g_{\alpha\beta}^{\text{effective}}$)

The mapping rules between the two problems (motivated by quantum considerations):

- the adiabatic invariants (the action variables) $I_i = \oint p_i \, \mathrm{d}q_i$ are identified in the two problems;
- the energies are mapped through a function f:

$$\mathcal{E}_{\text{effective}} = f(\mathcal{E}_{\text{real}}),$$

f is determined in the process of matching.

One looks for a metric $g_{\alpha\beta}^{\text{effective}}$ such that the energies of the bound states of a particle moving in $g_{\alpha\beta}^{\text{effective}}$ are in one-to-one correspondence with the energies of the two-body bound states:

$$\mathcal{E}_{\text{effective}}(I_i) = f\left(\mathcal{E}_{\text{real}}(I_i)\right).$$

- Static and spherically symmetric effective metric (in Schwarzschild-like coordinates):

$$\mathrm{d}s_{\mathrm{eff}}^2 = -A(q')\,\mathrm{d}t'^2 + \frac{D(q')}{A(q')}\,\mathrm{d}q'^2 + q'^2\,(\mathrm{d}\theta'^2 + \sin^2\theta'\,\mathrm{d}\varphi'^2),$$

$$A(q') = 1 + a_1\,\frac{M_0}{q'} + a_2\left(\frac{M_0}{q'}\right)^2 + a_3\left(\frac{M_0}{q'}\right)^3 + a_4\left(\frac{M_0}{q'}\right)^4 + \cdots,$$

$$D(q') = 1 + d_1\,\frac{M_0}{q'} + d_2\left(\frac{M_0}{q'}\right)^2 + d_3\left(\frac{M_0}{q'}\right)^3 + \cdots.$$

- The energy-map for the "non-relativistic" energies

$\mathcal{E}_{\rm eff}^{\rm NR} \equiv \mathcal{E}_{\rm eff}^{\rm R} - m_0$ and $\mathcal{E}_{\rm real}^{\rm NR} \equiv \mathcal{E}_{\rm real}^{\rm R} - M$:

$$\frac{\mathcal{E}_{\rm eff}^{\rm NR}}{m_0} = \frac{\mathcal{E}_{\rm real}^{\rm NR}}{\mu}\left(1 + \alpha_1 \frac{\mathcal{E}_{\rm real}^{\rm NR}}{\mu} + \alpha_2 \left(\frac{\mathcal{E}_{\rm real}^{\rm NR}}{\mu}\right)^2 + \alpha_3 \left(\frac{\mathcal{E}_{\rm real}^{\rm NR}}{\mu}\right)^3 + \cdots\right).$$

- It is natural to require that

$$m_0 \equiv \mu = m_1\, m_2/(m_1 + m_2),$$

$$M_0 \equiv M = m_1 + m_2,$$

with these choices the Newtonian limit tells us that $a_1 = -2$.

- At each PN level one has only three arbitrary coefficients:
  the 1PN level: a_2, d_1, and α_1;
  the 2PN level: a_3, d_2, and α_2;
  the 3PN level: a_4, d_3, and α_3.
- How many independent equations has to be satisfied when mapping the real problem onto the effective one?

- The structure of the nPN conservative relative-motion real Hamiltonian in the center-of-mass frame:

$$\widehat{H}^{\mathrm{NR}}_{n\mathrm{PN}}(\mathbf{q},\mathbf{p}) = p^{2(n+1)} + \frac{1}{q}\left(p^{2n} + p^{2(n-1)}(np)^2 + \cdots + (np)^{2n}\right)$$

$$+ \frac{1}{q^2}\left(p^{2(n-1)} + \cdots + (np)^{2(n-1)}\right) + \cdots + \frac{1}{q^{n+1}};$$

$$\widehat{H}^{\mathrm{NR}}_{n\mathrm{PN}}(\mathbf{q},\mathbf{p}) \quad \text{has} \quad C_H(n) = \frac{(n+1)(n+2)}{2} + 1 \quad \text{coefficients.}$$

- The identification of the action variables guarantees that the two problems are mapped by a canonical transformation, with generating function

$$\tilde{G}(q, p') = q^i p'_i + G(q, p'),$$

so the relation between the real phase-space coordinates (q^i, p_i) and the effective phase-space coordinates (q'^i, p'_i) reads

$$q'^i = q^i + \frac{\partial\, G(q, p')}{\partial\, p'_i}, \quad p_i = p'_i + \frac{\partial\, G(q, p')}{\partial\, q^i}.$$

- The structure of the nPN generating function:

$$G_{n\mathrm{PN}}(\mathbf{q},\mathbf{p}') = (\mathbf{q}\cdot\mathbf{p}')\left\{p'^{2n} + \frac{1}{q}\left(p'^{2(n-1)} + \cdots + (np')^{2(n-1)}\right) + \cdots + \frac{1}{q^n}\right\};$$

$$G_{n\mathrm{PN}}(\mathbf{q},\mathbf{p}') \quad \text{has} \quad C_G(n) = \frac{n(n+1)}{2} + 1 \quad \text{coefficients.}$$

- The difference $\Delta(n)$ between the number of equations to satisfy and the number of unknowns at the nPN level:

$$\Delta(n) = C_H(n) - (C_G(n) + 3) = n - 2\,;$$

$$\Delta(1) = -1\,,$$

$$\Delta(2) = 0\,,$$

$$\Delta(3) = +1\,.$$

2PN-accurate effective (geodesic) Hamiltonian

$$0 = \mu^2 + g_{\rm eff}^{\alpha\beta}(x')\, p'_\alpha\, p'_\beta$$

$$\Downarrow$$

$$\widehat{H}_{\rm eff}^{\rm R}(\mathbf{q}', \mathbf{p}') = \sqrt{A(q')\left(1 + \mathbf{p}'^2 + \left(\frac{A(q')}{D(q')} - 1\right)(\mathbf{n}' \cdot \mathbf{p}')^2\right)}.$$

Matching results at the 1PN level

- One unrestricted degree of freedom is used to impose the condition $d_1 = 0$, i.e. that the linearized effective metric coincides with the linearized Schwarzschild metric.
- Energy map coefficient: $\alpha_1 = \frac{1}{2}\nu$.

Matching results at the 1PN level

- One unrestricted degree of freedom is used to impose the condition $d_1 = 0$, i.e. that the linearized effective metric coincides with the linearized Schwarzschild metric.
- Energy map coefficient: $\alpha_1 = \frac{1}{2}\nu$.

Matching results at the 2PN level

- No degrees of freedom left.
- Energy map coefficient: $\alpha_2 = 0$.

3PN-accurate effective (nongeodesic) Hamiltonian

$$0 = \mu^2 + g_{\rm eff}^{\alpha\beta}(x')\, p'_\alpha\, p'_\beta + A^{\alpha\beta\gamma\delta}(x')\, p'_\alpha\, p'_\beta\, p'_\gamma\, p'_\delta$$

$$\Downarrow$$

$$\widehat{H}_{\rm eff}^{\rm R}(\mathbf{q}', \mathbf{p}') = \sqrt{A(q')\left(1 + \mathbf{p}'^2 + \left(\frac{A(q')}{D(q')} - 1\right)(\mathbf{n}'\cdot\mathbf{p}')^2 + Z(\mathbf{q}', \mathbf{p}')\right)},$$

where $\quad Z(\mathbf{q}', \mathbf{p}') \equiv \dfrac{1}{q'^2}\left(z_1(\mathbf{p}'^2)^2 + z_2\, \mathbf{p}'^2(\mathbf{n}'\cdot\mathbf{p}')^2 + z_3(\mathbf{n}'\cdot\mathbf{p}')^4\right).$

Matching results at the 3PN level

- The parameters z_1, z_2, and z_3 satisfy the linear constraint:

$$8z_1 + 4z_2 + 3z_3 = 6(4 - 3\nu)\nu \, .$$

- Energy map coefficient:

$$\alpha_3 = 0 \, .$$

To simplify the 3PN effective dynamics of circular orbits, one chooses

$$z_1 = 0 = z_2 \, , \quad z_3 = 2(4 - 3\nu)\nu \, .$$

- The values $\alpha_1 = \frac{1}{2}\nu$ at 1PN, $\alpha_2 = 0$ at 2PN, and $\alpha_3 = 0$ at 3PN correspond exactly to

$$\frac{\mathcal{E}^{\mathrm{R}}_{\mathrm{eff}}}{\mu} = \frac{(\mathcal{E}^{\mathrm{R}}_{\mathrm{real}})^2 - m_1^2 - m_2^2}{2m_1 m_2}.$$

- Solving above equation with respect to $\mathcal{E}^{\mathrm{R}}_{\mathrm{real}}$ one obtains the real EOB-improved 3PN-accurate Hamiltonian:

$$\boxed{H^{\mathrm{R}}_{\mathrm{real}}(\mathbf{q},\mathbf{p}) = M\sqrt{1 + 2\nu\frac{H^{\mathrm{R}}_{\mathrm{eff}}\big(\mathbf{q}'(\mathbf{q},\mathbf{p}),\mathbf{p}'(\mathbf{q},\mathbf{p})\big) - \mu}{\mu}}.}$$

Padé approximant of (k, l)-type (with $k + l = n$) for series $\sigma(x) = c_0 + c_1\, x + \cdots + c_n\, x^n$ (with $c_0 \neq 0$):

$$\mathrm{P}^k_l(\sigma(x)) \equiv \frac{N_k(x)}{D_l(x)},$$

where the polynomials N_k (of degree k) and D_l (of degree l) are such that the Taylor expansion of $\mathrm{P}^k_l(\sigma(x))$ coincides with $\sigma(x)$ up to $\mathcal{O}(x^n)$ terms.

- Let us consider the reduced angular momentum of the system in the center-of-mass reference frame:

$$j \equiv \frac{\mathcal{J}}{G m_1 m_2}.$$

- In the test-mass limit and along circular orbits:

$$j(x; \nu = 0) = \frac{1}{\sqrt{x(1-3x)}},$$

the pole $x = 1/3$ corresponds to 'light ring'
(or the last unstable circular orbit).

- As a result of the 3PN-accurate PN computations one gets

$$j^2(x;\nu) = \frac{1}{x}\left[1 + \frac{1}{3}(9+\nu)\,x + \frac{1}{36}(36-\nu)(9-4\nu)\,x^2 + \frac{16}{3}w_5(\nu)\,x^3\right],$$

where

$$w_5(\nu) \equiv \frac{81}{16} + \frac{1}{64}\left(41\pi^2 - \frac{7321}{6}\right)\nu + \frac{23}{64}\nu^2 + \frac{1}{216}\nu^3.$$

- One can construct the following sequence of near-diagonal Padés of $j^2(x)$:

$$j^2_{P_1}(x;\nu) \equiv \frac{1}{x}\mathrm{P}^0_1\left[1 + \frac{1}{3}(9+\nu)\,x\right],$$

$$j^2_{P_2}(x;\nu) \equiv \frac{1}{x}\mathrm{P}^1_1\left[1 + \frac{1}{3}(9+\nu)\,x + \frac{1}{36}(36-\nu)(9-4\nu)\,x^2\right],$$

$$j^2_{P_3}(x;\nu) \equiv \frac{1}{x}\mathrm{P}^2_1\left[1 + \frac{1}{3}(9+\nu)\,x + \frac{1}{36}(36-\nu)(9-4\nu)\,x^2 + \frac{16}{3}w_5(\nu)\,x^3\right].$$

- The results of computation read:

$$j^2_{P_1}(x;\nu) = \frac{1}{x\left(1-\left(3+\frac{1}{3}\nu\right)x\right)},$$

$$j^2_{P_2}(x;\nu) = \frac{1+\frac{1}{9}\nu+\frac{25}{12}\nu x}{x\left(1+\frac{1}{9}\nu-\left(3-\frac{17}{12}\nu+\frac{1}{27}\nu^2\right)x\right)},$$

$$j^2_{P_3}(x;\nu) = \frac{1+\left(3+\frac{1}{3}\nu-w_6(\nu)\right)x+\left(9-\frac{17}{4}\nu+\frac{1}{9}\nu^2-\left(3+\frac{1}{3}\nu\right)w_6(\nu)\right)x^2}{x\left(1-w_6(\nu)x\right)},$$

where

$$w_6(\nu) \equiv \frac{192}{(36-\nu)(9-4\nu)}\,w_5(\nu).$$

- The test-mass limit is recovered exactly,

$$\lim_{\nu\to 0} j^2_{P_1}(x;\nu) = \lim_{\nu\to 0} j^2_{P_2}(x;\nu) = \lim_{\nu\to 0} j^2_{P_3}(x;\nu) = \frac{1}{x(1-3x)}.$$

- The EOB metric coefficient $-g_{00}^{\text{eff}}(u) = A(u)$ (with $u \equiv 1/q'$) has the following 3PN-accurate truncated Taylor expansion:

$$A(u;\nu) = 1 - 2\,u + 2\nu\,u^3 + a_4(\nu)\,u^4 + \mathcal{O}\left(u^5\right)\,,$$

where the 3PN coefficient $a_4(\nu)$ reads

$$a_4(\nu) = \left(\frac{94}{3} - \frac{41}{32}\pi^2\right)\nu.$$

- We want now to factor a zero of $A(u;\nu)$ (and no longer a pole), which will be smoothly connected with the test-mass-limit (i.e. $\nu \to 0$) value $u = 1/2$, therefore the Padé-improved $A_{P_n}(u;\nu)$ of $A(u;\nu)$ we define at the nPN level as

$$A_{P_n}(u;\nu) \equiv \mathrm{P}^1_n\left[T_{n+1}[A(u;\nu)]\right].$$

EOB formalism $\longleftrightarrow$ synergy $\longleftrightarrow$ Numerical Relativity

Select sample of NR results

$\downarrow$

Calibrating EOB flexibility parameters

$\downarrow$

NR-improved EOB
(accurate 'analytical' waveform templates; 'analytical' predictions)

Flexibility parameters in the metric coefficient $-g_{00}^{\text{eff}}(u) = A(u)$

Instead of using (maybe in Padé-improved form)

$$A^{3\text{PN}}(u;\nu) = 1 - 2\,u + 2\nu\,u^3 + a_4(\nu)\,u^4,$$

one considers two-parameter class of extensions of $A^{3\text{PN}}(u;\nu)$ defined by

$$A(u;\nu,a_5,a_6) \equiv \mathrm{P}^1_5\left[A^{3\text{PN}}(u;\nu) + \nu\,a_5\,u^5 + \nu\,a_6\,u^6\right].$$

"Contrary to the hint of Mroué, Kidder, and Teukolsky (2008), that EOB is 'only ... a very good fitting model', we think that EOB incorporates a lot of the real physics of coalescing BBH, and that its parametric flexibility is a way to complete it with the 'missing' non-perturbative physics provided by NR simulations."

(T.Damour, 2008)