LÉVY PROCESSES, MARTINGALES, REVERSED MARTINGALES AND ORTHOGONAL POLYNOMIALS

PAWEŁJ. SZABŁOWSKI

We study class of Lévy processes having all moments. We define system of polynomial martingales $\{M_n(X_t,t),\mathcal{F}_{\leq t}\}_{n\geq 1}$, where $\mathcal{F}_{\leq t}$ is a suitable filtration defined below. We present some properties of these martingales. Among others we show that $M_1(X_t,t)/t$ is a reversed martingale and M_1 is a harness. We study also chances for martingales M_n multiplied by suitable functions of t to be a reversed martingales. We show that for $n\geq 3$ it is possible only when the Lévy process in question is Gaussian (i.e. is a Wiener process). For $n=2$ we present Lévy process different from the Gaussian one that has this property.

Theorem 1. *Let $\{X_t\}_{t\geq 0}$ be Lévy process with all moments existing. Let us denote $m_j(t)=EX_t^j$ and $M_2(X_t,t)=m_2(-t)+2m_1(-t)X_t+X_t^2$. Then $(M_2(X_t,t),\mathcal{F}_{\leq t})$ is a martingale. Moreover $(\mu(t)M_2(X_t,t),\mathcal{F}_{\geq t})$ is a reversed martingale with $\mu(t)=1/(2c_2^2t+c_4)$ iff either $c_4=0$ and the Lévy process is Gaussian or*
$c_4>0$ and

$$\exp(tf(x)=e^{c_1tx}(\cos(x\sqrt{\frac{c_4}{2c_2}}))^{-2tc_2^2/c_4}, \tag{1}$$

for $|x|<\frac{\pi}{2}\sqrt{\frac{2c_2}{c_4}}$. In particular assuming for simplicity that $c_1=0$ the distribution of X_t for $t=\frac{c_4}{2c_2^2}$ has density $h(y)$ equal to

$$h(y)=\frac{\sqrt{c_4}}{\sqrt{8c_2}\cosh(\frac{\pi y\sqrt{2c_2}}{2\sqrt{c_4}})};\ y\in\mathbb{R}. \tag{2}$$

and is identifiable by moments.

References

[1] Bryc, Włodzimierz, Wesołowski Jacek, Stitching pairs of Lévy processes into harnesses, *Stochastic Processes and their Applications*, **122**(2012), str. 2854 - 2869
[2] Bożejko, M. Bryc, W. (2006), On a Class of Free Levy Laws Related to a Regression Problem, *Journal of Functional Analysis* **236** , 59-77.
[3] Włodzimierz Bryc, Wojciech Matysiak, Jacek Wesołowski. Quadratic Harnesses, $q-$commutations, and orthogonal martingale polynomials. *Trans. Amer. Math. Soc.* **359** (2007), no. 11, 5449–5483
[4] Gniedenko, B. W. i Kolmogorov, A. N. Rozkłady graniczne sum zmiennych losowych niezależnych, *PWN*, Warszawa, 1957 (in Polish)
[5] Lin, T. F. Multiple integrals of a homogeneous process with independent increments. *Ann. Probab.* **9** (1981), no. 3, 529–532. MR0614639 (82h:60141)
[6] Sato, Ken-iti. Lévy processes and infinitely divisible distributions. Translated from the 1990 Japanese original. Revised by the author. *Cambridge Studies in Advanced Mathematics, 68. Cambridge University Press,* Cambridge, 1999. xii+486 pp. ISBN: 0-521-55302-4 MR1739520 (2003b:60064)
[7] Segall, Adrian; Kailath, Thomas. Orthogonal functionals of independent-increment processes. *IEEE Trans. Information Theory* IT-**22** (1976), no. 3, 287–298. MR0413257 (54 #1374)
[8] Solé, Josep Lluís; Utzet, Frederic. On the orthogonal polynomials associated with a Lévy process. *Ann. Probab.* **36** (2008), no. 2, 765–795. MR2393997 (2009d:60145)
[9] Schoutens, Wim; Teugels, Jozef L. Lévy processes, polynomials and martingales. Special issue in honor of Marcel F. Neuts. *Comm. Statist. Stochastic Models* **14** (1998), no. 1-2, 335–349. MR1617536 (99k:60118)

[10] Schoutens, Wim. Stochastic processes and orthogonal polynomials. *Lecture Notes in Statistics*, **146**. Springer-Verlag, New York, 2000. xiv+163 pp. ISBN: 0-387-95015-X MR1761401 (2001f:60095)

[11] Paweł J. Szabłowski, On Markov processes with polynomials conditional moments, http://arxiv.org/abs/1210.6055

[12] Szabłowski, Paweł J. $q-$Wiener and $(\alpha, q)-$ Ornstein–Uhlenbeck processes. A generalization of known processes, *Theory of Probability and Its Applications,* **56** (4), 2011, 742–772, http://arxiv.org/abs/math/0507303

[13] Simon, Barry. The classical moment problem as a self-adjoint finite difference operator. *Adv. Math.* **137** (1998), no. 1, 82–203. MR1627806 (2001e:47020)

[14] Wesołowski, Jacek. A martingale characterization of the Wiener process. *Statist. Probab. Lett.* **10** (1990), no. 3, 213–215. MR1072511 (91i:60211)

[15] Wesołowski, Jacek. A martingale characterization of the Poisson process. *Bull. Polish Acad. Sci. Math.* **38** (1990), no. 1-12, 49–53. MR1194245 (93g:60094)

[16] Yablonski, Aleh. The calculus of variations for processes with independent increments. *Rocky Mountain J. Math.* **38** (2008), no. 2, 669–701. MR2401574 (2009c:60143)