FUNDAMENTAL SOLUTIONS AND THEIR SHORT TIME ESTIMATES FOR JUMP-DIFFUSIONS ON MANIFOLDS

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We study the Cauchy problem on a manifold M associated with an integro-differential operator, which arises from a jump-diffusion determined by an SDE. Its generator is written as

$$A(t)f = V_0(t)f + \frac{1}{2}\sum_{j=1}^{m} V_j(t)^2 f + \lim_{\epsilon\to 0}\int_{U_\epsilon(z_0)^c} \{f(\phi_{t,z}(x)) - f(x)\}\nu(dz),$$

where $V_j(t), j = 0, ..., m$ are time-dependent smooth vector fields on M, $\{\phi_{t,z}, t \in [0,T], z \in M\}$ are diffeomorphisms of M, ν is a Lévy measure on M with center z_0 and $U_\epsilon(z_0)$ is an ϵ-neighborhood of z_0. For the Lévy measure ν, we assume that it has a weak drift and satisfies the order condition with exponent $0 < \alpha < 2$.

We show that if the operator $A(t)$ is nondegenerate (more generally hypoelliptic) and satisfies a certain dual condition, the fundamental solution of the following Cauchy problem exists.

$$\begin{cases} (A(t) + c(x,t) + \frac{\partial}{\partial t})u(x,t) = 0, & 0 < t < T,\ x \in M, \\ \lim_{t\uparrow T} u(x,t) = f(x), & \text{(terminal condition)} \end{cases}$$

Further, the short time estimates of the fundamental solution $p(s,x;t,y)$ are studied. Take any positive number γ less than $2-\alpha$. Then for any relatively compact subset K of M and multi-index $\mathbf{i} = (i_1, ..., i_d)$ of nonnegative integers, there exists a positive constant c such that

$$\sup_{x,y\in K} |D_y^{\mathbf{i}} p(s,x;t,y)| \le c(t-s)^{-(|\mathbf{i}|+d)/\gamma}, \quad \forall s < t,$$

where $D^{\mathbf{i}} = D^{i_1} \cdots D^{i_d}$ is the derivative operator.

As an application, we give a simple criterion that ensures the existence of the smooth density with respect to the Haar measure, for the law of a Lévy process on a Lie group.

Basic tools for the study are the Malliavin calculus on the Wiener-Poisson space and the dual property of the jump-diffusion.

References

[1] Y. Ishikawa and H. Kunita, Malliavin calculus on the Wiener-Poisson space and its application to canonical SDE with jumps, Stochastic processes and their applications, 116 (2006), 1743-1769.

[2] H. Kunita, Analysis of nondegenerate Wiener-Poisson functionals and its applications to Itô's SDE with jumps, Sankhya, The Indian Journal of Statistics, 73-A, Part 1 (2011), 1-45.

[3] H. Kunita, Nondegenerate SDE's with jumps and their hypoelliptic properties, J. Math. Soc. Japan (2012), to appear.

[4] J. Picard, Density in small time at accessible points for jump processes, Stochastic Process. Appl. 67 (1997), 251-279.

[5] J. Picard and C. Savona, Smoothness of the law of manifold-valued Markov processes with jumps, (2012)