

# FUNDAMENTAL SOLUTIONS AND THEIR SHORT TIME ESTIMATES FOR JUMP-DIFFUSIONS ON MANIFOLDS

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We study the Cauchy problem on a manifold  $M$  associated with an integro-differential operator, which arises from a jump-diffusion determined by an SDE. Its generator is written as

$$A(t)f = V_0(t)f + \frac{1}{2} \sum_{j=1}^m V_j(t)^2 f + \lim_{\epsilon \rightarrow 0} \int_{U_\epsilon(z_0)^c} \{f(\phi_{t,z}(x)) - f(x)\} \nu(dz),$$

where  $V_j(t)$ ,  $j = 0, \dots, m$  are time-dependent smooth vector fields on  $M$ ,  $\{\phi_{t,z}, t \in [0, T], z \in M\}$  are diffeomorphisms of  $M$ ,  $\nu$  is a Lévy measure on  $M$  with center  $z_0$  and  $U_\epsilon(z_0)$  is an  $\epsilon$ -neighborhood of  $z_0$ . For the Lévy measure  $\nu$ , we assume that it has a weak drift and satisfies the order condition with exponent  $0 < \alpha < 2$ .

We show that if the operator  $A(t)$  is nondegenerate (more generally hypoelliptic) and satisfies a certain dual condition, the fundamental solution of the following Cauchy problem exists.

$$\begin{cases} (A(t) + c(x, t) + \frac{\partial}{\partial t})u(x, t) = 0, & 0 < t < T, x \in M, \\ \lim_{t \uparrow T} u(x, t) = f(x), & \text{(terminal condition)} \end{cases}$$

Further, the short time estimates of the fundamental solution  $p(s, x; t, y)$  are studied. Take any positive number  $\gamma$  less than  $2 - \alpha$ . Then for any relatively compact subset  $K$  of  $M$  and multi-index  $\mathbf{i} = (i_1, \dots, i_d)$  of nonnegative integers, there exists a positive constant  $c$  such that

$$\sup_{x, y \in K} |D_y^{\mathbf{i}} p(s, x; t, y)| \leq c(t - s)^{-(|\mathbf{i}|+d)/\gamma}, \quad \forall s < t,$$

where  $D^{\mathbf{i}} = D^{i_1} \dots D^{i_d}$  is the derivative operator.

As an application, we give a simple criterion that ensures the existence of the smooth density with respect to the Haar measure, for the law of a Lévy process on a Lie group.

Basic tools for the study are the Malliavin calculus on the Wiener-Poisson space and the dual property of the jump-diffusion.

## REFERENCES

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