



Lehrstuhl für
Mathematische Statistik



Fluctuations of subexponential Lévy processes with infinite mean

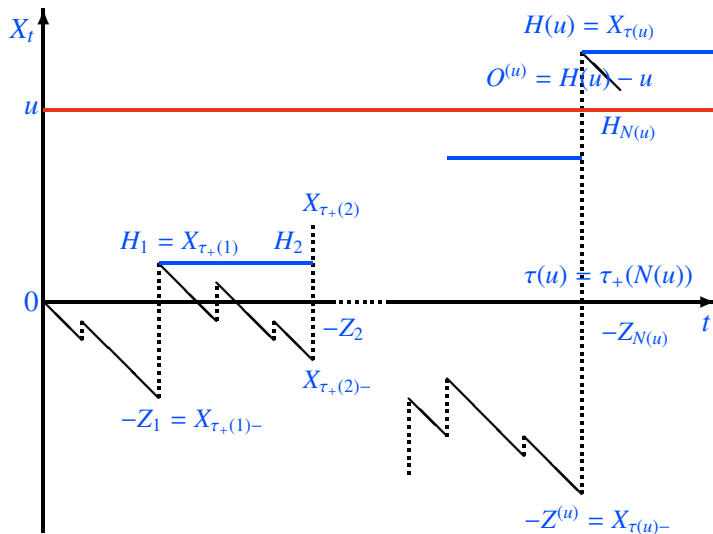
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Joint with Ron Doney & Ross Maller



Sample path: X CPP, finite mean

Pollaczek-Khinchine formula

Define $F_I(u) = \frac{1}{\mu} \int_0^u \bar{F}(s) ds$, then

$$\psi(u) = (1 - \rho) \sum_{n=1}^{\infty} \rho^n \bar{F}_I^{n*}(u) \quad u \geq 0.$$

For F_I subexponential ($F_I \in \mathcal{S}$),

Embrechts, Goldie and Veraverbeke (1979) proved:

$$F_I \in \mathcal{S} \quad \Leftrightarrow \quad \psi \in \mathcal{S} \quad \Leftrightarrow \quad \int_u^{\infty} \bar{F}(s) ds / \psi(u) \rightarrow \mu < \infty \text{ as } u \rightarrow \infty.$$

Undershoot $X_{\tau(u)-}$ and overshoot $X_{\tau(u)} - u$

Their asymptotics are given by classical extreme value theory:

Theorem $P^{(u)}(\cdot) := P(\cdot \mid \tau(u) < \infty)$

- ▶ If $F \in \text{MDA}(\Phi_{\alpha+1})$ ($\Leftrightarrow F \in \mathcal{R}(-(\alpha + 1))$) and $a(u) \sim u/\alpha$, then

$$\left(\frac{-X_{\tau(u)-}}{a(u)}, \frac{X_{\tau(u)} - u}{a(u)} \right) \xrightarrow{P^{(u)}\text{-dist.}} (Z, O)$$

with $P(Z > x, O > y) = \left(1 + \frac{x+y}{\alpha}\right)^{-\alpha}$, $x, y > 0$.

- ▶ If $F \in \text{MDA}(\Lambda)$ and $a(u) \sim \int_u^\infty \bar{F}(s) ds / \bar{F}(u)$, then

$$\left(\frac{-X_{\tau(u)-}}{a(u)}, \frac{X_{\tau(u)} - u}{a(u)} \right) \xrightarrow{P^{(u)}\text{-dist.}} (Z, O)$$

with $P(Z > x, O > y) = e^{-(x+y)}$, $x, y > 0$.

Theorem 1.1 [A&K 1996]

Assume that $F_I \in \mathcal{S}$ and
 $F \in \text{MDA}(\Phi_{\alpha+1})$ or $F \in \text{MDA}(\Lambda)$.

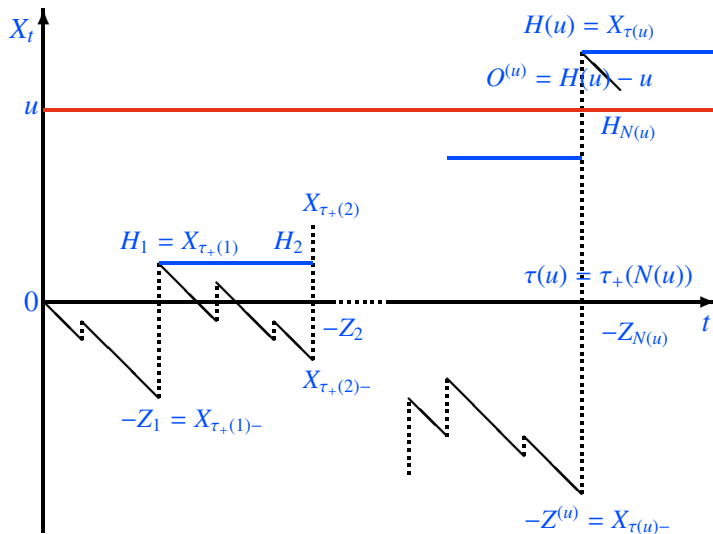
Then

$$\left(\frac{-X_{\tau(u)-}}{a(u)}, \frac{X_{\tau(u)} - u}{a(u)}, \frac{\tau(u)}{a(u)}, \left\{ \frac{-X_{s\tau(u)}}{\tau(u)} \right\}_{0 \leq s < 1} \right) \\ \xrightarrow{P^{(u)}\text{-dist.}} (Z, O, Z/\mu, \{\mu s\}_{0 \leq s < 1})$$

in $\mathbb{R}^3 \times \mathcal{D}[0, 1)$.



Sample path again:



The general set-up

$X = (X_t)_{t \geq 0}$ Lévy process with triplet $(\gamma, \sigma^2, \Pi_X)$,
 $\gamma \in \mathbb{R}$, $\sigma^2 \geq 0$, Π_X Lévy measure on \mathbb{R} ,

$$\lim_{t \rightarrow \infty} X_t = -\infty \quad \text{a.s.}$$

$(H_t)_{t \geq 0}$ **ascending ladder height** subordinator of X
 which is defective and links to a non-defective subordinator \mathcal{H} by

$$P(H_t \leq x) = P(H_t \leq x, t < L_\infty) = e^{qt} P(\mathcal{H}_t \leq x) \quad x > 0$$

where L_t is a local time of X .

$(H_t^*)_{t \geq 0}$ is the **descending ladder height process** of X ; i.e. the
 ascending ladder height subordinator of $X^* = -X$, which is proper
 corresponding to $q^* = 0$.

Heavy-tailed ascending and descending ladder heights

We denote $\Pi \in \mathcal{S}$ for any of our Lévy measures, iff the probability measure

$$\frac{\Pi(\cdot)\mathbf{1}_{\{x>1\}}}{\Pi([1, \infty))} \in \mathcal{S}.$$

For the ascending ladder height process we assume:

Assumption 1: $\Pi_{\mathcal{H}} \in \mathcal{S} \Rightarrow \bar{\Pi}_{\mathcal{H}}(u) \sim qP(\tau(u) < \infty)$

For the descending ladder height process we assume:

Assumption 2: $A_{H^*}(x) := \int_0^x \bar{\Pi}_{H^*}(y)dy \in \mathcal{R}(\gamma)$ for $\gamma \in [0, 1)$

Corresponding renewal measure $G^* \in \mathcal{R}(1 - \gamma)$

Theorem: Extend finite mean case of H^* to slow variation

Assume $\lim_{t \rightarrow \infty} X_t = -\infty$ a.s., $\Pi_{\mathcal{H}} \in \mathcal{S}$, and $A_{H^*} \in \mathcal{R}(0)$.

Throughout $0 < a(u) \rightarrow \infty$ as $u \rightarrow \infty$ is chosen appropriately.

The following are equivalent for $u \rightarrow \infty$:

- (a) $P^{(u)}(X_{\tau(u)} - u \in a(u)dx)$ has non-degenerate limit;
- (b) **Fréchet:** $\Pi_{\mathcal{H}} \in \mathcal{R}(1 - \beta)$ for some $\beta > 1$, or
Gumbel: $\Pi_{\mathcal{H}} \in \text{MDA}(\Lambda)$;
- (c) **Fréchet:** $\Pi_X^+ \in \mathcal{R}(-\beta)$ for some $\beta > 1$, or
Gumbel: $\Pi_X^+ \in \text{MDA}(\Lambda)$.

When (a)–(c) hold, then the process

$$\left(\frac{-X_{\tau(u)-}}{a(u)}, \frac{X_{\tau(u)} - u}{a(u)}, \frac{\tau_u}{b(a(u))}, \left(\frac{-X_{s\tau_u}}{a(u)} \right)_{0 \leq s < 1} \right) \\ \xrightarrow{P^{(u)\text{-dist.}}} (Z, O, Z, (\mathbf{D}^{(0)}(s))_{0 \leq s < 1})$$

in $\mathbb{R}^3 \times \mathbb{D}[0, 1)$ (weak convergence in the Skorokhod topology),

where $\mathbf{D}^{(0)}(s) = s$, and

for **Fréchet**:

$$P(Z \in dz, O \in y) = \frac{\beta(\beta - 1)}{(1 + z + y)^{\beta+1}} dz dy, \quad y, z > 0.$$

for **Gumbel**:

$$P(Z \in dz, O \in dy) = e^{-(z+y)} dz dy, \quad y, z > 0.$$

Skorohod convergence

If $A_{H^*} \in \mathcal{R}(0)$, then $X^* = -X$ is positively relatively stable.

Hence, for some $c(\cdot) \in \mathcal{R}(1)$ continuous and increasing,

$X_t^*/c(t) = -X_t/c(t) \xrightarrow{P} 1$ as $t \rightarrow \infty$, which implies that

$$(X_{us}^*/c(u))_{s \in [0,1]} = (-X_{us}/c(u))_{s \in [0,1]} \xrightarrow{P^{(u)\text{-dist.}}} \mathbf{D}^{(0)} \quad u \rightarrow \infty$$

in $\mathbb{D}[0, 1]$ with $\mathbf{D}^{(0)}(s) = s$.

If $A_{H^*} \in \mathcal{R}(\gamma)$ with $\gamma \in (0, 1)$, then $\Pi^-(x) \sim \gamma x^{-1} A_X^*(x) \in \mathcal{R}(\gamma - 1)$.

Denote by $\mathbf{D} = \mathbf{S}_{1-\gamma}$ a standard stable subordinator. Then $X^* \in \text{DA}(\mathbf{D})$ and for some continuous and increasing $c(\cdot) \in \mathcal{R}(1/(1 - \gamma))$,

$$(X_{sc(u)}^*/c(u))_{s>0} \xrightarrow{d} \mathbf{D} \quad u \rightarrow \infty$$

Let $\widehat{\mathbf{D}}_{t,z}$ be an associated “stable subordinator bridge” (a rescaled version of \mathbf{D}) satisfying

$$P(\widehat{\mathbf{D}}_{t,z} \in \cdot) = P((D_{ts})_{s \in [0,1]} \in \cdot \mid D_t = z).$$

Theorem: infinite mean case and regular variation of H^*

Assume $\lim_{t \rightarrow \infty} X_t = -\infty$ a.s., $\Pi_{\mathcal{H}} \in \mathcal{S}$, and $A_{H^*} \in \mathcal{R}(\gamma)$ ($\gamma \in (0, 1)$)

Throughout $0 < a(u) \rightarrow \infty$ as $u \rightarrow \infty$ chosen appropriately.

The following are equivalent for $u \rightarrow \infty$:

- (a) $P^{(u)}(X_{\tau(u)} - u \in a(u)dx)$ has a non-degenerate limit;
- (b) **Fréchet:** $\Pi_{\mathcal{H}} \in \mathcal{R}(1 - \gamma - \beta)$ for some $\beta > 1 - \gamma$, or
Gumbel: $\Pi_{\mathcal{H}} \in \text{MDA}(\Lambda)$;
- (c) **Fréchet:** $\Pi_X^+ \in \mathcal{R}(-\beta)$ for some $\beta > 1 - \gamma$, or
Gumbel: $\Pi_X^+ \in \text{MDA}(\Lambda)$.

When (a)–(c) hold and X_t has a non-lattice distribution for each $t > 0$.
Then

$$\left(\frac{-X_{\tau(u)-}}{a(u)}, \frac{X_{\tau(u)} - u}{a(u)}, \frac{\tau_u}{b(a(u))}, \left(\frac{-X_{s\tau_u}}{a(u)} \right)_{0 \leq s < 1} \right) \\ \xrightarrow{P^{(u)\text{-dist.}}} (Z, O, W, (\widehat{\mathbf{D}}_{W,Z}(s))_{0 \leq s < 1})$$

in $\mathbb{R}^3 \times \mathbb{D}[0, 1)$, where with $h_t(x)dx = P(D_t \in dx)$ and $t, y, z > 0$
for **Fréchet**:

$$P(Z \in dz, O \in dy, W \in dt) = \frac{\Gamma(\beta + 1)}{\Gamma(\gamma + \beta - 1)(1 + z + y)^{\beta+1}} h_t(z) dz dy dt,$$

for **Gumbel**:

$$P(Z \in dz, O \in dy, W \in dt) = e^{-(z+y)} h_t(z) dz dy dt.$$

Marginals, **Fréchet**:

$$P(Z \in dz, O \in dy) = \frac{\Gamma(\beta + 1)}{\Gamma(1 - \gamma)\Gamma(\gamma + \beta - 1)(1 + z + y)^{\beta+1}} dz dy, \quad y, z > 0,$$

and

$$P(W \in dt) = \frac{\Gamma(\beta)}{\Gamma(\gamma + \beta - 1)} \int_0^\infty \frac{h_1(z) dz}{(1 + t^{1/(1-\gamma)}z)} dt, \quad t > 0.$$

No pair of (Z, O, W) is independent.

Marginals, **Gumbel**:

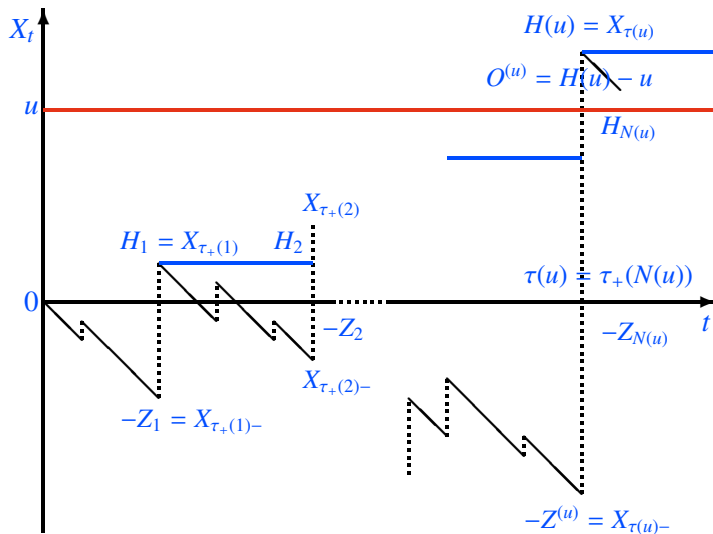
$$P(Z \in dz, O \in dy) = \frac{z^{-\gamma} e^{-(z+y)}}{\Gamma(1-\gamma)} dz dy, \quad y, z > 0,$$

and

$$P(W \in dt) = \int_0^\infty e^{-t^{1/(1-\gamma)} z} dz dt, \quad t > 0.$$

$Z \perp\!\!\!\perp O$, $O \perp\!\!\!\perp W$, but Z, W are dependent

Sample path



Preprints available at www-m4.ma.tum.de

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