

Functional Central Limit Theorem for Heavy Tailed Stationary Infinitely Divisible Processes Generated by Conservative Flows

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Let $\mathbf{X} = (X_1, X_2, \dots)$ be a stationary stochastic process.

A (functional) central limit theorem for such a process is a statement of the type

$$\left(\frac{1}{c_n} \sum_{k=1}^{\lceil nt \rceil} X_k - h_n t, 0 \leq t \leq 1 \right) \Rightarrow \left(Y(t), 0 \leq t \leq 1 \right).$$

- $\mathbf{Y} = \left(Y(t), 0 \leq t \leq 1 \right)$ is a non-degenerate process.
- By the Lamperti theorem, \mathbf{Y} is self-similar with stationary increments.

We consider a class of stationary symmetric infinitely divisible processes with regularly varying tails.

The main ergodic-theoretical property will be that of **pointwise dual ergodicity**.

The length of memory will be determined by the rate of growth of **wandering rate sequence**.

It will have one parameter, $0 < \beta < 1$, that will determine the limiting process **Y**.

The limiting process

Let $0 < \beta < 1$. We start with inverse process

$$M_\beta(t) = S_\beta^{\leftarrow}(t) = \inf\{u \geq 0 : S_\beta(u) \geq t\}, \quad t \geq 0.$$

- $(S_\beta(t), t \geq 0)$ is a (strictly) β -stable subordinator.
- $(M_\beta(t), t \geq 0)$ is called *the Mittag-Leffler process*.

The Mittag-Leffler process has a continuous and non-decreasing version.

- It is self-similar with exponent β .
- Its increments are neither stationary nor independent.
- All of its moments are finite.

$$E \exp\{\theta M_\beta(t)\} = \sum_{n=0}^{\infty} \frac{(\theta t^\beta)^n}{\Gamma(1 + n\beta)}, \quad \theta \in \mathbb{R}.$$

Define

$$Y_{\alpha,\beta}(t) = \int_{\Omega' \times [0,\infty)} M_{\beta}((t-x)_{+}, \omega') dZ_{\alpha,\beta}(\omega', x), \quad t \geq 0.$$

- $Z_{\alpha,\beta}$ is a S α S random measure on $\Omega' \times [0, \infty)$ with control measure $P' \times \nu$.
- ν a measure on $[0, \infty)$ given by $\nu(dx) = (1 - \beta)x^{-\beta} dx$.
- M_{β} is a Mittag-Leffler process defined on $(\Omega', \mathcal{F}', P')$.

The process $(Y_{\alpha,\beta}(t), t \geq 0)$ is a well defined S α S process with stationary increments.

It is self-similar with exponent of self-similarity

$$H = \beta + (1 - \beta)/\alpha.$$

We call it **the β -Mittag-Leffler (or β -ML) fractional S α S motion.**

A connection: $\hat{\beta}$ -stable local time fractional $S\alpha S$ motion.

Let $\hat{\beta} = (1 - \beta)^{-1} \in (1, \infty)$.

If $\hat{\beta} \in (1, 2)$, a $\hat{\beta}$ -stable local time fractional $S\alpha S$ motion was introduced in Dombry and Guillin-Plantard (2009).

$$\hat{Y}_{\alpha, \beta}(t) = \int_{\Omega' \times \mathbb{R}} L_t(x, \omega') d\hat{Z}_{\alpha}(\omega', x), \quad t \geq 0;$$

- \hat{Z}_{α} is a $S\alpha S$ random measure on $\Omega' \times \mathbb{R}$ with control measure $P' \times \text{Leb}$
- $(L_t(x), t \geq 0, x \in \mathbb{R})$ is a jointly continuous local time process of a symmetric $\hat{\beta}$ -stable Lévy process.

In this range, the ML fractional $S\alpha S$ motion coincides, distributionally, with the $\hat{\beta}$ -stable local time fractional $S\alpha S$ motion.

One can view the ML fractional $S\alpha S$ motion as

an extension of the $\hat{\beta}$ -stable local time fractional $S\alpha S$ motion from the range $1 < \hat{\beta} \leq 2$ to the range $1 < \hat{\beta} < \infty$.

A bit of ergodic theory

Let (E, \mathcal{E}, μ) be a σ -finite, **infinite** measure space.

Let $T : E \rightarrow E$ be a measurable map that preserves the measure μ .

When the entire sequence T, T^2, T^3, \dots of iterates of T is involved, we will sometimes refer to it as **a flow**.

The dual operator \widehat{T} is an operator $L^1(\mu) \rightarrow L^1(\mu)$ defined by

$$\widehat{T}f = \frac{d(\nu_f \circ T^{-1})}{d\mu},$$

with ν_f a signed measure on (E, \mathcal{E}) given by $\nu_f(A) = \int_A f d\mu$, $A \in \mathcal{E}$.

The dual operator satisfies the relation

$$\int_E \widehat{T}f \cdot g d\mu = \int_E f \cdot g \circ T d\mu$$

for $f \in L^1(\mu)$, $g \in L^\infty(\mu)$.

An ergodic conservative measure preserving map T is called **pointwise dual ergodic** if there is a sequence of positive constants $a_n \rightarrow \infty$ such that

$$\frac{1}{a_n} \sum_{k=1}^n \widehat{T}^k f \rightarrow \int_E f d\mu \text{ a.e.}$$

for every $f \in L^1(\mu)$.

Pointwise dual ergodicity rules out invertibility of the map T .

The stationary process X

We consider infinitely divisible processes of the form

$$X_n = \int_E f_n(x) dM(x), \quad n = 1, 2, \dots$$

- M is a homogeneous symmetric infinitely divisible random measure on a (E, \mathcal{E}) .
- μ has an infinite, σ -finite, control measure μ and local Lévy measure ρ : for every $A \in \mathcal{E}$ with $\mu(A) < \infty$, $u \in \mathbb{R}$,

$$Ee^{iuM(A)} = \exp \left\{ -\mu(A) \int_{\mathbb{R}} (1 - \cos(ux)) \rho(dx) \right\}.$$

The functions f_n , $n = 1, 2, \dots$ are deterministic functions of the form

$$f_n(x) = f \circ T^n(x) = f(T^n x), \quad x \in E, \quad n = 1, 2, \dots :$$

- $f : E \rightarrow \mathbb{R}$ is a measurable function, satisfying certain integrability assumptions;
- $T : E \rightarrow E$ a pointwise dual ergodic map.

We assume that the local Lévy measure ρ has a regularly varying tail with index $-\alpha$, $0 < \alpha < 2$:

$$\rho(\cdot, \infty) \in RV_{-\alpha} \text{ at infinity.}$$

With a proper integrability assumption on the function f :

the process \mathbf{X} has regularly varying finite-dimensional distributions, with the same tail exponent $-\alpha$.

Theorem

Assume that the normalizing sequence (a_n) in the pointwise dual ergodicity is regularly varying with exponent $0 < \beta < 1$ and that $\mu(f) = \int f d\mu \neq 0$. Then for some sequence (c_n) that is regularly varying with exponent $\beta + (1 - \beta)/\alpha$,

$$\frac{1}{c_n} \sum_{k=1}^{\lfloor n \cdot \rfloor} X_k \Rightarrow |\mu(f)| Y_{\alpha, \beta} \quad \text{in } D[0, \infty).$$

Example Consider an irreducible null recurrent Markov chain with state space \mathbb{Z} and transition matrix $P = (p_{ij})$.

Let $\{\pi_j, j \in \mathbb{Z}\}$ be the unique invariant measure of the Markov chain that satisfies $\pi_0 = 1$.

Define a σ -finite measure on $(E, \mathcal{E}) = (\mathbb{Z}^{\mathbb{N}}, \mathcal{B}(\mathbb{Z}^{\mathbb{N}}))$ by

$$\mu(\cdot) = \sum_{i \in \mathbb{Z}} \pi_i P_i(\cdot),$$

Let $T : \mathbb{Z}^{\mathbb{N}} \rightarrow \mathbb{Z}^{\mathbb{N}}$ be the left shift map

$$T(x_0, x_1, \dots) = (x_1, x_2, \dots) \text{ for } \{x_k, k = 0, 1, \dots\} \in \mathbb{Z}^{\mathbb{N}}.$$

Let $A = \{x \in \mathbb{Z}^{\mathbb{N}} : x_0 = 0\}$ and the corresponding first entrance time $\varphi(x) = \min\{n \geq 1 : x_n = 0\}$, $x \in \mathbb{Z}^{\mathbb{N}}$.

Assume that

$$P_0(\varphi \geq k) \in RV_{-\beta}.$$

The assumptions of the theorem hold, for example, if $f = \mathbf{1}_A$.

The length of memory in the process \mathbf{X} is quantified by the tail of the first return time φ .

In the general case, the length of memory is still quantified by a single parameter β . It is also related to return times.