Multivariate Generalized Ornstein-Uhlenbeck Processes

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A particle moving from left to right gets hit by more particles from the right than from the left side which results in a slowdown.
The velocity $v(t)$ of the particle is given by

$$m\,dv(t) = -\lambda v(t)dt + dB(t)$$

i.e.

$$v(t) = e^{-\lambda t/m} v(0) + e^{-\lambda t/m} \int_{(0,t]} e^{\lambda s/m} dB(s).$$

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This solution is called an **Ornstein-Uhlenbeck (OU) process**.
Setting $\lambda = 0$ yields the original formula by Einstein.

OU processes as AR(1) time series

For every $h > 0$ the Ornstein-Uhlenbeck process

$$V_t = e^{-\lambda t} V_0 + e^{-\lambda t} \int_{(0,t]} e^{\lambda s} dB_s, \quad t \geq 0,$$

fulfills the random recurrence equation

$$V_{nh} = e^{-\lambda h} V_{(n-1)h} + e^{-\lambda nh} \int_{((n-1)h, nh]} e^{\lambda s} dB_s, \quad n \in \mathbb{N}.$$

Hence it can be seen as a natural generalization in continuous time of the AR(1) time series

$$X_n = e^{-\lambda} X_{n-1} + Z_n, \quad n \in \mathbb{N},$$

with i.i.d. noise $(Z_n)_{n\in\mathbb{N}}$ such that $\mathcal{L}(Z_1) = \mathcal{L}(\int_{(0,1]} e^{-\lambda(1-s)} dB_s)$.

A more general AR(1) time series

By embedding the more general random sequence

$$Y_n = A_n Y_{n-1} + B_n, \quad n \in \mathbb{N},$$

with $(A_n, B_n)_{n\in\mathbb{N}}$ i.i.d., $A_1 > 0$ a.s., into a continuous time setting in 1989 De Haan and Karandikar introduced the generalized Ornstein-Uhlenbeck process

$$V_t = e^{-\xi_t} \left(V_0 + \int_{(0,t]} e^{\xi_{s-}} \, d\eta_s \right), \quad t \geq 0.$$

driven by a bivariate Lévy process $(\xi_t, \eta_t)_{t\geq 0}$ with starting random variable V_0.

Definition: Lévy processes

A Lévy process in $\mathbb{R}^d$ on a probability space $(\Omega, \mathcal{F}, P)$ is a stochastic process $X = (X_t)_{t \geq 0}$, $X_t : \Omega \to \mathbb{R}^d$ satisfying the following properties:

- $X_0 = 0$ a.s.
- X has independent increments, i.e. for all $0 \leq t_0 \leq t_1 \leq \ldots \leq t_n$ the random variables $X_{t_0}, X_{t_1} - X_{t_0}, \ldots, X_{t_n} - X_{t_{n-1}}$ are independent.
- X has stationary increments, i.e. for all $s, t \geq 0$ it holds $X_{s+t} - X_s \overset{d}{=} X_t$.
- X has a.s. càdlàg paths, i.e. for P-a.e. $\omega \in \Omega$ the path $t \mapsto X_t(\omega)$ is right-continuous in $t \geq 0$ and has left limits in $t > 0$.

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Elementary examples of Lévy processes include linear deterministic processes, Brownian motions as well as compound Poisson processes.

Definition: Generalized OU processes

The **generalized Ornstein-Uhlenbeck (GOU) process** $(V_t)_{t\geq 0}$ driven by the bivariate Lévy process $(\xi_t, \eta_t)_{t\geq 0}$ is given by

$$V_t = e^{-\xi_t} \left(V_0 + \int_{(0,t]} e^{\xi_{s-}} \, d\eta_s \right), \quad t \geq 0,$$

where V_0 is a finite random variable, usually chosen independent of (ξ, η).

In the case that $(\xi_t, \eta_t) = (\lambda t, \eta_t)$ with a Lévy process $(\eta_t)_{t\geq 0}$ and a constant $\lambda \neq 0$ the process $(V_t)_{t\geq 0}$ is called **Lévy-driven Ornstein-Uhlenbeck process** or **Ornstein-Uhlenbeck type process**.
Obviously, if additionally $(\eta_t)_{t\geq 0}$ is a Brownian motion, we get the classical Ornstein-Uhlenbeck process.

The generalized Ornstein-Uhlenbeck process: Applications

Example 1: $\xi_t = t$ deterministic

$$\Rightarrow V_t = e^{-t}\left(V_0 + \int_{(0,t]} e^s \, d\eta_s\right)$$

Lévy driven Ornstein-Uhlenbeck process, classical for $\eta_t = B_t$

- applications in storage theory
- stochastic volatility model of Barndorff-Nielsen and Shephard (2001): η_t subordinator, V_t squared volatility, price process G_t defined by $dG_t = (\mu + bV_t)dt + \sqrt{V_t}dB_t$ for constants μ, b.

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Example 2: $\eta_t = t$ deterministic.
Applications for Asian options, or COGARCH(1,1) model of Klüppelberg, L., Maller (2004).

The corresponding SDE

The generalized Ornstein-Uhlenbeck process driven by (ξ, η)

$$V_t = e^{-\xi_t} \left(V_0 + \int_{(0,t]} e^{\xi_{s-}} d\eta_s \right), t \geq 0$$

is the unique solution of the SDE

$$\boxed{dV_t = V_{t-} dU_t + dL_t, \; t \geq 0}$$

where $(U_t, L_t)_{t \geq 0}$ is a bivariate Lévy process completely determined by (ξ, η).

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In particular we have

$$\xi_t = -\log(\mathcal{E}(U)_t)$$

Definition: The Doléans-Dade Exponential

$$\mathcal{E}(U)_t = \exp\left(U_t - \frac{1}{2} t \sigma_U^2 \right) \prod_{s \leq t} ((1 + \Delta U_s) \exp(-\Delta U_s))$$

is the unique solution of the SDE $dZ_t = Z_{t-} dU_t, \; Z_0 = 1$ a.s.

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where $(U_t, L_t)_{t \geq 0}$ is a bivariate Lévy process completely determined by (ξ, η).
In particular we have

$$\xi_t = -\log(\mathcal{E}(U)_t)$$

and

$$\eta_t = L_t + \sum_{0<s\leq t} \frac{\Delta U_s \Delta L_s}{1 + \Delta U_s} - t\mathrm{Cov}\,(B_{U_1}, B_{L_1}).$$

Multivariate Generalized Ornstein-Uhlenbeck Processes

Recall: An AR(1) time series

The generalized Ornstein-Uhlenbeck process

$$V_t = e^{-\xi_t} \left(V_0 + \int_{(0,t]} e^{\xi_{s-}} \, d\eta_s \right), \quad t \geq 0.$$

driven by a bivariate Lévy process $(\xi_t, \eta_t)_{t \geq 0}$ with starting random variable V_0 had been derived by embedding the AR(1) time series

$$V_n = A_n V_{n-1} + B_n, \quad n \in \mathbb{N},$$

with $(A_n, B_n)_{n \in \mathbb{N}}$ i.i.d., $A_1 > 0$ a.s., into a continuous time setting.

Constructing a multivariate GOU

We aim to embed the random sequence

$$V_n = A_n V_{n-1} + B_n, \quad n \in \mathbb{N},$$

with $(A_n, B_n)_{n\in\mathbb{N}}$ i.i.d., $(A_n, B_n) \in \mathbb{R}^{m\times m} \times \mathbb{R}^m$, A_1 a.s. non-singular, into a continuous time setting.

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More precisely, we want to find all stochastic processes $(V_t)_{t\geq 0}$ such that

$$V_t = A_{s,t} V_s + B_{s,t}, \quad \forall\, 0 \leq s \leq t$$

and

$$(A_{(n-1)h,nh}, B_{(n-1)h,nh})_{n\in\mathbb{N}}$$

is i.i.d. for each $h > 0$.

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is i.i.d. for each $h > 0$.
Assuming slightly more leads to the following requirements:

Assumptions

For each $0 \le s \le t$ let $(A_{s,t}, B_{s,t}) \in \mathrm{GL}(\mathbb{R}, m) \times \mathbb{R}^m$ s.t.:
Assumption (a) For all $0 \le u \le s \le t$ almost surely

$$A_{u,t} = A_{s,t} A_{u,s} \quad \text{and} \quad B_{u,t} = A_{s,t} B_{u,s} + B_{s,t}.$$

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Assumption (b) The families of random matrices $\{(A_{s,t}, B_{s,t}), a \leq s \leq t \leq b\}$ and $\{(A_{s,t}, B_{s,t}), c \leq s \leq t \leq d\}$ are independent for $0 \leq a \leq b \leq c \leq d$.

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Assumption (c) For all $0 \leq s \leq t$ it holds

$$(A_{s,t}, B_{s,t}) \stackrel{d}{=} (A_{0,t-s}, B_{0,t-s}).$$

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Assumption (c) For all $0 \le s \le t$ it holds

$$(A_{s,t}, B_{s,t}) \stackrel{d}{=} (A_{0,t-s}, B_{0,t-s}).$$

Assumption (d) It holds

$$P - \lim_{t \downarrow 0} A_{0,t} = A_{0,0} = I \quad \text{and} \quad P - \lim_{t \downarrow 0} B_{0,t} = B_{0,0} = 0,$$

where I denotes the identity matrix and 0 the vector (or matrix) only having zero entries.

The Process $A_t := A_{0,t}$

Lemma: Every stochastic process $A_t := A_{0,t}$ in the autoregressive model above which fulfills Assumptions (a) to (d) has a version which is a multiplicative right Lévy process in the general linear group $\mathrm{GL}(\mathbb{R}, m)$ of order m.

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That means, $(A_t)_{t \geq 0}$ is a stochastic process with values in $\mathrm{GL}(\mathbb{R}, m)$ with the following properties:

- $A_0 = I$ a.s.
- it has independent left increments, i.e. for all $0 \leq t_1 \leq \ldots \leq t_n$, the random variables $A_0, A_{t_1} A_0^{-1}, \ldots, A_{t_n} A_{t_{n-1}}^{-1}$ are independent.
- it has stationary left increments, i.e. for all $s, t \geq 0$ it holds $A_{s+t} A_s^{-1} \stackrel{d}{=} A_t$.
- it has a.s. càdlàg paths, i.e. for P-a.e. $\omega \in \Omega$ the path $t \mapsto A_t(\omega)$ is right-continuous in $t \geq 0$ and has left limits in $t > 0$.

The Multivariate Stochastic Exponential I

Lemma: By an observation due to Skorokhod every right Lévy process in $(\mathrm{GL}(\mathbb{R}, m), \cdot)$ is the right stochastic exponential of a Lévy process in $(\mathbb{R}^{m\times m}, +)$.

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Definition: Let $(X_t)_{t\geq 0}$ be a semimartingale in $(\mathbb{R}^{m\times m}, +)$. Then its left stochastic exponential $\overleftarrow{\mathcal{E}}(X)_t$ is defined as the unique $\mathbb{R}^{m\times m}$-valued, adapted, càdlàg solution of the SDE

$$Z_t = I + \int_{(0,t]} Z_{s-} dX_s, \quad t \geq 0,$$

while the unique adapted, càdlàg solution of the SDE

$$Z_t = I + \int_{(0,t]} dX_s \, Z_{s-}, \quad t \geq 0,$$

will be called right stochastic exponential and denoted by $\overrightarrow{\mathcal{E}}(X)_t$.

The Multivariate Stochastic Exponential II

Let $(X_t)_{t\geq 0}$ be a semimartingale in $(\mathbb{R}^{m\times m}, +)$. Then we observe:

- A stochastic exponential of X is invertible for all $t \geq 0$ if and only if

$$\boxed{\det(I + \Delta X_t) \neq 0 \text{ for all } t \geq 0. \quad (*)}$$

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$$\boxed{\det(I + \Delta X_t) \neq 0 \text{ for all } t \geq 0. \quad (*)}$$

- Suppose $(X_t)_{t\geq 0}$ fulfills $(*)$. Then for $(U_t)_{t\geq 0}$ given by

$$U_t := -X_t + [X, X]_t^c + \sum_{0<s\leq t} \left((I + \Delta X_s)^{-1} - I + \Delta X_s\right), \ t \geq 0$$

  it holds

$$\boxed{[\overleftarrow{\mathcal{E}}(X)_t]^{-1} = \overrightarrow{\mathcal{E}}(U)_t, \ t \geq 0.}$$

Choice of $A_{s,t}$

We have

- Every stochastic process $A_t := A_{0,t}$ in the autoregressive model $V_t = A_{s,t} V_s + B_{s,t}$, $0 \leq s \leq t$, which fulfills Assumptions (a) to (d) is a multiplicative right Lévy process in $\mathrm{GL}(\mathbb{R}, m)$.

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- Every right Lévy process in $(\mathrm{GL}(\mathbb{R}, m), \cdot)$ is the right stochastic exponential of a Lévy process in $(\mathbb{R}^{m\times m}, +)$, i.e. we have $A_t = \overrightarrow{\mathcal{E}}(U)_t$.

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- Every right Lévy process in $(\mathrm{GL}(\mathbb{R}, m), \cdot)$ is the right stochastic exponential of a Lévy process in $(\mathbb{R}^{m\times m}, +)$, i.e. we have $A_t = \overrightarrow{\mathcal{E}}(U)_t$.
- There exists another Lévy process X in $(\mathbb{R}^{m\times m}, +)$ such that

$$A_t = \overleftarrow{\mathcal{E}}(X)^{-1}, \quad t \geq 0,$$

and the increments $A_{s,t} = A_t A_s^{-1}$ of A_t take the form

$$A_{s,t} = \overleftarrow{\mathcal{E}}(X)_t^{-1} \overleftarrow{\mathcal{E}}(X)_s, \quad 0 \leq s \leq t.$$

Choice of $(A_{s,t}, B_{s,t})_{0\le s\le t}$

Theorem: (i) Suppose $(X_t, Y_t)_{t\ge 0}$ to be a Lévy process in $(\mathbb{R}^{m\times m}\times\mathbb{R}^m, +)$ such that $\overleftarrow{\mathcal{E}}(X)$ is non-singular. For $0\le s\le t$ define

$$\begin{pmatrix} A_{s,t} \\ B_{s,t} \end{pmatrix} := \begin{pmatrix} \overleftarrow{\mathcal{E}}(X)_t^{-1}\overleftarrow{\mathcal{E}}(X)_s \\ \overleftarrow{\mathcal{E}}(X)_t^{-1}\int_{(s,t]}\overleftarrow{\mathcal{E}}(X)_{u-}\,dY_u \end{pmatrix}.$$

Then $(A_{s,t}, B_{s,t})_{0\le s\le t}$ satisfies Assumptions (a) to (d) above and for any starting random variable V_0 the process

$$V_t := \overleftarrow{\mathcal{E}}(X)_t^{-1}\left(V_0 + \int_{(0,t]}\overleftarrow{\mathcal{E}}(X)_{s-}\,dY_s\right)$$

satisfies $V_t = A_{s,t}V_s + B_{s,t}$, $0\le s\le t$.

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satisfies $V_t = A_{s,t}V_s + B_{s,t}$, $0\le s\le t$.
(ii) All processes satisfying $V_t = A_{s,t}V_s + B_{s,t}$, $0\le s\le t$, with $(A_{s,t}, B_{s,t})_{0\le s\le t}$ satisfying Assumptions (a) to (d), can be obtained in this way.

The Multivariate Generalized Ornstein-Uhlenbeck Process

Definition: Let $(X_t, Y_t)_{t\geq 0}$ be a Lévy process in $(\mathbb{R}^{m\times m} \times \mathbb{R}^m, +)$ such that $\det(I + \Delta X_t) \neq 0$ for all $t \geq 0$ and let V_0 be a random variable in $\mathbb{R}^m$. Then the process $(V_t)_{t\geq 0}$ in $\mathbb{R}^m$ given by

$$V_t := \overleftarrow{\mathcal{E}}(X)_t^{-1} \left(V_0 + \int_{(0,t]} \overleftarrow{\mathcal{E}}(X)_{s-} dY_s \right)$$

will be called **multivariate generalized Ornstein-Uhlenbeck (MGOU) process** driven by $(X_t, Y_t)_{t\geq 0}$.

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Remark:

- V_0 not a priori independent of (X, Y).

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$$V_t := \overleftarrow{\mathcal{E}}(X)_t^{-1} \left(V_0 + \int_{(0,t]} \overleftarrow{\mathcal{E}}(X)_{s-} dY_s \right)$$

will be called **multivariate generalized Ornstein-Uhlenbeck (MGOU) process** driven by $(X_t, Y_t)_{t\geq 0}$.

Remark:

- V_0 not a priori independent of (X, Y).
- $\overleftarrow{\mathcal{E}}(X)_t^{-1}$ may take negative (definite) values.

The corresponding SDE

Theorem: The MGOU process

$$V_t := \overleftarrow{\mathcal{E}}(X)_t^{-1} \left(V_0 + \int_{(0,t]} \overleftarrow{\mathcal{E}}(X)_{s-} dY_s \right)$$

driven by the Lévy process $(X_t, Y_t)_{t \geq 0}$ in $(\mathbb{R}^{m \times m} \times \mathbb{R}^m, +)$ is the unique solution of the SDE

$$\boxed{dV_t = dU_t V_{t-} + dL_t, \; t \geq 0,}$$

for the Lévy process $(U_t, L_t)_{t \geq 0}$ in $(\mathbb{R}^{m \times m} \times \mathbb{R}^m, +)$ given by

$$\begin{pmatrix} U_t \\ L_t \end{pmatrix} := \begin{pmatrix} -X_t + [X, X]_t^c + \sum_{0<s\leq t} \left((I + \Delta X_s)^{-1} - I + \Delta X_s \right) \\ Y_t + \sum_{0<s\leq t} \left((I + \Delta X_s)^{-1} - I \right) \Delta Y_s - [X, Y]_t^c \end{pmatrix},$$

for $t \geq 0$.

skip subsection

Stationary Solutions - Part 1

Theorem: Suppose $(V_t)_{t\geq 0}$ is a MGOU process driven by the Lévy process $(X_t, Y_t)_{t\geq 0}$ in $(\mathbb{R}^{m\times m} \times \mathbb{R}^m, +)$. Let $(U_t, L_t)_{t\geq 0}$ be the Lévy process defined as above.

(i) Suppose $\lim_{t\to\infty} \overleftarrow{\mathcal{E}}(U)_t = 0$ in probability, then:

A finite random variable V_0 can be chosen such that $(V_t)_{t\geq 0}$ is strictly stationary

$\Leftrightarrow$

$\int_{(0,t]} \overleftarrow{\mathcal{E}}(U)_{s-} dL_s$ converges in distribution.

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In this case, the distribution of the strictly stationary process $(V_t)_{t\geq 0}$ is uniquely determined and is obtained by choosing V_0 independent of $(X_t, Y_t)_{t\geq 0}$ as the distributional limit of $\int_{(0,t]} \overleftarrow{\mathcal{E}}(U)_{s-} dL_s$ as $t \to \infty$.

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(ii) Suppose $\lim_{t \to \infty} \overleftarrow{\mathcal{E}}(X)_t = 0$ in probability, then:

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A finite random variable V_0 can be chosen such that
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In this case the strictly stationary solution is unique and given by

$$V_t = -\overleftarrow{\mathcal{E}}(X)_t^{-1} \int_{(t,\infty)} \overleftarrow{\mathcal{E}}(X)_{s-} dY_s \quad \text{a.s. for all} \quad t \geq 0.$$

skip now

MGOU processes on affine subspaces

Definition: Suppose $(X_t, Y_t)_{t\geq 0}$ is a Lévy process in $(\mathbb{R}^{m\times m} \times \mathbb{R}^m, +)$ such that $\overleftarrow{\mathcal{E}}(X)$ is non-singular and define $(A_{s,t}, B_{s,t})_{0\leq s\leq t}$ by

$$\begin{pmatrix} A_{s,t} \\ B_{s,t} \end{pmatrix} := \begin{pmatrix} \overleftarrow{\mathcal{E}}(X)_t^{-1} \overleftarrow{\mathcal{E}}(X)_s \\ \overleftarrow{\mathcal{E}}(X)_t^{-1} \int_{(s,t]} \overleftarrow{\mathcal{E}}(X)_{u-} dY_u \end{pmatrix}.$$

Then an affine subspace H of $\mathbb{R}^m$ is called invariant under the autoregressive model $V_t = A_{s,t}V_s + B_{s,t}$, $0 \leq s \leq t$, if

$$A_{s,t}H + B_{s,t} \subseteq H \quad \text{almost surely,}$$

holds for all $0 \leq s \leq t$.
If $\mathbb{R}^m$ is the only invariant affine subspace, the model is called irreducible.

MGOU processes on affine subspaces

Theorem: The autoregressive model $V_t = A_{s,t} V_s + B_{s,t}$, $0 \le s \le t$, is irreducible if and only if there exists no pair (O, K) of an orthogonal transformation $O \in \mathbb{R}^{m \times m}$ and a constant $K = (k_1, \dots, k_d)^T \in \mathbb{R}^d$, $1 \le d \le m$, such that a.s.

$$OX_t O^{-1} = \begin{pmatrix} \mathcal{X}_t^1 & 0 \\ \mathcal{X}_t^2 & \mathcal{X}_t^3 \end{pmatrix} \quad \text{and} \quad OY_t = \begin{pmatrix} \mathcal{X}_t^1 K \\ \mathcal{Y}_t^2 \end{pmatrix}$$

where $\mathcal{X}_t^1 \in \mathbb{R}^{d \times d}$, $t \ge 0$. With $(U_t, L_t)_{t \ge 0}$ as defined above this is equivalent to

$$OU_t O^{-1} = \begin{pmatrix} \mathcal{U}_t^1 & 0 \\ \mathcal{U}_t^2 & \mathcal{U}_t^3 \end{pmatrix} \quad \text{and} \quad OL_t = \begin{pmatrix} -\mathcal{U}_t^1 K \\ \mathcal{L}_t^2 \end{pmatrix}$$

a.s. with $\mathcal{U}_t^1 \in \mathbb{R}^{d \times d}$.

Stationary Solutions of MGOU processes - Part 2

Theorem: Suppose $(V_t)_{t\geq 0}$ is a MGOU process driven by the Lévy process $(X_t, Y_t)_{t\geq 0}$ in $\mathbb{R}^{m\times m} \times \mathbb{R}^m$ such that the corresponding autoregressive model $V_t = A_{s,t}V_s + B_{s,t}$, $0 \leq s \leq t$, with $(A_{s,t}, B_{s,t})_{0\leq s\leq t}$ as defined before is irreducible. Let $(U_t, L_t)_{t\geq 0}$ be defined as above. Then

A finite random variable V_0, independent of $(X_t, Y_t)_{t\geq 0}$, can be chosen such that $(V_t)_{t\geq 0}$ is strictly stationary

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A similar result for strictly noncausal strictly stationary solutions of MGOU processes can be obtained, too.

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Extensions

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Often, in the one dimensional case volatilities are modeled as square-root process of a generalized Ornstein-Uhlenbeck process. Hence to construct a multivariate volatility model similarly, we have to ensure our processes to be positive semidefinite.

One possibility hereby is to consider processes which fulfill

$$V_t = A_{s,t} V_s A_{s,t}^T + B_{s,t}, \quad 0 \le s \le t$$

with $A_{s,t}$ in $\mathrm{GL}(\mathbb{R}, m)$ and $B_{s,t} \in \mathbb{R}^{m \times m}$ positive semidefinite.

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with $A_{s,t}$ in $\mathrm{GL}(\mathbb{R}, m)$ and $B_{s,t} \in \mathbb{R}^{m \times m}$ positive semidefinite. This is equivalent to

$$\text{vec}\, V_t = (A_{s,t} \otimes A_{s,t}) \text{vec}\, V_s + \text{vec}\, B_{s,t}.$$

A Construction

Arguing as above we see that the only process which fulfills the above random recurrence equation is given by

$$V_t = \overleftarrow{\mathcal{E}}(X)_t^{-1} \left(V_0 + \int_{(0,t]} \overleftarrow{\mathcal{E}}(X)_{s-} dY_s (\overleftarrow{\mathcal{E}}(X)_{s-})^T \right) (\overleftarrow{\mathcal{E}}(X)_t^{-1})^T,$$

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for a Lévy process $(X, Y) \in \mathbb{R}^{m \times m} \times \mathbb{R}^{m \times m}$ and that

$$\begin{aligned} \operatorname{vec} V_t &= \overleftarrow{\mathcal{E}}(X)_t^{-1} \otimes \overleftarrow{\mathcal{E}}(X)_t^{-1} \left(\operatorname{vec} V_0 + \int_0^t \overleftarrow{\mathcal{E}}(X)_{s-} \otimes \overleftarrow{\mathcal{E}}(X)_{s-} d\mathcal{Y}_s \right) \\ &= \overleftarrow{\mathcal{E}}(\mathcal{X})_t^{-1} \left(\operatorname{vec} V_0 + \int_0^t \overleftarrow{\mathcal{E}}(\mathcal{X})_{s-} d\mathcal{Y}_s \right), \quad t \geq 0 \end{aligned}$$

is a MGOU process driven by the Lévy process $(\mathcal{X}, \mathcal{Y}) \in \mathbb{R}^{m^2 \times m^2} \times \mathbb{R}^{m^2}$ with

$$\mathcal{X}_t = I \otimes X_t + X_t \otimes I + [X \otimes I, I \otimes X]_t, \quad t \geq 0$$

and $\mathcal{Y}_t = \operatorname{vec}(Y_t)$.

A Condition for Positive Semidefiniteness

The process

$$V_t = \overleftarrow{\mathcal{E}}(X)_t^{-1} \left(V_0 + \int_{(0,t]} \overleftarrow{\mathcal{E}}(X)_{s-} dY_s (\overleftarrow{\mathcal{E}}(X)_{s-})^T \right) (\overleftarrow{\mathcal{E}}(X)_t^{-1})^T,$$

is positive semidefinite for all $t \geq 0$ and all positive semidefinite starting random variables V_0 if and only if Y is a matrix subordinator.

Thank you for your attention!

Main references:

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