Parameter estimation for subcritical affine processes

Mátyás Barczy, Leif Döring, Zenghu Li and Gyula Pap

University of Debrecen, Universität Zürich, Beijing Normal University, University of Szeged

An affine two-factor model

For $a > 0$, $b, \theta, m \in \mathbb{R}$ and $\alpha \in (1, 2]$, let us consider the SDE

$$\begin{cases} \mathrm{d}Y_t = (a - bY_t)\,\mathrm{d}t + \sqrt[\alpha]{Y_{t-}}\,\mathrm{d}L_t, \\ \mathrm{d}X_t = (m - \vartheta X_t)\,\mathrm{d}t + \sqrt{Y_t}\,\mathrm{d}B_t, \end{cases} \qquad t \geq 0,$$

where $(L_t)_{t\geq 0}$ is a spectrally positive α-stable Lévy process with Lévy measure $C_\alpha z^{-1-\alpha}\mathbb{1}_{\{z>0\}}$ with $C_\alpha := (\alpha\Gamma(-\alpha))^{-1}$ in case $\alpha \in (1, 2)$, a Brownian motion in case $\alpha = 2$, and $(B_t)_{t\geq 0}$ is an independent Brownian motion.

Existence and uniqueness of a strong solution

Let (Y_0, X_0) be $\mathcal{F}_0$-measurable with $\mathbb{P}(Y_0 \geq 0) = 1$. Then there is a unique strong solution $(Y_t, X_t)_{t\geq 0}$ with $\mathbb{P}(Y_t \geq 0 \text{ for all } t \geq 0) = 1$. Moreover, $(Y_t, X_t)_{t\geq 0}$ is an affine process with infinitesimal generator

$$(\mathcal{A}f)(y, x) = (a - by)f_1'(y, x) + (m - \vartheta x)f_2'(y, x) + \frac{1}{2}yf_{2,2}''(y, x) + y\int_0^\infty \Big(f(y + z, x) - f(y, x) - zf_1'(y, x)\Big)\frac{C_\alpha}{z^{1+\alpha}}\,\mathrm{d}z$$

in case of $\alpha \in (1, 2)$, and

$$(\mathcal{A}f)(y, x) = (a - by)f_1'(y, x) + (m - \vartheta x)f_2'(y, x) + \frac{1}{2}y(f_{1,1}''(y, x) + f_{2,2}''(y, x))$$

in case of $\alpha = 2$, where $f \in \mathcal{C}_c^2(\mathbb{R}_+ \times \mathbb{R}, \mathbb{R})$.

Stationarity in the subcritical case

Let $b > 0$, $\vartheta > 0$ and $\mathbb{P}(Y_0 \geq 0) = 1$.

1. *Then $(Y_t, X_t) \xrightarrow{\mathcal{L}} (Y_\infty, X_\infty)$ as $t \to \infty$, and the distribution of (Y_∞, X_∞) is given by*

$$\mathbb{E}\left(\mathrm{e}^{-\lambda_1 Y_\infty + \mathrm{i}\lambda_2 X_\infty}\right) = \exp\left\{-a\int_0^\infty v_s(\lambda_1, \lambda_2)\,\mathrm{d}s + \mathrm{i}\frac{m}{\theta}\lambda_2\right\}$$

for $(\lambda_1, \lambda_2) \in \mathbb{R}_+ \times \mathbb{R}$, where $v_t(\lambda_1, \lambda_2)$, $t \geq 0$, is the unique non-negative solution of

$$\begin{cases} \frac{\partial v_t}{\partial t}(\lambda_1, \lambda_2) = -bv_t(\lambda_1, \lambda_2) - \frac{1}{\alpha}(v_t(\lambda_1, \lambda_2))^\alpha + \frac{1}{2}\mathrm{e}^{-2\vartheta t}\lambda_2^2, \qquad t \geq 0, \\ v_0(\lambda_1, \lambda_2) = \lambda_1. \end{cases}$$

2. *If $(Y_0, X_0) \stackrel{\mathcal{L}}{=} (Y_\infty, X_\infty)$, then $(Y_t, X_t)_{t\geq 0}$ is strictly stationary.*
3. *If $\alpha = 2$, then (Y_∞, X_∞) is absolutely continuous, and*

$$\mathbb{E}(\mathrm{e}^{-\lambda_1 Y_\infty}) = \left(1 + \frac{\lambda_1}{2b}\right)^{-2a}, \qquad \lambda_1 \in \mathbb{R}_+.$$

Ergodicity in the subcritical case if $\alpha = 2$

If $\alpha = 2$, $b > 0$, $\vartheta > 0$ and $\mathbb{P}(Y_0 \geq 0) = 1$ then

$$\mathbb{P}\left(\lim_{T\to\infty}\frac{1}{T}\int_0^T f(Y_s, X_s)\,\mathrm{d}s = \mathbb{E}\,f(Y_\infty, X_\infty)\right) = 1$$

for all Borel measurable functions $f : \mathbb{R}^2 \to \mathbb{R}$ with $\mathbb{E}\,|f(Y_\infty, X_\infty)| < \infty$.

It suffices to check a *Foster–Lyapunov criteria* :

- $(Y_t, X_t)_{t\geq 0}$ is a right process;
- all compact sets are petite for some skeleton chain;
- there exist $c > 0$ and $d \in \mathbb{R}$ such that

$$(\mathcal{A}_n V)(y, x) \leq -cV(y, x) + d, \qquad (y, x) \in O_n, \quad n \in \mathbb{N},$$

where $O_n := \{(y, x) \in \mathbb{R}_+ \times \mathbb{R} : \|(y, x)\| < n\}$ for each $n \in \mathbb{N}$,

$$V(y, x) := (y - c_1)^2 + (x - c_2)^2, \qquad (y, x) \in \mathbb{R}_+ \times \mathbb{R},$$

with some appropriate $c_1, c_2 \in \mathbb{R}$, and $\mathcal{A}_n$ denotes the extended generator of the stopped process $(Y_t^{(n)}, X_t^{(n)})_{t\geq 0}$ given by

$$(Y_t^{(n)}, X_t^{(n)}) := \begin{cases} (Y_t, X_t), & \text{for } t < \tau_n, \\ (0, n), & \text{for } t \geq \tau_n, \end{cases}$$

where the stopping time τ_n is defined by

$$\tau_n := \inf\{t \in \mathbb{R}_+ : (Y_t, X_t) \in (\mathbb{R}_+ \times \mathbb{R}) \setminus O_n\}.$$

Existence and uniqueness of MLE of (ϑ, m) if $\alpha = 2$

If $\alpha = 2$, $a \geq \frac{1}{2}$, $\mathbb{P}(Y_0 > 0) = 1$ and $T > 0$, then there exists a unique MLE of (ϑ, m) based on observations $(Y_t, X_t)_{t\in[0,T]}$, and it is given by

$$\begin{bmatrix} \widetilde{\vartheta}_T \\ \widetilde{m}_T \end{bmatrix} = \begin{bmatrix} \int_0^T \frac{X_s^2}{Y_s}\,\mathrm{d}s & -\int_0^T \frac{X_s}{Y_s}\,\mathrm{d}s \\ -\int_0^T \frac{X_s}{Y_s}\,\mathrm{d}s & \int_0^T \frac{1}{Y_s}\,\mathrm{d}s \end{bmatrix}^{-1} \begin{bmatrix} -\int_0^T \frac{X_s}{Y_s}\,\mathrm{d}X_s \\ \int_0^T \frac{1}{Y_s}\,\mathrm{d}X_s \end{bmatrix}, \qquad T > 0,$$

whenever $\int_0^T \frac{X_s^2}{Y_s}\,\mathrm{d}s \int_0^T \frac{1}{Y_s}\,\mathrm{d}s - \left(\int_0^T \frac{X_s}{Y_s}\,\mathrm{d}s\right)^2 > 0$.

Consistency of MLE in the subcritical case if $\alpha = 2$

If $\alpha = 2$, $a \geq \frac{1}{2}$, $b > 0$, $\vartheta > 0$ and $\mathbb{P}(Y_0 > 0) = 1$, then

$$\mathbb{P}\left(\lim_{T\to\infty}(\widetilde{\vartheta}_T, \widetilde{m}_T) = (\vartheta, m)\right) = 1.$$

Asymptotics of MLE in the subcritical case if $\alpha = 2$

If $\alpha = 2$, $a \geq \frac{1}{2}$, $b > 0$, $\vartheta > 0$ and $\mathbb{P}(Y_0 > 0) = 1$, then

$$\begin{bmatrix} \sqrt{T}(\widetilde{\vartheta}_T - \vartheta) \\ \sqrt{T}(\widetilde{m}_T - m) \end{bmatrix} \xrightarrow{\mathcal{L}} \mathcal{N}_2(0, \Sigma_{\mathrm{MLE}}) \qquad \text{as } T \to \infty,$$

with

$$\Sigma_{\mathrm{MLE}} := \frac{1}{\mathbb{E}\left(\frac{1}{Y_\infty}\right)\mathbb{E}\left(\frac{X_\infty^2}{Y_\infty}\right) - \left(\mathbb{E}\left(\frac{X_\infty}{Y_\infty}\right)\right)^2} \begin{bmatrix} \mathbb{E}\left(\frac{1}{Y_\infty}\right) & \mathbb{E}\left(\frac{X_\infty}{Y_\infty}\right) \\ \mathbb{E}\left(\frac{X_\infty}{Y_\infty}\right) & \mathbb{E}\left(\frac{X_\infty^2}{Y_\infty}\right) \end{bmatrix}.$$

LSE of (ϑ, m) based on discrete observations

LSE of (ϑ, m) based on observations $X_0, X_1, \ldots, X_n$, $n \in \mathbb{N}$, is defined as

$$\operatorname*{arg\,min}_{(\vartheta,m)\in\mathbb{R}^2} \sum_{i=1}^n (X_i - X_{i-1} - (m - \vartheta X_{i-1}))^2, \qquad n \in \mathbb{N},$$

and it is given by

$$\begin{bmatrix} \widehat{\vartheta}_T \\ \widehat{m}_T \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^n X_{i-1}^2 & -\sum_{i=1}^n X_{i-1} \\ -\sum_{i=1}^n X_{i-1} & n \end{bmatrix}^{-1} \begin{bmatrix} -\sum_{i=1}^n (X_i - X_{i-1})X_{i-1} \\ X_n - X_0 \end{bmatrix}, \qquad n \in \mathbb{N},$$

whenever $n\sum_{i=1}^n X_{i-1}^2 - (\sum_{i=1}^n X_{i-1})^2 > 0$.

LSE of (ϑ, m) based on continuous observations

LSE of (ϑ, m) based on observations $(X_t)_{t\in[0,T]}$, $T > 0$, is defined by

$$\begin{bmatrix} \widehat{\vartheta}_T \\ \widehat{m}_T \end{bmatrix} = \begin{bmatrix} \int_0^T X_s^2\,\mathrm{d}s & -\int_0^T X_s\,\mathrm{d}s \\ -\int_0^T X_s\,\mathrm{d}s & T \end{bmatrix}^{-1} \begin{bmatrix} -\int_0^T X_s\,\mathrm{d}X_s \\ X_T - X_0 \end{bmatrix}, \qquad T > 0,$$

whenever $T\int_0^T X_s^2 - \left(\int_0^T X_s\,\mathrm{d}s\right)^2 > 0$, which holds almost surely.

Consistency of LSE in the subcritical case if $\alpha = 2$

If $\alpha = 2$, $b > 0$, $\vartheta > 0$ and $\mathbb{P}(Y_0 > 0) = 1$, then

$$\mathbb{P}\left(\lim_{T\to\infty}(\widehat{\vartheta}_T, \widehat{m}_T) = (\vartheta, m)\right) = 1.$$

Asymptotics of LSE in the subcritical case if $\alpha = 2$

If $\alpha = 2$, $b > 0$, $\vartheta > 0$ and $\mathbb{P}(Y_0 > 0) = 1$, then

$$\begin{bmatrix} \sqrt{T}(\widehat{\vartheta}_T - \vartheta) \\ \sqrt{T}(\widehat{m}_T - m) \end{bmatrix} \xrightarrow{\mathcal{L}} \mathcal{N}_2(0, \Sigma_{\mathrm{LSE}}) \qquad \text{as } T \to \infty,$$

with some explicitly given Σ_{LSE}.

References

[1] M. Barczy, L. Döring, Z. Li and G. Pap (2013). Ergodicity for an affine two factor model. ArXiv: `http://arxiv.org/abs/1302.2534`

[2] M. Barczy, L. Döring, Z. Li and G. Pap (2013). Parameter estimation for an affine two factor model. ArXiv: `http://arxiv.org/abs/1302.3451`