Gradient estimates of harmonic functions for Lévy processes

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T. Kulczycki, M. Ryznar "*Gradient estimates of harmonic functions and transition densities for Lévy processes*"

$\{X_t\}_{t \geqslant 0}$ - a Lévy process in $\mathbb{R}^d$.

$$E^0 e^{i\langle \xi, X_t \rangle} = e^{-t\psi(\xi)}, \quad t \geqslant 0,\ \xi \in \mathbb{R}^d.$$

$$\psi(x) = -i\langle x, \gamma \rangle + \langle x, Ax \rangle - \int_{\mathbb{R}^d} \left(e^{i\langle x,z \rangle} - 1 - i\langle x, z \rangle 1_{|z|<1} \right) \nu(dz)$$

$\nu(dz)$ - a Lévy measure, $\nu(\{0\}) = 0$, $\int_{\mathbb{R}^d} (1 \wedge |z|^2)\, \nu(dz) < \infty$.
$D \subset \mathbb{R}^d$ an open set, $\tau_D = \inf\{t \geqslant 0 : X_t \notin D\}$ - the first exit time from D.

Definition

A Borel function $f : \mathbb{R}^d \to [0, \infty)$ is said to be *harmonic* with respect to $\{X_t\}$ in an open set $D \subset \mathbb{R}^d$ if for any open, bounded set B such that $\overline{B} \subset D$

$$f(x) = E^x f(X(\tau_B)), \qquad x \in B.$$

Example

$\{X_t\}_{t\geqslant 0}$ - the symmetric α-stable process in $\mathbb{R}$, $\alpha \in (0,2)$.

$$E^0 e^{i\langle \xi, X_t\rangle} = e^{-t|\xi|^\alpha}, \quad t \geqslant 0,\ \xi \in \mathbb{R} \qquad \psi(\xi) = |\xi|^\alpha.$$

$D = (0,2)$, $f(x) = 0$ for $x \in (-\infty, 0]$, $f(x) = 1$ for $x \in [2,\infty)$,
$f(x) = E^x[f(X(\tau_D))] = P^x[X(\tau_D) \in (2,\infty)]$, $x \in (0,2)$.
For $x \in D$ we have

$$f(x) = \int_{D^c} P_D(x,z) f(z)\, dz = c_\alpha \int_2^\infty \frac{(1-(x-1)^2)^{\alpha/2}}{((z-1)^2-1)^{\alpha/2}|x-z|}\, dz.$$

For $x \in (0,1)$ we have

$$\begin{aligned} c_1 x^{\alpha/2} &\leqslant f(x) \leqslant c_2 x^{\alpha/2}, \\ c_3 x^{\alpha/2-1} &\leqslant (-f'(x)) \leqslant c_4 x^{\alpha/2-1}, \qquad -f'(x) \approx \frac{f(x)}{x}. \end{aligned}$$

Main result

Theorem 1

Let $\{X_t\}$ be a Lévy process in $\mathbb{R}^d$ satisfying assumptions A, (formulated below). Let $D \subset \mathbb{R}^d$ be an open, nonempty set and let $f : \mathbb{R}^d \to [0, \infty)$ be a function which is harmonic with respect to $\{X_t\}$ in D. Then $\nabla f(x)$ exists for any $x \in D$ and we have

$$|\nabla f(x)| \leqslant c \frac{f(x)}{\delta_D(x) \wedge 1}, \qquad x \in D,$$

where $\delta_D(x) = \text{dist}(x, \partial D)$ and c is a constant depending only on the process $\{X_t\}$.

Remark. When X_t is the Brownian motion or the symmetric α-stable process the result is well known.

Assumptions A

$\{X_t\}_{t\geqslant 0}$ - a Lévy process in $\mathbb{R}^d$.
$E^0 e^{i\langle \xi, X_t\rangle} = e^{-t\psi(\xi)}, \quad t \geqslant 0,\ \xi \in \mathbb{R}^d$.

$$\psi(x) = -i\langle x, \gamma\rangle + \langle x, Ax\rangle - \int_{\mathbb{R}^d} \left(e^{i\langle x,z\rangle} - 1 - i\langle x, z\rangle 1_{|z|<1}\right) \nu(dz)$$

We assume: $\gamma = 0$, $A = 0$, $\nu(dz) = \nu(z)\,dz = \nu(|z|)\,dz$, $\nu(\mathbb{R}^d) = \infty$ ($\{X_t\}$ is a pure jump, isotropic Lévy process with an infinite Lévy measure).
We further assume:
(A1) Scaling properties of ψ. There exist $\alpha_1 > 0$, $\alpha_2 \in (0, 2)$, $\theta_0 \geqslant 0$, $c_1 > 0$, $c_2 > 0$

$$c_1 \lambda^{\alpha_1} \psi(\theta) \leqslant \psi(\lambda\theta) \leqslant c_2 \lambda^{\alpha_2} \psi(\theta), \qquad \lambda \geqslant 1,\ \theta \geqslant \theta_0.$$

(A2) $\nu(r) \in C^1(0, \infty)$ is decreasing, convex and satisfies

$$a\nu(r) \leqslant \nu(r + 1), \qquad r \geqslant 1.$$

Examples

1. $\{X_t\}$ - the relativistic process in $\mathbb{R}^d$ with a generator $m - \sqrt{m^2 - \Delta}$, $m > 0$. $\{X_t\}$ satisfies assumptions A.
2. $X_t = B_{S_t}$ - a subordinated Brownian motion in $\mathbb{R}^d$. $\{B_t\}$ the Brownian motion in $\mathbb{R}^d$ (with a generator Δ), $\{S_t\}$ - an independent subordinator (a non-decreasing Lévy process starting from 0.) The Laplace transform of S_t is of the form

$$Ee^{-\lambda S_t} = e^{-t\phi(\lambda)}, \qquad \lambda \geqslant 0,\ t \geqslant 0.$$

$(\psi(\xi) = \phi(|\xi|^2)$, $E^0 e^{i\langle \xi, X_t \rangle} = e^{-t\psi(\xi)}$.)
We assume that the Levy measure of $\{S_t\}$ is infinite, ϕ is a complete Bernstein function and it satisfies

$$c_1 \lambda^{\alpha/2} \ell(\lambda) \leqslant \phi(\lambda) \leqslant c_2 \lambda^{\alpha/2} \ell(\lambda), \qquad \lambda \geqslant 1,$$

where $0 < \alpha < 2$, ℓ varies slowly at infinity, i.e. $\forall x > 0$ $\lim_{\lambda \to \infty} \frac{\ell(\lambda x)}{\ell(\lambda)} = 1$.
$\{X_t\}$ satisfies assumptions A.

Examples

3. Let $\{X_t\}$ be the pure jump isotropic Lévy process in $\mathbb{R}^d$ with the Lévy measure $\nu(dx) = \nu(|x|)\,dx$ given by the formula

$$\nu(r) = \begin{cases} \mathcal{A}_{d,\alpha} r^{-d-\alpha} & \text{for} \quad r \in [0,1] \\ c_1 e^{-c_2 r} & \text{for} \quad r \in (1,\infty) \end{cases}$$

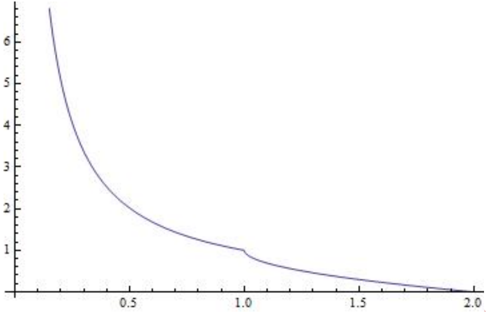
where $\mathcal{A}_{d,\alpha} r^{-d-\alpha}$ is the density of the Lévy measure for the symmetric α-stable process in $\mathbb{R}^d$, $\alpha \in (0,2)$ and $c_1 > 0$, $c_2 > 0$ are chosen so that $\nu(r) \in C^1(0,\infty)$. ($c_2 = d + \alpha$, $c_1 = \mathcal{A}_{d,\alpha} e^{d+\alpha}$). $\{X_t\}$ satisfies assumptions A.

Counterexample

Let $\{X_t\}$ be the pure jump isotropic Lévy process in $\mathbb{R}$ with the Lévy measure $\nu(dx) = \nu(|x|)\,dx$ given by the formula

$$\nu(r) = \begin{cases} \mathcal{A}_\alpha r^{-1-\alpha} & \text{for} \quad r \in (0,1], \\ \mathcal{A}_\alpha(1-(r-1)^\gamma) & \text{for} \quad r \in (1,2], \\ 0 & \text{for} \quad r \in (2,\infty), \end{cases}$$

where $\alpha \in (0, 1/2)$, $\gamma \in (1/2, 1)$, $\alpha + \gamma < 1$, $\mathcal{A}_\alpha |x|^{-1-\alpha}$ is the density of the Lévy measure for symmetric α-stable process.



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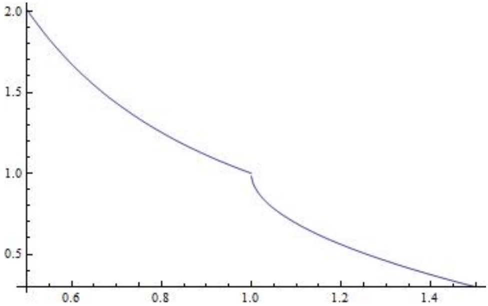


There exists a nonnegative function f which is harmonic in $(-1/2,1/2)$ with respect to $\{X_t\}$ for which $f'(0)$ does not exist.

Ideas used in the proof

1. Gradient estimates of transition densities for Lévy processes.

2. A concept of "a difference process".

3. Results obtained in papers:

T. Grzywny *"On Harnack inequality and Hölder regularity for isotropic unimodal Lévy processes"* (2013),

K. Bogdan, T. Grzywny, M. Ryznar *"Density and tails of unimodal convolution semigroups"* (2013),

K. Bogdan, T. Grzywny, M. Ryznar *"Barriers, exit time and survival probability for unimodal Lévy processes"* (2013),

P. Kim, A. Mimica *"Harnack Inequalities for Subordinate Brownian Motions"* (2012).

Theorem 2

Let $\{X_t\}$ be a Lévy process in $\mathbb{R}^d$ satisfying assumptions A, with the symbol ψ, the transition density $p_t(x) = p_t(|x|)$ and the density of Lévy measure $\nu(x) = \nu(|x|)$. Then there exists a Lévy process $X_t^{(d+2)}$ in $\mathbb{R}^{d+2}$ with the characteristic exponent ψ and the **radially decreasing transition density** $p_t^{(d+2)}(x) = p_t^{(d+2)}(|x|)$ and the Lévy measure $\nu^{(d+2)}(x) = \nu^{(d+2)}(|x|)$ satisfying

$$p_t^{(d+2)}(r) = \frac{-1}{2\pi r}\frac{d}{dr}p_t(r), \quad \nu^{(d+2)}(r) = \frac{-1}{2\pi r}\frac{d}{dr}\nu(r), \quad r > 0,\ t > 0.$$

Corollary

If additionally ψ is strictly increasing then for $t \in (0, T]$, $r \in (0, R]$

$$-\frac{d}{dr}p_t(r) \approx \min\left\{ r\left[\psi^{-1}\left(\frac{1}{t}\right)\right]^{d+2}, \frac{t\psi\left(\frac{1}{r}\right)}{r^{d+1}} \right\}.$$

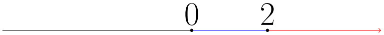
Corollary

If additionally ψ is strictly increasing then for $r \in (0, R]$

$$-\frac{d}{dr}\nu(r) \approx \frac{\psi\left(\frac{1}{r}\right)}{r^{d+1}}.$$

The idea of the proof

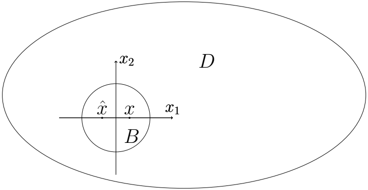
Example. $\{X_t\}_{t\geqslant 0}$ - the symmetric α-stable process in $\mathbb{R}$, $\alpha \in (0,2)$.
$D=(0,2)$, $f(x)=0$ for $x\in(-\infty,0]$, $f(x)=1$ for $x\in[2,\infty)$,
$f(x)=E^x\left[f(X(\tau_D))\right]=P^x[X(\tau_D)\in(2,\infty)]$, $x\in(0,2)$.



For $x \in D$ we have

$$f(x)=\int_{D^c} P_D(x,z)f(z)\,dz = c_\alpha \int_{D^c} \frac{(1-(x-1)^2)^{\alpha/2}}{((z-1)^2-1)^{\alpha/2}|x-z|} f(z)\,dz.$$

Let $\{X_t\}$ be a Lévy process satisfying assumptions A and $f : \mathbb{R}^d \to [0, \infty)$ be harmonic in an open set $D \subset \mathbb{R}^d$ with respect to $\{X_t\}$.



$$r = \frac{\delta_D(0)}{2} \wedge r_0, \quad B = B(0, r).$$

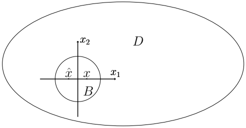
$$h \in (0, r),\ x = (h, 0, \ldots, 0).$$

$$f(x) = E^x f(X(\tau_B)) = \int_{B^c} \left(\int_B G_B(x, y) \nu(y - z)\, dy \right) f(z)\, dz.$$

Notation: $\hat{y} = (-y_1, y_2, \ldots, y_d)$ for $y = (y_1, y_2, \ldots, y_d)$, $B_+ = \{(y_1, \ldots, y_d) \in B :\ y_1 > 0\}$.

$$\frac{f(x) - f(\hat{x})}{2h} = \frac{1}{2h} \int_{B_+} (G_B(x, y) - G_B(\hat{x}, y)) \int_{B^c} (\nu(y - z) - \nu(\hat{y} - z)) f(z) dz dy$$

The idea of the proof



$h \in (0, r)$, $x = (h, 0, \dots, 0)$.
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$y = (y_1, y_2, \dots, y_d)$

$$\frac{f(x) - f(\hat{x})}{2h} = \frac{1}{2h} \int_{B_+} (G_B(x,y) - G_B(\hat{x},y)) \int_{B^c} (\nu(y-z) - \nu(\hat{y}-z)) f(z)\, dz\, dy.$$

$$\begin{aligned} G_B(x,y) &= \int_0^\infty (p_t(x-y) - E^x(p_t(X(\tau_B) - y), \tau_B < t))\, dt \\ &\leqslant \int_0^\infty p_t(x,y)\, dt, \qquad x, y \in B. \end{aligned}$$

$$G_B(x,y) - G_B(\hat{x},y) \leqslant \int_0^\infty (p_t(x-y) - p_t(\hat{x}-y))\, dt.$$

For $t > 0$, $x, y \in \mathbb{R}^d_+$ we define sub-Markov transition densities $\tilde{p}_t(x,y) = p_t(x-y) - p_t(\hat{x}-y)$. There exists a Hunt process $\{\tilde{X}_t\}_{t \geqslant 0}$ on R^d_+ with transition densities $\tilde{p}_t(x,y)$ (*the difference process*).

$$\frac{f(x)-f(\hat{x})}{2h} = \frac{1}{2h}\int_{B_+}(G_B(x,y)-G_B(\hat{x},y))\int_{B^c}(\nu(y-z)-\nu(\hat{y}-z))f(z)dzdy.$$

$h \in (0,r)$, $x=(h,0,\dots,0)$, $|y|>2h$.

$$\begin{aligned}
|G_B(x,y)-G_B(\hat{x},y)| &\leqslant \left|\int_0^\infty (p_t(x-y)-p_t(\hat{x}-y))\,dt\right| \\
&= |x-\hat{x}|\left|\int_0^\infty \frac{\partial}{\partial x_1}p_t(\xi-y)\,dt\right| \\
&= 2\pi|x-\hat{x}|\left|\int_0^\infty (\xi-y)p_t^{(d+2)}(\xi-y)\,dt\right| \\
&\leqslant ch\int_0^\infty |x-y|p_t^{(d+2)}\left(\frac{|x-y|}{2}\right)\,dt \\
&\leqslant \frac{ch}{\psi(|x-y|^{-1})|x-y|^{d+1}}.
\end{aligned}$$

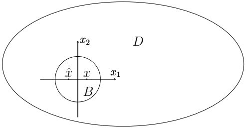
ξ is a point between x and $\hat{x}$.

$$\frac{f(x)-f(\hat{x})}{2h} = \frac{1}{2h}\int_{B_+}(G_B(x,y)-G_B(\hat{x},y))\int_{B^c}(\nu(y-z)-\nu(\hat{y}-z))f(z)dzdy.$$

By the scale invariant Harnack inequality we get

$$\begin{aligned}
&\left|\int_{B(0,2r)\setminus B(0,r)}(\nu(y-z)-\nu(\hat{y}-z))f(z)\,dz\right| \\
\leqslant\quad & cf(0)\int_{B(0,2r)\setminus B(0,r)}|\nu(|y-z|)-\nu(|\hat{y}-z|)|\,dz \\
\leqslant\quad & cf(0)|y-\hat{y}|\int_{B(0,2r)\setminus B(0,r)}|\nu'(\xi)|\,dz \\
\leqslant\quad & \frac{cf(0)|y|\psi(r^{-1})}{r}.
\end{aligned}$$

ξ is a point between $|y-z|$ and $|\hat{y}-z|$



$h \in (0, r)$, $x = (h, 0, \ldots, 0)$.
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$y = (y_1, y_2, \ldots, y_d)$.
$r = \frac{\delta_D(0)}{2} \wedge r_0$.

$$
\begin{aligned}
& \frac{f(x) - f(\hat{x})}{2h} \\
= \; & \frac{1}{2h} \int_{B_+} (G_B(x, y) - G_B(\hat{x}, y)) \int_{B^c} (\nu(y - z) - \nu(\hat{y} - z)) f(z) dz dy \\
\leqslant \; & \frac{cf(0)}{r}.
\end{aligned}
$$