

# ON RANDOM WALKS THAT AVOID A STRIPE

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Let  $S_n = x + X_1 + \dots + X_n$  be a centered random walk, and let  $\tau_B := \min\{k \geq 0 : S_k \in B\}$  be the hitting time of a Borel set  $B$ . We consider the case that  $B$  is bounded, and our main question is to study the asymptotic of  $\mathbb{P}_x(\tau_B > n)$  as  $n \rightarrow \infty$ , where  $\mathbb{P}_x$  is the distribution of the walk starting at  $x$ . This problem was completely solved in dimension one and two in the 1960s (see [1]) for integer-valued random walk, essentially, under no additional assumptions. The proof of [1] is by induction in the number of lattice points in  $B$  and it uses the methods of potential theory. Thus the problem simplifies to the study of  $\mathbb{P}_x(\tau_{\{0\}} > n)$  which may be done with the use of theory of recurrent events. Of course this does not work in the general case, and we use an entirely different approach that forces us to work in one dimension and assume finiteness of variance.

We start with the basic case that  $B$  is an interval of the form  $(-d, d)$  for some  $d > 0$ , and in this case the event  $\{\tau_B > n\}$  simply means that the trajectory of the walk avoids the stripe  $(-d, d) \times [1, \infty)$  up to the time  $n$ . The main idea is that a typical trajectory upcrosses and downcrosses the stripe only for a few times as this is exponentially costly in the number of crossings. These crossings are performed at the very first steps and then the walk stays either to the right or to the left of  $B$  all the time up to  $n$ .

Define the moments when the random walk jumps over the edges of  $B$  from the outside as  $T_0 := 0$  and

$$T_{k+1} := \min\{n > T_k : S_{n-1} \geq d, S_n < d\} \wedge \min\{n > T_k : S_{n-1} \leq -d, S_n > -d\}$$

for  $k \geq 0$ . Then  $\tau_{(-d,d)} = T_\kappa$ , where  $\kappa := \min\{k \geq 1 : |S_{T_k}| < d\}$  is the number of jumps over the stripe by the time it gets hit. Put  $V_d(x) := \mathbb{E}_x \sum_{i=1}^{\kappa} |S_{T_i} - S_{T_{i-1}}|$ . Our main result is as follows.

**Theorem 1.** *Let  $S_n$  be a random walk with  $\mathbb{E}X_1 = 0$  and  $\text{Var}(X_1^2) =: \sigma^2 \in (0, \infty)$ . Then for any  $d > 0$  and  $|x| \geq d$ ,*

$$\mathbb{P}_x(\tau_{(-d,d)} > n) \sim \frac{\sqrt{2}V_d(x)}{\sigma\sqrt{\pi n}}, \quad n \rightarrow \infty.$$

*Moreover, this relation holds uniformly as  $|x|/\sqrt{n} \rightarrow 0$ . The function  $V_d(x)$  is finite for any  $x$ , strictly positive for  $|x| \geq d$ , and  $V_d(x) \sim |x|$  as  $|x| \rightarrow \infty$ .*

We will also discuss an immediate application of the theorem that gives a tight estimate of the size of the largest gap  $G_n$  in the range of a random walk  $S_n$ , that is

$$G_n := \max_{1 \leq k \leq n-1} S_{(k+1,n)} - S_{(k,n)},$$

where  $S_{(1,n)} \leq S_{(2,n)} \leq \dots \leq S_{(n,n)}$  denote the elements of  $S_1, \dots, S_n$  arranged in the weakly ascending order.

Finally, we provide a generalization of Theorem 1 for general bounded sets.

## REFERENCES

- [1] H. Kesten and F. Spitzer. *Ratio theorems for random walks I*, *J. Anal. Math.*, 11, 285–322 (1963).