A central limit theorem for the sample autocorrelation of a Lévy driven moving average process

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Summary

Introduction : Motivations

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Goal Estimate d (resp. H) based on observations $X_1, X_2, \ldots, X_n$

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$$\mathbb{E}(X_{t+h}X_t) = \gamma_X(h) = \frac{\sigma_0^2}{2}\left(|h+1|^{2d+1} - 2|h|^{2d+1} + |h-1|^{2d+1}\right) \tag{3}$$

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Idea Estimate $\gamma_X(h),\ h=0,\dots,\bar{h}$ by sample autocovariance.

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$$\widehat{\rho}_X(h) = \frac{\widehat{\gamma}_X(h)}{\widehat{\gamma}_X(0)}, \quad \widehat{d} = \frac{1}{2}\left(1 + \frac{\log(\widehat{\rho}_X(1)+1)}{\log 2}\right) \tag{7}$$

of $\rho_X(1)$, d.

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almost surely and in L^1 , so $\widehat{d}$ is strongly consistent. But $(X_t)_{t\in\mathbb{Z}}$ is not strongly mixing and we don't have a central limit theorem.

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1. $F \in L^2([0,1]) \Rightarrow f \in L^1(\mathbb{R})$: the theorem cannot be applied to fractional Lévy noise.
2. $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ does not imply $\sum_{j=-\infty}^{\infty} |\gamma_X(j)| < \infty$.

Bartlett's formula

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where $W = (w_{ij})_{i,j=1,\ldots,n}$,

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$$\begin{aligned} w_{ij} &= \sum_{k=-\infty}^{\infty} \big(\rho(k+i)\rho(k+j) + \rho(k-i)\rho(k+j) + 2\rho(i)\rho(j)\rho(k)^2 \\ &\quad -2\rho(i)\rho(k)\rho(k+j) - 2\rho(j)\rho(k)\rho(k+i)\big) \end{aligned}$$

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Theorem

If $f \in L^2(\mathbb{R}) \cap L^4(\mathbb{R})$, $\sigma_L^2 = \mathbb{E}L_1^2 < \infty$, *and* $\eta := \sigma_L^{-4}\mathbb{E}L_1^4 < \infty$.

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Denote $g_q(u) = \sum_{j=-\infty}^{\infty} f(u+j)f(u+j+q)$

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If $f \in L^2(\mathbb{R}) \cap L^4(\mathbb{R})$, $\sigma_L^2 = \mathbb{E}L_1^2 < \infty$, *and* $\eta := \sigma_L^{-4}\mathbb{E}L_1^4 < \infty$. *Suppose* $\sum_{j=-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(s)f(s+j)\,ds \right)^2 < \infty$. *Let* $g_0(u) = \sum_{j=-\infty}^{\infty} f(u+j)^2$ *and suppose* $g_0 \in L^2([0,1])$. *Denote* $g_q(u) = \sum_{j=-\infty}^{\infty} f(u+j)f(u+j+q)$ *then*

$$\sqrt{n}(\widehat{\rho}(h) - \rho(h))_{h=1,\dots,n} \xrightarrow{d} N(0, \widetilde{W}).$$

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$$\widetilde{w}_{ij} = w_{ij} + \frac{(\eta-3)\sigma_L^4}{\gamma(0)^2} \int_0^1 (g_i(u) - \rho(i)g_0(u))(g_j(u) - \rho(j)g_0(u))\, du.$$

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Back to fractional Lévy noise

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