Lehrstuhl für Mathematische Statistik

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Fluctuations of subexponential Levy processes with infinite mean

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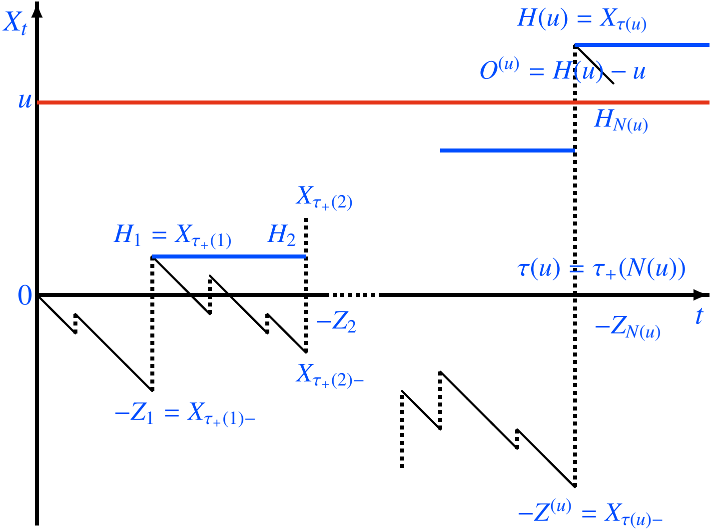
Joint with Ron Doney & Ross Maller

Sample path: X CPP, finite mean



Pollaczek-Khinchine formula

Define $F_I(u) = \frac{1}{\mu}\int_0^u \overline{F}(s)ds$, then

$$\psi(u) \quad = \quad (1-\rho)\sum_{n=1}^{\infty}\rho^n\overline{F_I^{n*}}(u) \quad u \geq 0.$$

For F_I subexponential ($F_I \in \mathcal{S}$),
Embrechts, Goldie and Veraverbeke (1979) proved:

$$F_I \in \mathcal{S} \quad \Leftrightarrow \quad \psi \in \mathcal{S} \quad \Leftrightarrow \quad \int_u^{\infty}\overline{F}(s)ds/\psi(u) \to \mu < \infty \text{ as } u \to \infty.$$

Undershoot $X_{\tau(u)-}$ and overshoot $X_{\tau(u)} - u$

Their asymptotics are given by classical extreme value theory:

Theorem $P^{(u)}(\cdot) := P(\cdot \mid \tau(u) < \infty)$

- If $F \in \mathrm{MDA}(\Phi_{\alpha+1})$ ($\Leftrightarrow$ $F \in \mathcal{R}(-(\alpha+1))$) and $a(u) \sim u/\alpha$, then

$$\left(\frac{-X_{\tau(u)-}}{a(u)} , \frac{X_{\tau(u)} - u}{a(u)} \right) \xrightarrow{P^{(u)}-dist.} (Z\,,\, O)$$

  with $P(Z > x, O > y) = \left(1 + \frac{x+y}{\alpha}\right)^{-\alpha}, \quad x, y > 0.$

- If $F \in \mathrm{MDA}(\Lambda)$ and $a(u) \sim \int_u^\infty \overline{F}(s)ds / \overline{F}(u)$, then

$$\left(\frac{-X_{\tau(u)-}}{a(u)} , \frac{X_{\tau(u)} - u}{a(u)} \right) \xrightarrow{P^{(u)}-dist.} (Z\,,\, O)$$

  with $P(Z > x, O > y) = e^{-(x+y)}, \quad x, y > 0.$

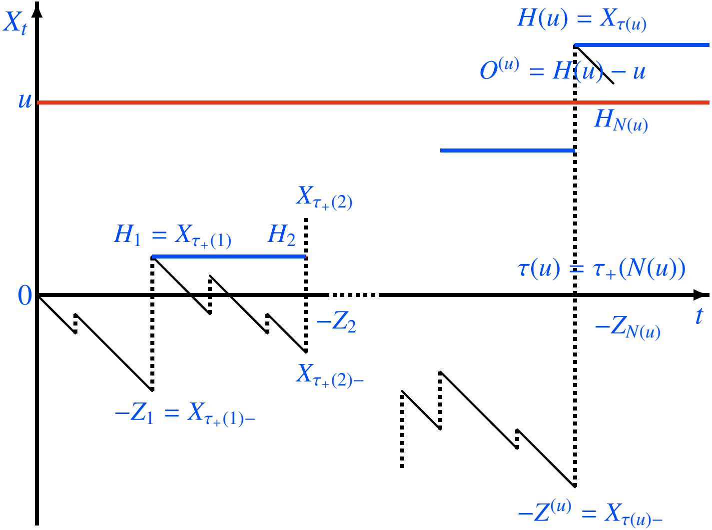
Theorem 1.1 [A&K 1996]

Assume that $F_I \in \mathcal{S}$ and
$F \in \mathrm{MDA}(\Phi_{\alpha+1})$ or $F \in \mathrm{MDA}(\Lambda)$.
Then

$$\left(\frac{-X_{\tau(u)-}}{a(u)}, \frac{X_{\tau(u)} - u}{a(u)}, \frac{\tau(u)}{a(u)}, \left\{\frac{-X_{s\tau(u)}}{\tau(u)}\right\}_{0\le s<1}\right) \stackrel{P^{(u)}-dist.}{\longrightarrow} (Z, O, Z/\mu, \{\mu s\}_{0\le s<1})$$

in $\mathbb{R}^3 \times \mathbb{D}[0,1)$. □

Sample path again:



The general set-up

$X = (X_t)_{t\geq 0}$ Lévy process with triplet $(\gamma, \sigma^2, \Pi_X)$,
$\gamma \in \mathbb{R}$, $\sigma^2 \geq 0$, Π_X Lévy measure on $\mathbb{R}$,

$$\lim_{t\to\infty} X_t = -\infty \quad \text{a.s.}$$

$(H_t)_{t\geq 0}$ **ascending ladder height** subordinator of X
which is defective and links to a non-defective subordinator $\mathcal{H}$ by

$$P(H_t \leq x) = P(H_t \leq x, t < L_\infty) = e^{qt} P(\mathcal{H}_t \leq x) \quad x > 0$$

where L_t is a local time of X.
$(H_t^*)_{t\geq 0}$ is the **descending ladder height process** of X; i.e. the ascending ladder height subordinator of $X^* = -X$, which is proper corresponding to $q^* = 0$.

Heavy-tailed ascending and descending ladder heights

We denote $\Pi \in \mathcal{S}$ for any of our Lévy measures, iff the probability measure

$$\frac{\Pi(\cdot)\mathbf{1}_{\{x>1\}}}{\Pi([1,\infty))} \in \mathcal{S}.$$

For the ascending ladder height process we assume:

Assumption 1: $\Pi_{\mathcal{H}} \in \mathcal{S} \Rightarrow \overline{\Pi}_{\mathcal{H}}(u) \sim qP(\tau(u) < \infty)$

For the descending ladder height process we assume:

Assumption 2: $A_{H^*}(x) := \int_0^x \overline{\Pi}_{H^*}(y)dy \in \mathcal{R}(\gamma)$ for $\gamma \in [0,1)$

Corresponding renewal measure $G^* \in \mathcal{R}(1-\gamma)$

Theorem: Extend finite mean case of H^* to slow variation

Assume $\lim_{t\to\infty} X_t = -\infty$ a.s., $\Pi_{\mathcal{H}} \in \mathcal{S}$, and $A_{H^*} \in \mathcal{R}(0)$.
Throughout $0 < a(u) \to \infty$ as $u \to \infty$ is chosen appropriately.

The following are equivalent for $u \to \infty$:

(a) $P^{(u)}(X_{\tau(u)} - u \in a(u)dx)$ has non-degenerate limit;

(b) **Fréchet:** $\Pi_{\mathcal{H}} \in \mathcal{R}(1-\beta)$ for some $\beta > 1$, or
Gumbel: $\Pi_{\mathcal{H}} \in \mathrm{MDA}(\Lambda)$;

(c) **Fréchet:** $\Pi_X^+ \in \mathcal{R}(-\beta)$ for some $\beta > 1$, or
Gumbel: $\Pi_X^+ \in \mathrm{MDA}(\Lambda)$.

When (a)–(c) hold, then the process

$$\left(\frac{-X_{\tau(u)-}}{a(u)}, \frac{X_{\tau(u)} - u}{a(u)}, \frac{\tau_u}{b(a(u))}, \left(\frac{-X_{s\tau_u}}{a(u)} \right)_{0 \le s < 1} \right) \xrightarrow{P^{(u)}-dist.} \left(Z, O, Z, (Z\mathbf{D}^{(0)}(s))_{0 \le s < 1} \right)$$

in $\mathbb{R}^3 \times \mathbb{D}[0,1)$ (weak convergence in the Skorokhod topology),
where $\mathbf{D}^{(0)}(s) = s$, and
for **Fréchet:**

$$P(Z \in dz, O \in y) = \frac{\beta(\beta-1)}{(1+z+y)^{\beta+1}} dz dy, \quad y, z > 0.$$

for **Gumbel:**

$$P(Z \in dz, O \in dy) = e^{-(z+y)} dz dy, \quad y, z > 0.$$

Skorohod convergence

If $A_{H^*} \in \mathcal{R}(0)$, then $X^* = -X$ is positively relatively stable. Hence, for some $c(\cdot) \in \mathcal{R}(1)$ continuous and increasing, $X_t^*/c(t) \ = \ -X_t/c(t) \ \stackrel{P}{\to} \ 1$ as $t \to \infty$, which implies that

$$(X_{us}^*/c(u))_{s\in[0,1)} \ = \ (-X_{us}/c(u))_{s\in[0,1)} \quad \xrightarrow{P^{(u)}-dist.} \quad \mathbf{D}^{(0)} \quad u \to \infty$$

in $\mathbb{D}[0,1)$ with $\mathbf{D}^{(0)}(s) = s$.

If $A_{H^*} \in \mathcal{R}(\gamma)$ with $\gamma \in (0,1)$, then $\Pi^-(x) \sim \gamma x^{-1} A_X^*(x) \in \mathcal{R}(\gamma - 1)$. Denote by $\mathbf{D} = \mathbf{S}_{1-\gamma}$ a standard stable subordinator. Then $X^* \in \mathrm{DA}(\mathbf{D})$ and for some continuous and increasing $c(\cdot) \in \mathcal{R}(1/(1-\gamma))$,

$$(X_{sc(u)}^*/c(u))_{s>0} \quad \stackrel{d}{\to} \quad \mathbf{D} \quad u \to \infty$$

Let $\widehat{\mathbf{D}}_{t,z}$ be an associated "stable subordinator bridge" (a rescaled version of $\mathbf{D}$) satisfying

$$P(\widehat{\mathbf{D}}_{t,z} \in \cdot) = P((D_{ts})_{s\in[0,1)} \in \cdot \mid D_t = z).$$

Theorem: infinite mean case and regular variation of H^*

Assume $\lim_{t\to\infty} X_t = -\infty$ a.s., $\Pi_{\mathcal{H}} \in \mathcal{S}$, and $A_{H^*} \in \mathcal{R}(\gamma)$ $(\gamma \in (0,1))$

Throughout $0 < a(u) \to \infty$ as $u \to \infty$ chosen appropriately.

The following are equivalent for $u \to \infty$:

(a) $P^{(u)}(X_{\tau(u)} - u \in a(u)dx)$ has a non-degenerate limit;

(b) **Fréchet:** $\Pi_{\mathcal{H}} \in \mathcal{R}(1-\gamma-\beta)$ for some $\beta > 1-\gamma$, or
Gumbel: $\Pi_{\mathcal{H}} \in \mathrm{MDA}(\Lambda)$);

(c) **Fréchet:** $\Pi_X^+ \in \mathcal{R}(-\beta)$ for some $\beta > 1-\gamma$, or
Gumbel: $\Pi_X^+ \in \mathrm{MDA}(\Lambda)$.

When (a)–(c) hold and X_t has a non-lattice distribution for each $t > 0$. Then

$$\left(\frac{-X_{\tau(u)-}}{a(u)}, \frac{X_{\tau(u)}-u}{a(u)}, \frac{\tau_u}{b(a(u))}, \left(\frac{-X_{s\tau_u}}{a(u)}\right)_{0\le s<1}\right) \xrightarrow{P^{(u)}-dist.} \left(Z, O, W, (\widehat{\mathbf{D}}_{W,Z}(s))_{0\le s<1}\right)$$

in $\mathbb{R}^3 \times \mathbb{D}[0,1)$, where with $h_t(x)dx = P(D_t \in dx)$ and $t, y, z > 0$ for **Fréchet:**

$$P\left(Z \in dz, O \in dy, W \in dt\right) = \frac{\Gamma(\beta+1)}{\Gamma(\gamma+\beta-1)(1+z+y)^{\beta+1}} h_t(z)dz\,dy\,dt,$$

for **Gumbel:**

$$P\left(Z \in dz, O \in dy, W \in dt\right) = e^{-(z+y)} h_t(z)dz\,dy\,dt.$$

Marginals, **Fréchet:**

$$P(Z \in dz, O \in dy) = \frac{\Gamma(\beta+1)}{\Gamma(1-\gamma)\Gamma(\gamma+\beta-1)(1+z+y)^{\beta+1}} dz\, dy,\ \ y, z > 0,$$

and

$$P(W \in dt) = \frac{\Gamma(\beta)}{\Gamma(\gamma+\beta-1)} \int_0^\infty \frac{h_1(z)dz}{(1+t^{1/(1-\gamma)}z)}\, dt,\ \ t > 0.$$

No pair of (Z, O, W) is independent.

Marginals, **Gumbel:**

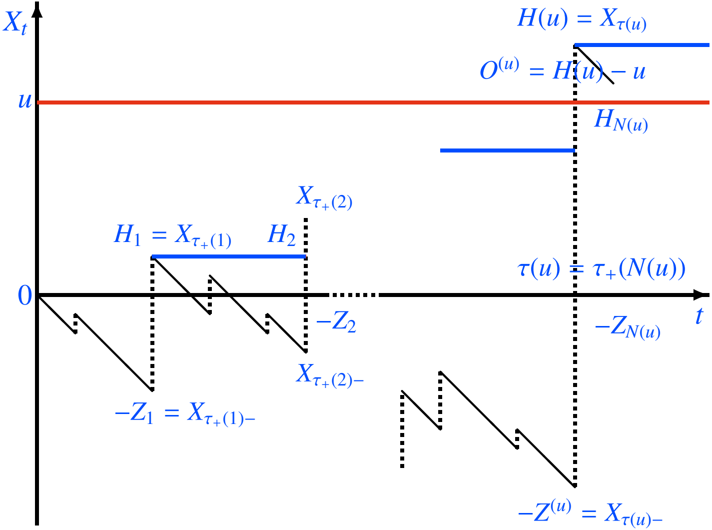
$$P\,(Z \in dz, O \in dy) = \frac{z^{-\gamma}e^{-(z+y)}}{\Gamma(1-\gamma)}dz\,dy,\ \ y,z > 0,$$

and

$$P\,(W \in dt) = \int_0^\infty e^{-t^{1/(1-\gamma)}z}dz\,dt,\ \ t > 0.$$

$Z \perp\!\!\!\perp O$, $O \perp\!\!\!\perp W$, but Z, W are dependent

Sample path


X_t
u
0
t
$H(u) = X_{\tau(u)}$
$O^{(u)} = H(u) - u$
$H_{N(u)}$
$X_{\tau_+(2)}$
$H_1 = X_{\tau_+(1)}$
H_2
$\tau(u) = \tau_+(N(u))$
$-Z_2$
$-Z_{N(u)}$
$X_{\tau_+(2)-}$
$-Z_1 = X_{\tau_+(1)-}$
$-Z^{(u)} = X_{\tau(u)-}$

Preprints available at www-m4.ma.tum.de

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