

Gradient estimates of harmonic functions for Lévy processes

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T. Kulczycki, M. Ryznar "*Gradient estimates of harmonic functions and transition densities for Lévy processes*"

$\{X_t\}_{t \geq 0}$ - a Lévy process in \mathbb{R}^d .

$$E^0 e^{i\langle \xi, X_t \rangle} = e^{-t\psi(\xi)}, \quad t \geq 0, \xi \in \mathbb{R}^d.$$

$$\psi(x) = -i\langle x, \gamma \rangle + \langle x, Ax \rangle - \int_{\mathbb{R}^d} \left(e^{i\langle x, z \rangle} - 1 - i\langle x, z \rangle 1_{|z| < 1} \right) \nu(dz)$$

$\nu(dz)$ - a Lévy measure, $\nu(\{0\}) = 0$, $\int_{\mathbb{R}^d} (1 \wedge |z|^2) \nu(dz) < \infty$.

$D \subset \mathbb{R}^d$ an open set, $\tau_D = \inf\{t \geq 0 : X_t \notin D\}$ - the first exit time from D .

Definition

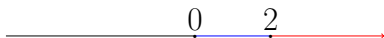
A Borel function $f : \mathbb{R}^d \rightarrow [0, \infty)$ is said to be *harmonic* with respect to $\{X_t\}$ in an open set $D \subset \mathbb{R}^d$ if for any open, bounded set B such that $\overline{B} \subset D$

$$f(x) = E^x f(X(\tau_B)), \quad x \in B.$$

Example

$\{X_t\}_{t \geq 0}$ - the symmetric α -stable process in \mathbb{R} , $\alpha \in (0, 2)$.

$$E^0 e^{i\langle \xi, X_t \rangle} = e^{-t|\xi|^\alpha}, \quad t \geq 0, \xi \in \mathbb{R} \quad \psi(\xi) = |\xi|^\alpha.$$



$D = (0, 2)$, $f(x) = 0$ for $x \in (-\infty, 0]$, $f(x) = 1$ for $x \in [2, \infty)$,
 $f(x) = E^x [f(X(\tau_D))] = P^x [X(\tau_D) \in (2, \infty)]$, $x \in (0, 2)$.

For $x \in D$ we have

$$f(x) = \int_{D^c} P_D(x, z) f(z) dz = c_\alpha \int_2^\infty \frac{(1 - (x - 1)^2)^{\alpha/2}}{((z - 1)^2 - 1)^{\alpha/2} |x - z|} dz.$$

For $x \in (0, 1)$ we have

$$c_1 x^{\alpha/2} \leq f(x) \leq c_2 x^{\alpha/2},$$
$$c_3 x^{\alpha/2-1} \leq (-f'(x)) \leq c_4 x^{\alpha/2-1}, \quad -f'(x) \approx \frac{f(x)}{x}.$$

Theorem 1

Let $\{X_t\}$ be a Lévy process in \mathbb{R}^d satisfying assumptions A, (formulated below). Let $D \subset \mathbb{R}^d$ be an open, nonempty set and let $f : \mathbb{R}^d \rightarrow [0, \infty)$ be a function which is harmonic with respect to $\{X_t\}$ in D . Then $\nabla f(x)$ exists for any $x \in D$ and we have

$$|\nabla f(x)| \leq c \frac{f(x)}{\delta_D(x) \wedge 1}, \quad x \in D,$$

where $\delta_D(x) = \text{dist}(x, \partial D)$ and c is a constant depending only on the process $\{X_t\}$.

Remark. When X_t is the Brownian motion or the symmetric α -stable process the result is well known.

Assumptions A

$\{X_t\}_{t \geq 0}$ - a Lévy process in \mathbb{R}^d .
 $E^0 e^{i\langle \xi, X_t \rangle} = e^{-t\psi(\xi)}, \quad t \geq 0, \xi \in \mathbb{R}^d.$

$$\psi(x) = -i\langle x, \gamma \rangle + \langle x, Ax \rangle - \int_{\mathbb{R}^d} \left(e^{i\langle x, z \rangle} - 1 - i\langle x, z \rangle 1_{|z| < 1} \right) \nu(dz)$$

We assume: $\gamma = 0, A = 0, \nu(dz) = \nu(z) dz = \nu(|z|) dz,$
 $\nu(\mathbb{R}^d) = \infty$ ($\{X_t\}$ is a pure jump, isotropic Lévy process with an infinite Lévy measure).

We further assume:

(A1) Scaling properties of ψ . There exist $\alpha_1 > 0, \alpha_2 \in (0, 2),$
 $\theta_0 \geq 0, c_1 > 0, c_2 > 0$

$$c_1 \lambda^{\alpha_1} \psi(\theta) \leq \psi(\lambda\theta) \leq c_2 \lambda^{\alpha_2} \psi(\theta), \quad \lambda \geq 1, \theta \geq \theta_0.$$

(A2) $\nu(r) \in C^1(0, \infty)$ is decreasing, convex and satisfies

$$a\nu(r) \leq \nu(r+1), \quad r \geq 1.$$

Examples

1. $\{X_t\}$ - the relativistic process in \mathbb{R}^d with a generator $m - \sqrt{m^2 - \Delta}$, $m > 0$. $\{X_t\}$ satisfies assumptions A.
2. $X_t = B_{S_t}$ - a subordinated Brownian motion in \mathbb{R}^d . $\{B_t\}$ the Brownian motion in \mathbb{R}^d (with a generator Δ), $\{S_t\}$ - an independent subordinator (a non-decreasing Lévy process starting from 0.) The Laplace transform of S_t is of the form

$$Ee^{-\lambda S_t} = e^{-t\phi(\lambda)}, \quad \lambda \geq 0, t \geq 0.$$

($\psi(\xi) = \phi(|\xi|^2)$, $E^0 e^{i\langle \xi, X_t \rangle} = e^{-t\psi(\xi)}$.)

We assume that the Levy measure of $\{S_t\}$ is infinite, ϕ is a complete Bernstein function and it satisfies

$$c_1 \lambda^{\alpha/2} \ell(\lambda) \leq \phi(\lambda) \leq c_2 \lambda^{\alpha/2} \ell(\lambda), \quad \lambda \geq 1,$$

where $0 < \alpha < 2$, ℓ varies slowly at infinity, i.e. $\forall x > 0$

$$\lim_{\lambda \rightarrow \infty} \frac{\ell(\lambda x)}{\ell(\lambda)} = 1.$$

$\{X_t\}$ satisfies assumptions A.

3. Let $\{X_t\}$ be the pure jump isotropic Lévy process in \mathbb{R}^d with the Lévy measure $\nu(dx) = \nu(|x|) dx$ given by the formula

$$\nu(r) = \begin{cases} \mathcal{A}_{d,\alpha} r^{-d-\alpha} & \text{for } r \in [0, 1] \\ c_1 e^{-c_2 r} & \text{for } r \in (1, \infty) \end{cases}$$

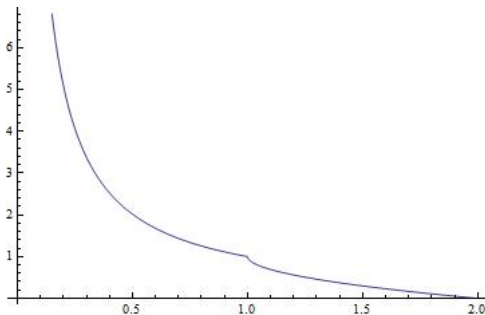
where $\mathcal{A}_{d,\alpha} r^{-d-\alpha}$ is the density of the Lévy measure for the symmetric α -stable process in \mathbb{R}^d , $\alpha \in (0, 2)$ and $c_1 > 0$, $c_2 > 0$ are chosen so that $\nu(r) \in C^1(0, \infty)$. ($c_2 = d + \alpha$, $c_1 = \mathcal{A}_{d,\alpha} e^{d+\alpha}$). $\{X_t\}$ satisfies assumptions A.

Counterexample

Let $\{X_t\}$ be the pure jump isotropic Lévy process in \mathbb{R} with the Lévy measure $\nu(dx) = \nu(|x|) dx$ given by the formula

$$\nu(r) = \begin{cases} \mathcal{A}_\alpha r^{-1-\alpha} & \text{for } r \in (0, 1], \\ \mathcal{A}_\alpha(1 - (r - 1)^\gamma) & \text{for } r \in (1, 2], \\ 0 & \text{for } r \in (2, \infty), \end{cases}$$

where $\alpha \in (0, 1/2)$, $\gamma \in (1/2, 1)$, $\alpha + \gamma < 1$, $\mathcal{A}_\alpha|x|^{-1-\alpha}$ is the density of the Lévy measure for symmetric α -stable process.

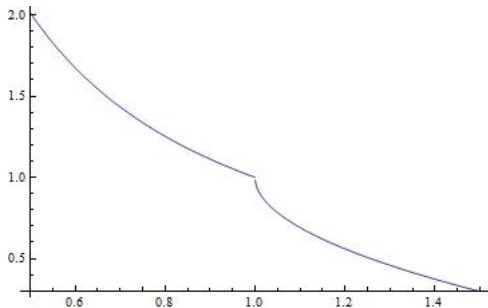


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There exists a nonnegative function f which is harmonic in $(-1/2, 1/2)$ with respect to $\{X_t\}$ for which $f'(0)$ does not exist.

1. Gradient estimates of transition densities for Lévy processes.
2. A concept of "a difference process".
3. Results obtained in papers:

T. Grzywny "*On Harnack inequality and Hölder regularity for isotropic unimodal Lévy processes*" (2013),

K. Bogdan, T. Grzywny, M. Ryznar "*Density and tails of unimodal convolution semigroups*" (2013),

K. Bogdan, T. Grzywny, M. Ryznar "*Barriers, exit time and survival probability for unimodal Lévy processes*" (2013),

P. Kim, A. Mimica "*Harnack Inequalities for Subordinate Brownian Motions*" (2012).

Theorem 2

Let $\{X_t\}$ be a Lévy process in \mathbb{R}^d satisfying assumptions A, with the symbol ψ , the transition density $p_t(x) = p_t(|x|)$ and the density of Lévy measure $\nu(x) = \nu(|x|)$. Then there exists a Lévy process $X_t^{(d+2)}$ in \mathbb{R}^{d+2} with the characteristic exponent ψ and the **radially decreasing transition density** $p_t^{(d+2)}(x) = p_t^{(d+2)}(|x|)$ and the Lévy measure $\nu^{(d+2)}(x) = \nu^{(d+2)}(|x|)$ satisfying

$$p_t^{(d+2)}(r) = \frac{-1}{2\pi r} \frac{d}{dr} p_t(r), \quad \nu^{(d+2)}(r) = \frac{-1}{2\pi r} \frac{d}{dr} \nu(r), \quad r > 0, t > 0.$$

Corollary

If additionally ψ is strictly increasing then for $t \in (0, T]$, $r \in (0, R]$

$$-\frac{d}{dr} p_t(r) \approx \min \left\{ r \left[\psi^{-1} \left(\frac{1}{t} \right) \right]^{d+2}, \frac{t\psi \left(\frac{1}{r} \right)}{r^{d+1}} \right\}.$$

Corollary

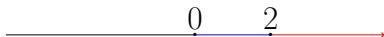
If additionally ψ is strictly increasing then for $r \in (0, R]$

$$-\frac{d}{dr}\nu(r) \approx \frac{\psi\left(\frac{1}{r}\right)}{r^{d+1}}.$$

The idea of the proof

Example. $\{X_t\}_{t \geq 0}$ - the symmetric α -stable process in \mathbb{R} ,
 $\alpha \in (0, 2)$.

$D = (0, 2)$, $f(x) = 0$ for $x \in (-\infty, 0]$, $f(x) = 1$ for $x \in [2, \infty)$,
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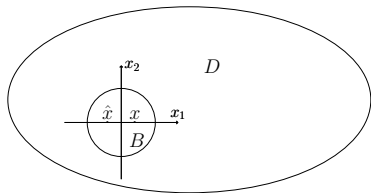


For $x \in D$ we have

$$f(x) = \int_{D^c} P_D(x, z) f(z) dz = c_\alpha \int_{D^c} \frac{(1 - (x - 1)^2)^{\alpha/2}}{((z - 1)^2 - 1)^{\alpha/2} |x - z|} f(z) dz.$$

The idea of the proof

Let $\{X_t\}$ be a Lévy process satisfying assumptions A and $f : \mathbb{R}^d \rightarrow [0, \infty)$ be harmonic in an open set $D \subset \mathbb{R}^d$ with respect to $\{X_t\}$.



$$r = \frac{\delta_D(0)}{2} \wedge r_0, \quad B = B(0, r).$$

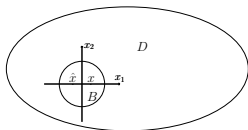
$$h \in (0, r), \quad x = (h, 0, \dots, 0).$$

$$f(x) = E^x f(X(\tau_B)) = \int_{B^c} \left(\int_B G_B(x, y) \nu(y - z) dy \right) f(z) dz.$$

Notation: $\hat{y} = (-y_1, y_2, \dots, y_d)$ for $y = (y_1, y_2, \dots, y_d)$,
 $B_+ = \{(y_1, \dots, y_d) \in B : y_1 > 0\}$.

$$\frac{f(x) - f(\hat{x})}{2h} = \frac{1}{2h} \int_{B_+} (G_B(x, y) - G_B(\hat{x}, y)) \int_{B^c} (\nu(y - z) - \nu(\hat{y} - z)) f(z) dz dy$$

The idea of the proof



$$h \in (0, r), x = (h, 0, \dots, 0).$$

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$$\frac{f(x) - f(\hat{x})}{2h} = \frac{1}{2h} \int_{B_+} (G_B(x, y) - G_B(\hat{x}, y)) \int_{B^c} (\nu(y-z) - \nu(\hat{y}-z)) f(z) dz dy.$$

$$\begin{aligned} G_B(x, y) &= \int_0^\infty (p_t(x-y) - E^x(p_t(X(\tau_B) - y), \tau_B < t)) dt \\ &\leq \int_0^\infty p_t(x, y) dt, \quad x, y \in B. \end{aligned}$$

$$G_B(x, y) - G_B(\hat{x}, y) \leq \int_0^\infty (p_t(x-y) - p_t(\hat{x}-y)) dt.$$

For $t > 0$, $x, y \in \mathbb{R}_+^d$ we define sub-Markov transition densities $\tilde{p}_t(x, y) = p_t(x-y) - p_t(\hat{x}-y)$. There exists a Hunt process $\{\tilde{X}_t\}_{t \geq 0}$ on \mathbb{R}_+^d with transition densities $\tilde{p}_t(x, y)$ (*the difference process*).

$$\frac{f(x) - f(\hat{x})}{2h} = \frac{1}{2h} \int_{B_+} (G_B(x, y) - G_B(\hat{x}, y)) \int_{B^c} (\nu(y-z) - \nu(\hat{y}-z)) f(z) dz dy.$$

$h \in (0, r)$, $x = (h, 0, \dots, 0)$, $|y| > 2h$.

$$\begin{aligned} |G_B(x, y) - G_B(\hat{x}, y)| &\leq \left| \int_0^\infty (p_t(x-y) - p_t(\hat{x}-y)) dt \right| \\ &= |x - \hat{x}| \left| \int_0^\infty \frac{\partial}{\partial x_1} p_t(\xi - y) dt \right| \\ &= 2\pi |x - \hat{x}| \left| \int_0^\infty (\xi - y) p_t^{(d+2)}(\xi - y) dt \right| \\ &\leq ch \int_0^\infty |x - y| p_t^{(d+2)} \left(\frac{|x - y|}{2} \right) dt \\ &\leq \frac{ch}{\psi(|x - y|^{-1}) |x - y|^{d+1}}. \end{aligned}$$

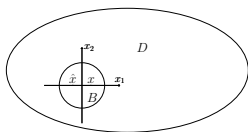
ξ is a point between x and \hat{x} .

$$\frac{f(x) - f(\hat{x})}{2h} = \frac{1}{2h} \int_{B_+} (G_B(x, y) - G_B(\hat{x}, y)) \int_{B^c} (\nu(y-z) - \nu(\hat{y}-z)) f(z) dz dy.$$

By the scale invariant Harnack inequality we get

$$\begin{aligned} & \left| \int_{B(0,2r) \setminus B(0,r)} (\nu(y-z) - \nu(\hat{y}-z)) f(z) dz \right| \\ & \leq c f(0) \int_{B(0,2r) \setminus B(0,r)} |\nu(|y-z|) - \nu(|\hat{y}-z|)| dz \\ & \leq c f(0) |y - \hat{y}| \int_{B(0,2r) \setminus B(0,r)} |\nu'(\xi)| dz \\ & \leq \frac{c f(0) |y| \psi(r^{-1})}{r}. \end{aligned}$$

ξ is a point between $|y-z|$ and $|\hat{y}-z|$



$h \in (0, r)$, $x = (h, 0, \dots, 0)$.

$\hat{y} = (-y_1, y_2, \dots, y_d)$ for

$y = (y_1, y_2, \dots, y_d)$.

$r = \frac{\delta_D(0)}{2} \wedge r_0$.

$$\begin{aligned} & \frac{f(x) - f(\hat{x})}{2h} \\ = & \frac{1}{2h} \int_{B_+} (G_B(x, y) - G_B(\hat{x}, y)) \int_{B^c} (\nu(y - z) - \nu(\hat{y} - z)) f(z) dz dy \\ \leq & \frac{cf(0)}{r}. \end{aligned}$$