FINE REGULARITY OF LÉVY PROCESSES AND LINEAR FRACTIONAL STABLE MOTION

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The multifractal nature of Lévy processes has first been investigated by Jaffard [7]. This seminal article, later extended in [4, 5], shows that the singularity spectrum of an $\mathbf{R}^d$-valued Lévy process X, characterized by the Lévy-Khintchine triplet $(a, Q = 0, \pi)$, is almost surely equal to

$$d_X(h) = \begin{cases} \beta h & \text{if } h \in [0, \frac{1}{\beta}]; \\ -\infty & \text{if } h \in (\frac{1}{\beta}, \infty] \end{cases} \text{ where } \beta = \inf\Big\{ \delta \geq 0 : \int_{\mathbf{R}^d} \Big(1 \wedge \|x\|^\delta \Big) \, \pi(\mathrm{d}x) < \infty \Big\}. \quad (1)$$

Our work [1] investigates the generalization of the aforementioned result within the 2-microlocal analysis framework. The later provides a tool, named the *2-microlocal frontier* and denoted $s' \mapsto \sigma_{X,t}(s')$, which gives a finer description of the local regularity of a stochastic process (as described in [6, 2]). In particular, it embraces the common pointwise and local Hölder exponents, since respectively $\alpha_{X,t} = -\inf\{s' : \sigma_{X,t}(s') \geq 0\}$ and $\widetilde{\alpha}_{X,t} = \sigma_{X,t}(0)$.

Classic multifractal analysis studies geometrical properties of the iso-Hölder sets $E_h = \{t \in \mathbf{R} : \alpha_{X,t} = h\}$. Hence, from a 2-microlocal point of view, it is natural to consider the following extensions of $(E_h)_{h \in \mathbf{R}_+}$

$$\widetilde{E}_h = \{ t \in E_h : \forall s' \in \mathbf{R};\ \sigma_{X,t}(s') = (h + s') \wedge 0 \} \quad \text{and} \quad \widehat{E}_h = E_h \setminus \widetilde{E}_h.$$

Theorem 1. *Sample paths of a Lévy process X characterized by $(a, Q = 0, \pi)$ a.s. satisfy*

$$\dim_{\mathcal{H}}(\widetilde{E}_h) = \begin{cases} \beta h & \text{if } h \in [0, \frac{1}{\beta}]; \\ -\infty & \text{if } h \in (\frac{1}{\beta}, \infty] \end{cases} \text{ and } \dim_{\mathcal{H}}(\widehat{E}_h) \leq \begin{cases} 2\beta h - 1 & \text{if } h \in (\frac{1}{2\beta}, \frac{1}{\beta}); \\ -\infty & \text{if } h \in [0, \frac{1}{2\beta}] \cup [\frac{1}{\beta}, \infty]. \end{cases}$$

The generalization of Equation (1) obtained in Theorem 1 happens to be interesting for the study of another class of stochastic processes, named linear fractional stable motion (LFSM) and defined by

$$X_t = \int_{\mathbf{R}} \Big\{ (t-u)_+^{H-1/\alpha} - (-u)_+^{H-1/\alpha} \Big\} \, M_{\alpha,\beta}(\mathrm{d}u),$$

where $M_{\alpha,\beta}$ is an α-stable random measure and $H \in (0,1)$.

Theorem 2. *Let X be a LFSM parametrized by $\alpha \in (1,2)$ and $H \in (\frac{1}{\alpha}, 1)$. Then, its multifractal spectrum is almost surely equal to*

$$d_X(h) = \begin{cases} \alpha(h - H) + 1 & \text{if } h \in [H - \frac{1}{\alpha}, H]; \\ -\infty & \text{otherwise.} \end{cases}$$

The later result highly relies on the stability of the 2-microlocal frontier under the action of fractional operators.

References

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