We study the Cauchy problem on a manifold $M$ associated with an integro-differential operator, which arises from a jump-diffusion determined by an SDE. Its generator is written as

$$A(t)f = V_0(t)f + \frac{1}{2} \sum_{j=1}^{m} V_j(t)^2 f + \lim_{\epsilon \to 0} \int_{U_\epsilon(z_0)^c} \{f(\phi_{t,z}(x)) - f(x)\} \nu(dz),$$

where $V_j(t), j = 0, ..., m$ are time-dependent smooth vector fields on $M$, $\{\phi_{t,z}, t \in [0, T], z \in M\}$ are diffeomorphisms of $M$, $\nu$ is a Lévy measure on $M$ with center $z_0$ and $U_\epsilon(z_0)$ is an $\epsilon$-neighborhood of $z_0$. For the Lévy measure $\nu$, we assume that it has a weak drift and satisfies the order condition with exponent $0 < \alpha < 2$.

We show that if the operator $A(t)$ is nondegenerate (more generally hypoelliptic) and satisfies a certain dual condition, the fundamental solution of the following Cauchy problem exists.

$$\begin{cases} (A(t) + c(x,t) + \frac{\partial}{\partial t})u(x,t) = 0, & 0 < t < T, \ x \in M, \\ \lim_{t \uparrow T} u(x,t) = f(x), & \text{(terminal condition)} \end{cases}$$

Further, the short time estimates of the fundamental solution $p(s, x; t, y)$ are studied. Take any positive number $\gamma$ less than $2 - \alpha$. Then for any relatively compact subset $K$ of $M$ and multi-index $i = (i_1, ..., i_d)$ of nonnegative integers, there exists a positive constant $c$ such that

$$\sup_{x, y \in K} |D^i_y p(s, x; t, y)| \leq c(t - s)^{-(|i|+d)/\gamma}, \ \forall s < t,$$

where $D^i = D^{i_1} \cdots D^{i_d}$ is the derivative operator.

As an application, we give a simple criterion that ensures the existence of the smooth density with respect to the Haar measure, for the law of a Lévy process on a Lie group.

Basic tools for the study are the Malliavin calculus on the Wiener-Poisson space and the dual property of the jump-diffusion.

REFERENCES


