ROGERS FUNCTIONS AND FLUCTUATION THEORY

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Let X_t be a Lévy process and Ψ be the Lévy–Khintchine exponent of X_t. The *Wiener–Hopf factors* $\kappa(\tau;\xi)$ and $\hat{\kappa}(\tau;\xi)$ of $\tau+\Psi(\xi)$ are closely related to the Laplace transforms of distributions of the supremum and the infimum of X_t over $t \in [0,T]$. More precisely, $\kappa(\tau;\xi)$ and $\hat{\kappa}(\tau;\xi)$ are (bivariate) Laplace exponents of the ascending and the descending ladder processes. The Wiener–Hopf factors are fundamental objects in fluctuation theory of Lévy processes. Their name corresponds to the identity $\tau+\Psi(\xi)=\kappa(\tau,-i\xi)\hat{\kappa}(\tau,i\xi)$.

A function $f:(0,\infty)\to[0,\infty)$ is said to be a complete Bernstein function (CBF) if it extends to a holomorphic function on $\mathbf{C}\setminus(-\infty,0]$ such that $\operatorname{Im} f(\xi)\ge 0$ if $\operatorname{Im}\xi>0$. This class of functions is particularly well-suited for the inversion of Laplace transforms. For this reason one is interested whether the Wiener–Hopf factors are CBFs.

Thirty years ago L.C.G. Rogers published a paper [3], where he proved the following theorem: $\kappa(\tau;\xi)$ and $\hat{\kappa}(\tau;\xi)$ are CBFs of ξ if and only if X_t has *completely monotone jumps*, that is, the Lévy measures of X_t and the dual process $\hat{X}_t$ have completely monotone density functions on $(0,\infty)$. Under additional assumptions (which, I believe, are of purely technical nature), this result can be strengthened as follows.

Theorem. *If X_t has completely monotone jumps and it is* balanced*, then $\kappa(\tau;\xi)$ and $\hat{\kappa}(\tau;\xi)$ are CBFs of both τ and ξ. Furthermore, in this case*

$$\frac{\kappa(\tau;\xi_1)}{\kappa(\tau;\xi_2)}, \qquad \frac{\hat{\kappa}(\tau;\xi_1)}{\hat{\kappa}(\tau;\xi_2)}, \qquad \frac{\kappa(\tau_1;\xi)}{\kappa(\tau_2;\xi)}, \qquad \frac{\hat{\kappa}(\tau_1;\xi)}{\hat{\kappa}(\tau_2;\xi)}$$

are all CBFs of ξ or τ when $\tau_1<\tau_2$ and $\xi_1<\xi_2$. Finally, $\kappa(\tau;\xi_1)\hat{\kappa}(\tau;\xi_2)$ is a CBF of τ for all ξ_1, ξ_2.

The notion of a balanced Lévy process is defined in terms of the Lévy–Khintchine exponent Ψ. For example, all strictly stable Lévy processes which are neither increasing nor decreasing are balanced, and this condition is preserved by subordination.

For the proof of the above result, the theory of Lévy–Khintchine exponents Ψ of Lévy processes with completely monotone jumps is developed. Because these functions were first considered in [3], the name *Rogers functions* is proposed.

In order to explain the role of the theorem stated above, I shall begin the talk with a very short introduction to fluctuation theory of Lévy processes. Then I shall move to the definition and properties of Rogers functions. In this part the precise meaning of the word *balanced* will be given, and various forms of Wiener–Hopf factorization of Rogers functions will be presented. At the end of the talk I shall discuss possible applications and extensions.

For the related research for *symmetric* Lévy processes, see [1, 2]

References

[1] M. Kwaśnicki, J. Małecki, M. Ryznar, *First passage times for subordinate Brownian motions.* Stoch. Proc. Appl 123 (2013): 1820–1850.

[2] M. Kwaśnicki, J. Małecki, M. Ryznar, *Suprema of Lévy processes.* Ann. Probab. 41(3B) (2013): 2047–2065.

[3] L.C.G. Rogers, *Wiener–Hopf factorization of diffusions and Lévy processes.* Proc. London Math. Soc. 47(3) (1983): 177–191.