Lévy Processes with Two-Sided Reflection

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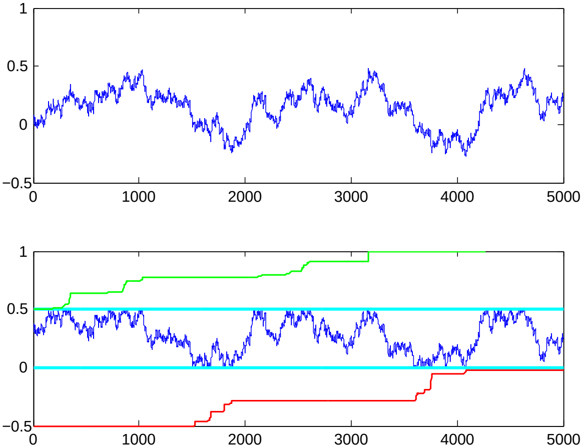
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Contents and Key Words

Existence and construction
- Direct
- Skorokhod problem

Stationary distribution
- Two-sided exit
- Scale function
- Explicit case

Loss rate
- Kella-Whitt martingale
- Itô

Large buffer asymptotics
- Light tails
- Heavy tails
- Functional limits

SA
Lars Nørvang Andersen
Peter W. Glynn
Michel Mandjes
Mats Pihlsgård

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L.N. Andersen (2011/12) Subexponential loss rate asymptotics for Lévy processes. *Mathematical Methods of Operations Research*

SA & L.N. Andersen (2011/12) Local time asymptotics for centered Lévy processes with two-sided reflection. *Stochastic Models*

L.N. Andersen & M. Mandjes (2009) Structural properties of reflected Lévy processes. *Queueing Systems*

P.W. Glynn & M. Pihlsgård (2012/13) Loss rates for two-sided reflected semimartingales via Itô's formula. *Journal of Applied Probability*

SA & M. Pihlsgård (2007) Loss rates for Lévy processes with two reflecting barriers. *Mathematics of Operations Research*

M. Pihlsgård (2005) Loss rate asymptotics in a $GI/G/1$ queue with finite buffer. *Stochastic Models*

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$X = \big(X(t);\ t \geq 0\big)$: real-valued Lévy process
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Construction straightforward in discrete time

$X_n = Y_1 + \cdots + Y_n$ random walk

$$V^b_{n+1} = \mathcal{R}^b_0(V^b_n + Y_n), \quad \mathcal{R}^b_0(z) = \begin{cases} z & 0 \leq z \leq b \\ 0 & z < 0 \\ b & z > b \end{cases}$$

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If $V^\infty(0) = 0$: $\;V^\infty(t) \;=\; X(t) - \min_{s \leq t} X(s)$

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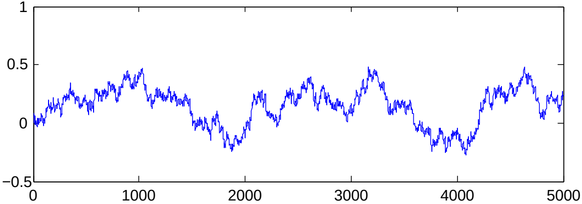
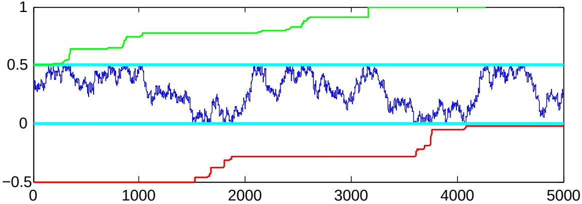
Two-sided in continuous time??

Pragmatic approach

Glue together segments with one-sided reflection





Formal mathematical approach

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Skorokhod problem

$$V^b(t) = X(t) + L(t) - U(t) \in [0, b] \text{ for all } t,$$

$$\int_0^\infty V^b(t)\,\mathrm{d}L(t) = 0 \quad \text{and} \quad \int_0^\infty (b - V^b(t))\,\mathrm{d}U(t) = 0\,.$$

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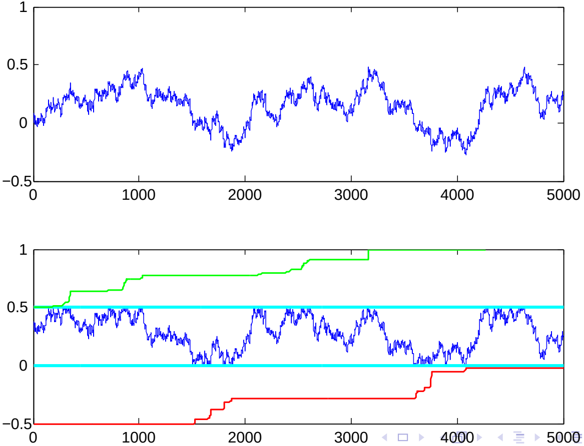

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$V^b(t)$ is given by

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Some simplification by Andersen & Mandjes

The stationary distribution

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stationary distribution π^{∞} of V^{∞} exists if drift $\mu = \mathbb{E}X(1) < 0$

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D. Lindley (1952) The theory of a queue with a single server. *Proceedings of the Cambridge Philosophical Society*

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But a handful more found recently

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PH: passage time of finite Markov process
Solution in terms of zeros of a polynomial

  or equivalently the analytic continuation
  of the Lévy exponent $\kappa(\alpha) = \log \mathbb{E}\mathrm{e}^{\alpha X(1)}$

Finite set of linear equations
Optional stopping at τ

  Formally Wald martingale
  Rigorously (analytic continuation) Kella-Whitt martingale

The loss rate

Applied literature: finite buffer, finite capacity problems

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How much is lost?? (flows over etc.)
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Loss at time n is $(V_n^b + Y_n - b)^+$

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What is ℓ^b in the Lévy case?
SA-Pihlsgård 2007

$$\ell^b = \frac{1}{2b}\left\{2\mu\,\mathbb{E}_{\pi^b} V^b + \sigma^2 + \int_0^b \pi(\mathrm{d}x)\int_{-\infty}^{\infty} \varphi(x,y)\nu(\mathrm{d}y)\right\},$$

where

$$\varphi(x,y) = \begin{cases} -(x^2+2xy) & \text{if } y+x \le 0, \\ y^2 & \text{if } 0 < x+y < b, \\ 2y(b-x) - (b-x)^2 & \text{if } x+y \ge b. \end{cases}$$

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Intuition ??

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Remark on conservation laws

One-sided reflection:

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Stationary expectations at $t = 1$ (requires $\mu < 0$):

$\overline{v}^\infty = \overline{v}^\infty + \mu + \ell_0$

$\overline{v}^\infty = \mathbb{E}_{\pi^\infty} V^\infty(1)$, $\mu = \mathbb{E}X(1)$, $\ell_0 = \mathbb{E}_{\pi^\infty} L(1)$

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$V^b(t) = V^b(0) + X(t) + L(t) - U(t)$

$\boxed{\ell_0 - \ell^b = -\mu}$

Kella-Whitt martingale

Brownian motion μ, σ^2

$\kappa(\alpha) = \alpha\mu + \alpha^2\sigma^2/2 = \log \mathbb{E}\mathrm{e}^{\alpha X(1)}$

$$M(t) = \kappa(\alpha)\int_0^t \mathrm{e}^{\alpha V^b(s)}\mathrm{d}s + \mathrm{e}^{\alpha V^b(0)} - \mathrm{e}^{\alpha V^b(t)} + \alpha L(t) - \alpha \mathrm{e}^{\alpha b} U(t)$$

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$\alpha = \gamma = -2\mu/\sigma^2$ (Cramér-Lundberg root), stationary expectations

$0 = 0 + c - c + \gamma\ell_0 - \gamma\mathrm{e}^{\gamma b}$; $\ell_0 - \mathrm{e}^{\gamma b}\ell^b = 0$

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Brownian motion μ, σ^2

$\kappa(\alpha) = \alpha\mu + \alpha^2\sigma^2/2 = \log \mathbb{E}\mathrm{e}^{\alpha X(1)}$

$$M(t) = \kappa(\alpha)\int_0^t \mathrm{e}^{\alpha V^b(s)}\mathrm{d}s + \mathrm{e}^{\alpha V^b(0)} - \mathrm{e}^{\alpha V^b(t)} + \alpha L(t) - \alpha \mathrm{e}^{\alpha b} U(t)$$

$\alpha = \gamma = -2\mu/\sigma^2$ (Cramér-Lundberg root), stationary expectations

$0 = 0 + c - c + \gamma\ell_0 - \gamma \mathrm{e}^{\gamma b}$; $\boxed{\ell_0 - \mathrm{e}^{\gamma b}\ell^b = 0}$

Conservation law: $\boxed{\ell_0 - \ell^b = -\mu}$

General Lévy process: $V = X + B$, B adapted with bd variation

$M(t) = \int_0^t \mathrm{e}^{\alpha V(s-)}\,\mathrm{d}W(s)$, $\quad W(t) = \mathrm{e}^{\alpha X(t) - t\kappa(\alpha)}$ Wald MG.
After rewriting (Itô), the K-W MG $M(t)$ becomes

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Loss rate: $B = L - U$. Take $\alpha = \gamma$ where $\kappa(\gamma) = 0$:

$\boxed{\ell^b = \dfrac{1}{\mathrm{e}^{\gamma b} - 1}\left\{\mathrm{e}^{\gamma b} I_1 + I_2 - \mu\right\}}$ where

$$I_1 = \int_0^b \pi(\mathrm{d}x)\int_{b-x}^{\infty}\big((y - b + x) + \gamma^{-1}(1 - \mathrm{e}^{\gamma(y-b+x)})\big)\nu(\mathrm{d}y)$$

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Requires light tails. Instead, $\alpha \to 0$ and work!

Itô approach to ℓ^b

Glynn & Pihlsgård

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Glynn & Pihlsgård

If X semimartingale:

$$
\begin{aligned}
U(t) = (2b)^{-1}\{V(0)^2 - V(t)^2 + \int_{0+}^{t} 2V(s-)\mathrm{d}X(s) + [X,X]^c(t) \\
+ \sum_{0<s\leq t} \{-2V(s-)\min(\Delta X(s) + V(s-), 0) \\
+ 2(b - V(s-))\max(\Delta X(s) - b + V(s-), 0) \\
+ (\max(\min(\Delta X(s), b - V(s-)), 0) \\
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$[X,X]^c$ the continuous part of the quadratic variation.

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ℓ^b then easy

Light tailed asymptotics

(exponential moments exist)

Traditional one sided asymptotics

Cramér-Lundberg (Bertoin-Doney):

$\mathbb{P}(V^\infty > x) \sim C_1 \mathrm{e}^{-\gamma x}$ as $x \to \infty$ in stationarity

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Pihlsgård: $\ell^b \sim C_2 \mathrm{e}^{-\gamma b}$ where

$$C = -\mathbb{E}X(1) + \mathbb{E}_\gamma \mathrm{e}^{-\gamma B(\infty)} \int_0^\infty \mathrm{e}^{\gamma x} \mathbb{P}_\gamma(\tau_-(-x) = \infty) \int_x^\infty (1 - \mathrm{e}^{\gamma(y-x)}) \nu(\mathrm{d}y)\mathrm{d}x$$

$$+ \int_{-\infty}^0 (y + \gamma^{-1}(1 - \mathrm{e}^{\gamma y}))\nu(\mathrm{d}y) + \int_0^\infty \mathbb{P}(\tau_+^w(x) < \infty) \int_{-\infty}^{-x} (1 - \mathrm{e}^{\gamma(x+y)})\nu(\mathrm{d}y)\mathrm{d}x.$$

Not easier in discrete time

Heavy tailed asymptotics

$\mathbb{E}\mathrm{e}^{\alpha X(1)} = \infty$ for all $\alpha > 0$.
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Finite buffer GI/G/1 waiting times: Jelenkovic

Drift zero

$\mu = \mathbb{E}X(1) = \kappa'(0) = 0$
SA-Lars Nørvang Andersen

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Infinite variance

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Finite variance, $\mathbb{V}\mathrm{ar}\big(X(1)\big) = \kappa''(0) < \infty$

Functional CLT: $\left\{\frac{1}{b}\big(X(tb^2)\big)\right\}_{t\geq 0} \to$ BM.

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Functional LT: X properly scaled $\to$ stable process
under regular variation conditions.

Guess: $\ell^b \sim$ loss rate for the stable process scaled back

Correct

Assume that for some $1 < \alpha < 2$, there exist slowly varying functions $L_1(x)$ and $L_2(x)$ such that

$$\overline{\nu}(x) = x^{-\alpha} L_1(x), \quad \nu(-x) = x^{-\alpha} L_2(x) \qquad \lim_{x\to\infty} \frac{L_1(x)}{L(x)} = \frac{\beta+1}{2}$$

where $L = L_1 + L_2$, $\nu(x) = \nu(-\infty, x]$, $\overline{\nu}(x) := \nu([x, \infty))$.

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$\rho = 1/2 + (\pi\alpha)^{-1} \arctan(\beta \tan(\pi\alpha/2))$, $d = (\beta+1)/2$, $c = (1-\beta)/2$,

$$\eta = \frac{cB(2-\alpha\rho, \alpha\rho) + dB(2-\alpha(1-\rho), \alpha(1-\rho))}{B(\alpha\rho, \alpha(1-\rho))(\alpha-1)(2-\alpha)}$$

($B(x,y) = \Gamma(x)\Gamma(y)/\Gamma(x+y)$ the Beta function)

Proofs

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Uniform integrability, continuity of π^b, ℓ^b (scaled)

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Uniform integrability, continuity of π^b, ℓ^b (scaled)
ℓ^b needs to be computed for stable process

Extensions

Open problems

Extensions to Markov additive processes

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Finite ergodic background Markov process J
X evolves as $X^{(i)}$ on intervals where $J \equiv i$
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Main application example fBM
Non-Markovian
Stochastic monotonicity properties ??
π^b ?? ℓ^b ??

Thank you !!!