The distribution of the Rosenblatt process
– Examples of distributions in the Thorin class –

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Outline

1. The Rosenblatt process
2. The Thorin class
3. Results
4. The non-symmetric Rosenblatt process
5. The unimodality of the Rosenblatt distribution
6. Stochastic integral representations with respect to Lévy process of the Rosenblatt distribution
7. Other recent examples of GGC distributions

The Rosenblatt process

Let $0 < D < \frac{1}{2}$. The Rosenblatt process is defined, for $t \geq 0$, as

$$Z_D(t) = C(D) \int'_{\mathbb{R}^2} \left(\int_0^t (u - s_1)_+^{-(1+D)/2} (u - s_2)_+^{-(1+D)/2} du \right) dB(s_1) dB(s_2),$$

where $\{B(s), s \in \mathbb{R}\}$ is a standard Brownian motion, $\int'_{\mathbb{R}^2}$ is the integral over $\mathbb{R}^2$ except the hyperplane $s_1 = s_2$ and $C(D)$ is a normalizing constant. The distribution of $Z_D(1)$ is called the Rosenblatt distribution.

The Rosenblatt process is H-selfsimilar with $H = 1 - D$ and has stationary increments.
The Rosenblatt process lives in the so-called second Wiener chaos.
Consequently, it is not a Gaussian process.
In the last few years, this stochastic process has been the object of several papers. (See Pipiras-Taqqu(2010), Tudor(2008), Tudor-Viens(2009), Veillette-Taqqu(2012) among others.)

The Rosenblatt distribution is the first example of non-Gaussian limiting distribution of normalized partial sums of some strongly dependent stationary random variables discovered by Rosenblatt(1961), but very few things had been known about its distributional properties.

The Rosenblatt distribution is known to be infinitely divisible.
Recently, in the work by Veillette-Taqqu(2012), they computed, among others, its Lévy measure, its cumulants and moments. They also derived numerically the shape of the density.

The Thorin class (I)

We first define the Thorin class of probability distributions on $\mathbb{R}_+$. Originally this class was studied by Thorin(1977, two papers) when he wanted to prove the infinite divisibility of the Pareto distribution and of the lognormal distribution. The Thorin class on $\mathbb{R}_+$, denoted by $T(\mathbb{R}_+)$, is the smallest class of distributions on $\mathbb{R}_+$ that contains all gamma distributions and is closed under convolution and weak convergence.
A probability distribution in $T(\mathbb{R}_+)$ is called generalized gamma convolution (GGC).

This class was extended to $\mathbb{R}^d$ by Barndorff-Nielsen-M.-Sato(2006) as follows: Call Γx an elementary gamma random variable in $\mathbb{R}^d$ if x is a non-random non-zero vector in $\mathbb{R}^d$ and Γ is a gamma random variable on $\mathbb{R}_+$. Then the Thorin class on $\mathbb{R}^d$, denoted by $T(\mathbb{R}^d)$, is defined as the smallest class of distributions on $\mathbb{R}^d$ that contains all elementary gamma distributions on $\mathbb{R}^d$ and is closed under convolution and weak convergence. (The Thorin class on $\mathbb{R}$ is already defined in Thorin(1978) as the name of the extended generalized gamma convolutions (EGGC).)

This class is a subclass of the class of selfdecomposable distributions, but *it is surprisingly rich*.

1. L. Bondesson, Generalized Gamma Convolutions and Related Classes of Distributions and Densities. Lecture Notes in Statistics **76**, Springer, 1992.
2. L.F. James, B. Roynette and M. Yor, Generalized gamma convolutions, Dirichlet means, Thorin measures, with explicit examples, *Probability Surveys* **5** (2008), 341-415.

The Thorin class (II)

For any infinitely divisible distribution μ on $\mathbb{R}^d$, we have the following Lévy-Khintchine representation of the characteristic function $\widehat{\mu}(z), z \in \mathbb{R}^d$, of μ:

$$\widehat{\mu}(z) = \exp\left\{ -\tfrac{1}{2}\langle z, Az\rangle + i\langle \gamma, z\rangle + \int_{\mathbb{R}^d} \left(e^{i\langle z,x\rangle} - 1 - \frac{i\langle z,x\rangle}{1+|x|^2} \right) \nu(dx) \right\},$$

where A is a symmetric nonnegative-definite $d \times d$ matrix, ν is a measure on $\mathbb{R}^d$ (called Lévy measure) satisfying

$$\nu(\{0\}) = 0 \quad \text{and} \quad \int_{\mathbb{R}^d} (1 \wedge |x|^2)\nu(dx) < \infty,$$

and $\gamma \in \mathbb{R}^d$.

• Polar decomposition of Lévy measure ν

$$\nu(B) = \int_S \lambda(d\xi) \int_0^\infty 1_B(r\xi)\nu_\xi(dr), \quad B \in \mathcal{B}(\mathbb{R}^d \setminus \{0\}),$$

where λ is a measure on $S = \{\xi \in \mathbb{R}^d : |\xi| = 1\}$,
$\{\nu_\xi : \xi \in S\}$ is a family of positive measures on $(0, \infty)$,
We call ν_ξ the radial component of ν.

When ν_ξ is differentiable, its density $\ell_\xi(r)$ $(\nu_\xi(dr) = \ell_\xi(r)dr)$ is called the Lévy density.

$\mu \in T(\mathbb{R}^d)$ if and only if $r\ell_\xi(r)$ is completely monotone in $r \in (0, \infty)$.

Namely,

$$\ell_\xi(r) = \frac{k_\xi(r)}{r}$$

and $k_\xi(r)$ is completely monotone.

Representation of $Z_D(t)$

Let W be a complex-valued Gaussian random measure on $\mathbb{R}$ such that for Borel sets in $\mathbb{R}$, A, B, A_j, $E[W(A)] = 0, E[W(A)\overline{W(B)}] =$ the Lebesgue measure of $A \cap B$, $W\left(\bigcup_{j=1}^n A_j\right) = \sum_{j=1}^n W(A_j)$ for mutually disjoint sets $A_1, ..., A_n$ and $W(A) = \overline{W(-A)}$.
Let

$$\mathcal{H}_D = \Big\{ h : h \text{ is a complex-valued function on } \mathbb{R}, h(x) = \overline{h(-x)}, \\ \int_{\mathbb{R}} h(x)^2 |x|^{D-1} dx < \infty \Big\}$$

and for every $t \geq 0$ define an integral operator A_t by

$$A_t h(x) = C(D) \int_{-\infty}^{\infty} \frac{e^{it(x-y)} - 1}{i(x-y)} h(y)|y|^{D-1} dy, \quad h \in \mathcal{H}_D.$$

Since A_t is a self-adjoint Hilbert-Schmidt operator (see Dobrushin-Major(1979)), all eigenvalues $\lambda_n(t), n = 1, 2, ...,$ are real and satisfy $\sum_{n=1}^{\infty} \lambda_n^2(t) < \infty$.

Our first theorem is as follows.

Theorem (1)

For every $t_1, ..., t_d \geq 0$*,*

$$(Z_D(t_1), ..., Z_D(t_d)) \stackrel{\mathrm{d}}{=} \left(\sum_{n=1}^{\infty} \lambda_n(t_1)(\varepsilon_n^2 - 1), ..., \sum_{n=1}^{\infty} \lambda_n(t_d)(\varepsilon_n^2 - 1) \right),$$

where $\{\varepsilon_n\}$ *are i.i.d.* $N(0,1)$ *random variables.*

The case $d = 1$ was shown by Taqqu (see Proposition 2 of Dobrushin-Major(1979)).

The proof is enough to extend the idea of Taqqu from one dimension to multi-dimension.

Theorem (2)

For every $t_1, ..., t_d \geq 0$, the law of $(Z_D(t_1), ..., Z_D(t_d))$ belongs to $T(\mathbb{R}^d)$.

Proof. By Theorem (1),

$$
\begin{aligned}
(Z_D(t_1),&..., Z_D(t_d)) \\
&\stackrel{\mathrm{d}}{=} \left(\sum_{n=1}^{\infty} \lambda_n(t_1)(\varepsilon_n^2 - 1), ..., \sum_{n=1}^{\infty} \lambda_n(t_d)(\varepsilon_n^2 - 1)\right) \\
&= \sum_{n=1}^{\infty} \varepsilon_n^2(\lambda_n(t_1), ..., \lambda_n(t_d)) - \left(\sum_{n=1}^{\infty} \lambda_n(t_1), ..., \sum_{n=1}^{\infty} \lambda_n(t_d)\right),
\end{aligned}
$$

where $\varepsilon_n^2(\lambda_n(t_1), ..., \lambda_n(t_d)), n = 1, 2, ...,$ are the elementary gamma random variables in $\mathbb{R}^d$. Since they are independent, by the properties of the class $T(\mathbb{R}^d)$ that the class is closed under convolution and weak convergence, we see that $\sum_{n=1}^{\infty} \varepsilon_n^2(\lambda_n(t_1), ..., \lambda_n(t_d))$ belongs to $T(\mathbb{R}^d)$, and so does $(Z_D(t_1), ..., Z_D(t_d))$. This completes the proof. $\square$

The non-symmetric Rosenblatt process

In M.-Tudor(2012), we extended the Rosenblatt process and introduced the non-symmetric Rosenblatt process as follows.
Let $0 < D_1 \neq D_2 < \frac{1}{2}$. The non-symmetric Rosenblatt process is defined as

$$Z_{D_1,D_2}(t) = C(D_1, D_2) \int_{\mathbb{R}^2}' \left(\int_0^t (u - s_1)_+^{-(1+D_1)/2} (u - s_s)_+^{-(1+D_2)/2} du \right) dB(s_1) dB(s_2).$$

This process is selfsimilar with stationary increments in the second Wiener chaos. When $D_1 = D_2 = D$, it is the Rosenblatt process $Z_D(t)$.

Theorem

For every $t_1, ..., t_d \geq 0$, the law of $(Z_{D_1,D_2}(t_1), ..., Z_{D_1,D_2}(t_d))$ belongs to $T(\mathbb{R}^d)$.

Proof. For $\alpha_1, ...\alpha_d \in \mathbb{R}$,

$$\begin{aligned}
&\alpha_1 Z_{D_1,D_2}(t_1) + \cdots + \alpha_d Z_{D_1,D_2}(t_d) \\
&\quad \stackrel{\mathrm{d}}{=} \int''_{\mathbb{R}^2} \left(\alpha_1 \frac{e^{it_1(x_1+x_2)} - 1}{i(x_1+x_2)} + \cdots + \alpha_d \frac{e^{it_d(x_1+x_2)} - 1}{i(x_1+x_2)} \right) \\
&\qquad\qquad |x_1|^{(D-1)/2} |x_2|^{(D-1)/2} W(dx_1) W(dx_2) \\
&\quad \stackrel{\mathrm{d}}{=} \sum_{n=1}^{\infty} \lambda_n^{(D_1,D_2)}(t_1, ..., t_d)(\varepsilon_n^2 - 1),
\end{aligned}$$

where $\int''_{\mathbb{R}^2}$ is the integral over $\mathbb{R}^2$ except the hyperplanes $x_1 = \pm x_2$ and $\lambda_n^{(D_1,D_2)}(t_1, ..., t_d)$ are eigenvalues of some integral operator. This can be shown by extending the idea for $d = 1$ of Proposition 2 of Dobrushin-Major(1979) to the case $d \geq 1$.

The unimodality of the Rosenblatt distribution

Let us consider the Rosenblatt distribution when $d = 1$.
(The following are the same for the non-symmetric Rosenblatt distribution.)

We have seen that their distributions belong to the Thorin class $T(\mathbb{R})$.

Thus, the Rosenblatt distribution is selfdecomposable, and hence it has the density and *unimodal*, which was suggested numerically by Veillette-Taqqu(2012).

Stochastic integral representations with respect to Lévy process of the Rosenblatt distribution

The Rosenblatt distribution is represented by double Wiener-Itô integral. However, the distributions in $T(\mathbb{R})$ have several stochastic integral representations with respect to Lévy processes.
We regard them as members of the class of selfdecomposable distributions, which is a larger class than the Thorin class.
This allows us to obtain a new result related to the Rosenblatt distribution.

We know that any selfdecomposable random variable X has the stochastic integral representation with respect to some Lévy process $\{Z_t\}$ in law. Namely, $X \stackrel{\mathrm{d}}{=} \int_0^\infty e^{-t} dZ_t$. However, for the Rosenblatt distribution, we can give an explicit form of $\{Z_t\}$.

Aoyama-M.-Ueda(2011) showed that if $\{\gamma_{t,\lambda}, t \geq 0\}$ is a gamma process with parameter $\lambda > 0$, $\{N(t), t \geq 0\}$ is a Poisson process with unit rate and they are independent, then

$$\gamma_{\frac{1}{2},\frac{1}{2}} \stackrel{\mathrm{d}}{=} \int_0^\infty e^{-t} d\gamma_{N(\frac{1}{2}t),\frac{1}{2}}.$$

Let

$$Y_t = \gamma_{N(\frac{1}{2}t),\frac{1}{2}} - t.$$

Note that $\{Y_t, t \geq 0\}$ is a Lévy process. Then we have

$$\varepsilon_n^2 - 1 \stackrel{\mathrm{d}}{=} \gamma_{\frac{1}{2},\frac{1}{2}} - 1 \stackrel{\mathrm{d}}{=} \int_0^\infty e^{-t} dY_t,$$

Let $\{Y_t^{(n)}\}$ be independent copies of $\{Y_t\}$. Then

$$Z_D \stackrel{\mathrm{d}}{=} \sum_{n=1}^\infty \lambda_n(\varepsilon_n^2 - 1) \stackrel{\mathrm{d}}{=} \int_0^\infty e^{-t} d\left(\sum_{n=1}^\infty \lambda_n Y_t^{(n)}\right) =: \int_0^\infty e^{-t} dZ_t.$$

Remark

$\sum_{n=1}^{\infty} \lambda_n Y_t^{(n)}$ is convergent a.s. and in L^2 because

$$\sum_{n=1}^{\infty} E\left[\left(\lambda_n Y_t^{(n)}\right)^2\right] = E\left[Y_t^2\right] \sum_{n=1}^{\infty} \lambda_n^2 < \infty.$$

Remark

Since $\{Y_t^{(n)}\}, n = 1, 2, \ldots$, are independent and identically distributed Lévy processes, their infinite weighted sum $\{Z_t\}$ is a Lévy process.

We thus finally have the following theorem.

Theorem

$$Z_D \stackrel{\mathrm{d}}{=} \int_0^{\infty} e^{-t} dZ_t,$$

where $\{Z_t\}$ is a Lévy process defined above.

Other recent examples of GGC.

Bertoin-Fujita-Roynette-Yor(2006) (Random excursion of Bessel processes)

Let $\{R_t, t \geq 0\}$ be a Bessel process with $R_0 = 0$, with dimension $d = 2(1-\alpha)$. $(0 < \alpha < 1$, equivalently $0 < d < 2.)$
When $\alpha = \frac{1}{2}$, $\{R_t\}$ is a Brownian motion. Let

$$g_t^{(\alpha)} := \sup\{s \leq t : R_s = 0\},$$

$$d_t^{(\alpha)} := \inf\{s \geq t : R_s = 0\}$$

and

$$\Delta_t^{(\alpha)} := d_t^{(\alpha)} - g_t^{(\alpha)},$$

which is the length of the excursion above 0, straddling t, for the process $\{R_u, u \geq 0\}$, and let ε be a standard exponential random variable independent of $\{R_u, u \geq 0\}$. Let $\Delta_\alpha := \Delta_\varepsilon^{(\alpha)}$. Then

$$\mathcal{L}(\Delta_\alpha) \in T(\mathbb{R}_+)(\subset L(\mathbb{R}_+)).$$

In Bertoin-Fujita-Roynette-Yor(2006), only "$\in L(\mathbb{R})$" is mentioned. However, they actually showed that

$$E\left[e^{-\lambda\Delta_\alpha}\right] = \exp\left\{-(1-\alpha)\int_0^\infty \left(1-e^{-\lambda x}\right)\frac{E[e^{-xG_\alpha}]}{x}dx\right\},\ \lambda > 0,$$

with a random variable G_α. (The density function of G_α is explicitly given.) Since $k(x) := E[e^{-xG_\alpha}]$ is completely monotone by Bernstein's theorem, $\mathcal{L}(\Delta_\alpha)$ belongs not only to $L(\mathbb{R}_+)$ but also to $T(\mathbb{R}_+)$.

Handa(2012)
(Continuous state branching processes with immigration)

CBCI-process with quadruplet (a, b, n, δ) :

Continuous state **B**ranching process with **C**ontinuous **I**mmigration with the generator

$$L_\delta f(x) = axf''(x) - bxf'(x) + x\int_0^\infty [f(x+y) - f(x) - yf'(x)]n(dy) + \delta f'(x),$$

where n is a measure on $(0,\infty)$ satisfying $\int_0^\infty (y \wedge y^2)n(dy) < \infty$.

If we restrict ourselves to GGC on $\mathbb{R}_+$, $\mu \in T(\mathbb{R}_+)$ has the Laplace transform:

$$\pi(s) := \int_0^\infty e^{-sx}\mu(dx), \quad s > 0,$$
$$= \exp\left\{-\gamma s - \int_0^\infty (1 - e^{-sx})\nu(dx)\right\},$$

where $\gamma \geq 0$ and $\int_0^\infty (1 \wedge x)\nu(dx) < \infty$.

Note that

$\mu \in T(\mathbb{R}_+) \Leftrightarrow \nu(dx) = \ell(x)dx$ and $x\ell(x)$ is completely monotone on $(0, \infty)$.

Since the completely monotone function is the Laplace transform of some σ-finite and positive measure (let's say, σ) by Bernstein's theorem, we have

$$x\ell(x) = \int_0^\infty e^{-xy}\sigma(dy).$$

When ℓ is the Lévy density of GGC, we call σ the Thorin measure.

Recall

$$\begin{aligned}\pi(s) :=& \int_0^\infty e^{-sx}\mu(dx), \quad s>0,\\ =& \exp\left\{-\gamma s - \int_0^\infty (1-e^{-sx})\nu(dx)\right\}\\ =& \exp\left\{-\gamma s - \int_0^\infty (1-e^{-sx})\frac{1}{x}\left(\int_0^\infty e^{-xy}\sigma(dy)\right)dx\right\}.\end{aligned}$$

Since GGC on $\mathbb{R}_+$ is determined by γ and σ, we call it the GGC with pair (γ, σ).

Theorem

Let $\gamma \geq 0$ and suppose that σ is a non-zero Thorn measure.

(1) There exist (a, b, M) such that

$$\gamma + \int \frac{1}{s+u}\sigma(du) = \frac{1}{as + b + \int \frac{s}{s+u} M(du)}, s > 0.$$

(2) Any GGC with pair (γ, σ) is a unique stationary solution of CBCI-process with quadruplet $(a, b, n, 1)$, where n is a measure on $(0, \infty)$ defined by

$$n(dy) = \left(\int_0^\infty u^2 e^{-yu} M(du) \right) dy.$$

Takemura-Tomisaki(2012) (Lévy density of inverse local time of some diffusion processes)

Example (Also, Shilling-Song-Vondraček(2010),p.201)

Let $I = (0, \infty)$ and $-1 < \nu < 0$.
Let $\mathcal{G}^{(\nu)} = \frac{1}{2}\frac{d^2}{dx^2} + \frac{2\nu+1}{2x}\frac{d}{dx}$.
Assume 0 is reflecting.
$\mathbb{D}^{(\nu)}$: the diffusion process on I with the generator $\mathcal{G}^{(\nu)}$
$n^{(\nu)}$: the Lévy density of the inverse local time at 0 for $\mathbb{D}^{(\nu)}$

$$\Longrightarrow \qquad n^{(\nu)}(x) = C\frac{1}{x}x^{-|\nu|} \quad \text{(GGC)}$$

Example

Let $I = (0, \infty)$ and $-1 < \nu < 0$.

$\mathcal{G}^{(\nu)} = 2x\frac{d^2}{dx^2} + (2\nu + 2)\frac{d}{dx}$

$\mathbb{D}^{(\nu)}$: the diffusion process with the generator $\mathcal{G}^{(\nu)}$
and the end point 0 being reflecting.

$n^{(\nu)}$: the Lévy density of the inverse local time at 0 for $\mathbb{D}^{(\nu)}$

$$\Longrightarrow \qquad n^{(\nu)}(x) = C\frac{1}{x}x^{-|\nu|} \quad \text{(GGC)}$$

Example

Let $-1 < \nu < 1$ and $\beta > 0$. Let

$$\mathcal{G}^{(\nu,\beta)} = \frac{1}{2}\frac{d^2}{dx^2} + \left\{ \frac{1}{2x} + \sqrt{2\beta}\frac{K'_\nu(\sqrt{2\beta}x)}{K_\nu(\sqrt{2\beta}x)} \right\} \frac{d}{dx},$$

where $K_\nu(x)$ is the modified Bessel function.

$\mathbb{D}^{(\nu,\beta)}$: the diffusion process on I with the generator $\mathcal{G}^{(\nu,\beta)}$ and the end point 0 being reflecting.

$n^{(\nu,\beta)}$: the Lévy density of the inverse local time at 0 for $\mathbb{D}^{(\nu,\beta)}$

$$\Longrightarrow \qquad n^{(\nu,\beta)}(x) = C\tfrac{1}{x}x^{-|\nu|}e^{-\beta x}. \quad \text{(GGC)}$$

(When $\nu = 0$, Shilling-Song-Vondraček(2010),p.202.)

Example (Shilling-Song-Vondraček(2010),p.201)

Let $0 < \nu < 1$ and $\beta > 0$. Let

$$\mathcal{G}^{(\nu,\beta)} = \frac{1}{2}\frac{d^2}{dx^2} + \left\{ \frac{\beta - 1}{2x} + \sqrt{2\beta}\frac{K_\nu'(\sqrt{2\beta}x)}{K_\nu(\sqrt{2\beta}x)} \right\} \frac{d}{dx},$$

where $K_\nu(x)$ is the modified Bessel function.

$\mathbb{D}^{(\nu,\beta)}$: the diffusion process on I with the generator $\mathcal{G}^{(\nu,\beta)}$ and the end point 0 being reflecting.

$n^{(\nu,\beta)}$: the Lévy density of the inverse local time at 0 for $\mathbb{D}^{(\nu,\beta)}$

$$\Longrightarrow \qquad n^{(\nu,\beta)}(x) = C\tfrac{1}{x}x^{-\nu}e^{-\beta x}. \quad \text{(GGC)}$$

Example

Let $-1 < \nu < 1$ and $\beta > 0$. Let

$$\mathcal{G}^{(\nu,\beta)} = 2x\frac{d^2}{dx^2} + 2\left\{1 + \sqrt{2\beta x}\frac{K_\nu'(\sqrt{2\beta x})}{K_\nu(\sqrt{2\beta x})}\right\}\frac{d}{dx}.$$

$\mathbb{D}^{(\nu,\beta)}$: the diffusion process with the generator $\mathcal{G}^{(\nu,\beta)}$
and the end point 0 being reflecting.

$n^{(\nu,\beta)}$: the Lévy density of the inverse local time at 0 for $\mathbb{D}^{(\nu,\beta)}$

$$\Longrightarrow \qquad n^{(\nu,\beta)}(x) = C\tfrac{1}{x}x^{-|\nu|}e^{-\beta x}. \quad \text{(GGC)}$$

Thank you very much!