

Asymptotic behaviour of the first passage time distribution for subordinators

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Let X be a real valued Lévy process with characteristic exponent Ψ , associated to the characteristics (a, σ^2, Π) ,

$$\mathbb{E}(\exp\{i\lambda X_t\}) = \exp\{-t\Psi(\lambda)\},$$

$$\Psi(\lambda) = -ia\lambda + \frac{\lambda^2\sigma^2}{2} + \int_{\mathbb{R}\setminus\{0\}} \left(1 - e^{i\lambda x} + i\lambda x 1_{\{|x|<1\}}\right) \Pi(dx), \quad \lambda \in \mathbb{R}$$

$a \in \mathbb{R}$, $\sigma \in \mathbb{R}$, and $\int_{\mathbb{R}\setminus\{0\}} 1 \wedge x^2 \Pi(dx) < \infty$.

If X is a **subordinator**, i.e. has non-decreasing paths, then it is characterized via its Laplace exponent ψ

$$\mathbb{E}(\exp\{-\lambda X_t\}) = \exp\{-t\psi(\lambda)\},$$

$$\psi(\lambda) = b\lambda + \int_0^\infty (1 - e^{-\lambda x}) \Pi(dx), \quad \lambda \geq 0.$$

For $x > 0$, let $T_x = \inf\{t \geq 0 : X_t > x\}$, be the first passage time above level x for X .

Aim: under some assumptions on X , describe the local behaviour of the first passage time distribution for X , i.e. obtain estimates for

$$\mathbb{P}(T_x \in (t, t + \Delta]),$$

uniform in $0 < \Delta \leq \Delta_0$, and in x in different regimes, when either $t \rightarrow \infty$ or $t \rightarrow 0$.

More precisely, we will describe the contributions

$$h_x^C(t, \Delta) = \mathbb{P}(T_x \in (t, t + \Delta], X_{T_x} = x), \quad \text{creeping,}$$

and

$$h_x^J(t, \Delta) = \mathbb{P}(T_x \in (t, t + \Delta], X_{T_x-} < x), \quad \text{discontinuous crossing.}$$

There are just a few cases where the law of T_x is known explicitly.

Example

If X is a stable subordinator with index α , $\psi(\lambda) = \lambda^\alpha$,

$$T_x \stackrel{\text{Law}}{\equiv} x^\alpha X_1^{-\alpha} \sim x^\alpha \text{Mittag-Leffler}(\alpha).$$

In a work with R. Doney, we described the case of a non-monotone process when $\exists c : [0, \infty) \rightarrow [0, \infty)$ s.t. $\frac{X_t}{c(t)} \xrightarrow[t \rightarrow \infty]{\text{Law}} Y_1$, with Y an α -stable process.

- We obtained estimates for

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- We proved that uniformly in x

$$\lim_{t \rightarrow \infty} \frac{h_x^C(t, \Delta)}{h_x^J(t, \Delta)} = 0,$$

unless Y is spectrally negative and the upward ladder height subordinator of X has finite mean, in which case both contributions are equivalent.

We say that a subordinator is *stochastically compact or belongs to the Feller class* at infinity, respectively at 0,

[SC] if

$$\limsup \frac{y^2 \bar{\Pi}(y)}{\int_0^y z^2 \Pi(dz)} < \infty$$

as $y \rightarrow \infty$, respectively as $y \rightarrow 0+$, where

$$\bar{\Pi}(y) := \Pi(y, \infty), \quad y > 0.$$

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[SC'] equivalently, if $\exists \alpha \in (0, 2]$ and $c \geq 1$ such that

$$\limsup \frac{\int_0^{\lambda z} y \bar{\Pi}(y) dy}{\int_0^z y \bar{\Pi}(y) dy} \leq c \lambda^{2-\alpha} \text{ for } \lambda > 1, \text{ as } z \rightarrow \infty,$$

respectively as $z \rightarrow 0+$;

Example:

- $\Pi(x, \infty) = x^{-\alpha} \ell(x)$, where ℓ is slowly varying at infinity, resp. at 0, and $\alpha \in (0, 1)$. Hence X is in the domain of attraction of an α -stable law. Taking $c(t)$ s.t.

$$\int_0^\infty (1 - e^{-y/c(t)}) \Pi(dy) = \frac{1}{t},$$

$$\frac{X_t - bt}{c(t)} \xrightarrow[t \rightarrow \infty (t \rightarrow 0)]{\text{Law}} \mathcal{S}_\alpha \sim \alpha - \text{stable}$$

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- $\Pi(x, \infty) = x^{-\beta} \ell(x)$, where ℓ is a function that is slowly varying at infinity and $\beta \in [1, 2)$. There exists a function $c : [0, \infty) \rightarrow (0, \infty)$ s.t.

$$\frac{X_t - t\mu}{c(t)} \xrightarrow[t \rightarrow \infty]{} \tilde{\mathcal{S}}_\beta \sim \text{SP} - \beta - \text{stable},$$

with $\mu = \mathbb{E}(X_1)$,

- $\Pi(dx) = \pi(x)dx$, with $\pi(x) \asymp x^{-\alpha-1}\ell(x)$, for $\alpha \in (0, 1)$ for large (small) values of x

Theorem (Maller and Mason (2009, 2010))

If X is stochastically compact at infinity (respectively at zero),
 $\exists c : [0, \infty) \rightarrow (0, \infty)$ and $\mathbf{b} : [0, \infty) \rightarrow [0, \infty)$ s.t. for any sequence
 $(t_k, k \geq 0)$ tending towards infinity (respectively, towards 0) there is
 a subsequence $(t'_k, k \geq 0)$ such that

$$\frac{X_{t'_k} - \mathbf{b}(t'_k)}{c(t'_k)} \xrightarrow[k \rightarrow \infty]{\text{Law}} Y', \quad (1)$$

where Y' is a real valued non-degenerate random variable, whose law may depend on the subsequence taken. The functions may be taken to solve, for all $t > 0$

$$\frac{1}{t} = 2(c(t))^{-2} \int_0^{c(t)} y \bar{\Pi}(y) dy, \quad \mathbf{b}(t) = t \left(b + \int_0^{c(t)} y \Pi(dy) \right)$$

Theorem (Jain and Pruitt (1987))

Let $x > 0$, $b < x_t := x/t < \mu =: \mathbb{E}(X_1)$. Assume either

(SC₀-I) the Lévy measure Π satisfies the condition SC_0 , $t \rightarrow \infty$, $x_t \rightarrow b$;

(SC₀-II) the Lévy measure Π satisfies the condition SC_0 , $t \rightarrow 0$, $x_t \rightarrow b$, and “a technical condition on x ”.

(SC_∞-I) the Lévy measure Π satisfies the condition SC_∞ , $t \rightarrow \infty$, $x_t \rightarrow \mu$, and “a technical condition on x ”.

(G) $t \rightarrow \infty$ and $b < \liminf_{t \rightarrow \infty} \frac{x}{t} \leq \limsup_{t \rightarrow \infty} \frac{x}{t} < \mu$, and X is non-lattice.

With $\psi'(\rho_t) = x/t$, $H(\rho_t) = \psi(\rho_t) - \rho_t \psi'(\rho_t)$,

$\sigma^2(\rho_t) = \int_0^\infty y^2 e^{-\rho_t y} \Pi(dy)$, we have $tH(\rho_t) \rightarrow \infty$ and

$$\mathbb{P}(X_t \leq x) = \mathbb{P}(T_x \geq t) \sim \frac{e^{-tH(\rho_t)}}{\sqrt{2\pi t} \sigma(\rho_t) \rho_t}.$$

The “technical condition on x ” is that if

$$\limsup_{y \rightarrow 0(y \rightarrow \infty)} \frac{y(b + \int_0^y z \Pi(dz))}{\int_0^y z^2 \Pi(dz)} < \infty, \quad (2)$$

equivalently

$$\limsup_{t \rightarrow 0(t \rightarrow \infty)} \frac{\mathbf{b}(t)}{c(t)} < \infty. \quad (3)$$

then

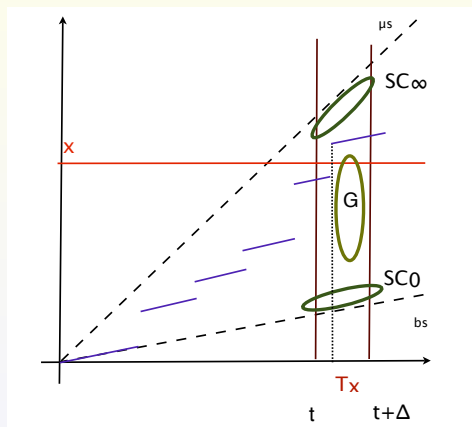
$$\frac{x - bt}{c(t)} \rightarrow 0 \quad \text{as } t \rightarrow 0(t \rightarrow \infty)$$

If (2) does not hold, then

$$\frac{x - \mathbf{b}(t)}{c(t)} \rightarrow -\infty \quad \text{as } t \rightarrow 0(t \rightarrow \infty)$$

We will say that we are in the situation $x - bt = o(c(t))$

Barriers and hypotheses



$$\mathbb{P}(T_x \in (t, t+\Delta], X_{T_x} = x) \sim ?$$

$$\mathbb{P}(T_x \in (t, t+\Delta], X_{T_x-} < x) \sim ?$$

Lemma (First passage time density, M. Meerschaert, H. Scheffler 2008, Doney & R. 2013, Kesten 1969, Winkel 2005, Griffin & Maller 2011)

$$\frac{\mathbb{P}(T_x \in dt, X_{T_x-} < x)}{dt} = \int_0^x \mathbb{P}(X_t \in dy) \bar{\Pi}(x - y), \quad x > 0, t > 0.$$

If $b > 0$,

$$\int_0^\infty dt \mathbb{P}(X_t \in dx) = u(x)dx, \quad \mathbb{P}(X_{T_x} = x) = bu(x),$$

$$\mathbb{P}(T_x \leq t, X_{T_x} = x) = b\partial_z \int_0^t ds \mathbb{P}(X_s \leq z)|_{z=x}, \quad x > 0, t \geq 0.$$

If X_s has a density $f_s(\cdot)$, $s \geq 0$, then

$$\mathbb{P}(T_x \in dt, X_{T_x} = x) = bf_t(x), \quad x > 0, t \geq 0.$$

Theorem

Suppose that X is a subordinator which has drift $b \geq 0$ and Lévy measure Π . For $b < x/t < \mu := \mathbb{E}(X_1)$, define $x_t := x/t$ and $\rho_t := \rho(x/t)$, that is $\psi'(\rho_t) = x/t$, $H(u) = \psi(u) - u\psi'(u)$, and $\sigma^2(u) = \int_0^\infty y^2 e^{-uy} \Pi(dy)$.

- (i) If X is stochastically compact at 0, the unidimensional law of X admits a density, say $\mathbb{P}(X_t \in dy) = f_t(y)dy$, $y \geq 0$, and such that $f_t \in C^\infty(\mathbb{R})$.
- (ii) In the settings (SC₀-(I-II)) we have the estimate

$$\sqrt{t}\sigma(\rho_t)f_t(z) = \left(\phi((z-x)/\sqrt{t}\sigma(\rho_t)) + o(1) \right) e^{-tH(\rho_t)} e^{\rho_t(z-x)}, \quad (4)$$

uniformly in $z > 0$ and x , with ϕ the standard normal density.

To prove the first claim we establish that there is $0 < \beta < 2$ and a constant c_1 s.t.

$$\Re(\psi(-i\theta)) = \int_0^\infty (1 - \cos(\theta y))\Pi(dy) \geq c_1|\theta|^\beta.$$

$$\int_{\mathbb{R}} |\theta|^n e^{-\Re(\psi(-i\theta))} d\theta < \infty, \quad \forall n \geq 1.$$

To prove the second claim we introduce a change of measure. Let $(Y_s, s \geq 0)$, a subordinator whose Laplace exponent is given by ψ_{ρ_t} ,

$$\psi_{\rho_t}(\lambda) = \psi(\rho_t + \lambda) - \psi(\rho_t) = b\lambda + \int_{(0, \infty)} (1 - e^{-\lambda y}) e^{-\rho_t y} \Pi(dy), \quad \lambda \geq 0. \quad (5)$$

In particular we have the following key relation:

$$\mathbb{P}(Y_t \in dy) = e^{tH(\rho_t)} e^{-\rho_t(y - tx_t)} \mathbb{P}(X_t \in dy), \quad y \in \mathbb{R}^+. \quad (6)$$

Y has moment of all orders, we use a Normal approximation

To prove the second claim we introduce a change of measure. Let $(Y_s^{(x_t)}, s \geq 0)$, a subordinator whose Laplace exponent is given by ψ_{ρ_t} ,

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$Y^{(x_t)}$ has moment of all orders, we use a Normal approximation

Lemma (Petrov 1975)

Let Z_1, Z_2, \dots, Z_n be independent rvs having finite 3rd moments, write $\mathbb{E}Z_r = \mu_r$, $\text{Var}(Z_r) = \sigma_r^2$, and $\mathbb{E}|Z_r - \mu_r|^3 = \nu_r$, and put $W = \sum_1^n Z_r$, $m = \mathbb{E}(W) = \sum_1^n \mu_r$, and $s^2 = \text{Var}(W) = \sum_1^n \sigma_r^2$. Assume further that $\int_{-\infty}^{\infty} |\tilde{\Psi}(u)| du < \infty$, where $\tilde{\Psi}(u) = \mathbb{E}(e^{iuW})$, and denote by f and ϕ the pdf of W and the standard Normal pdf. Then there is an absolute constant A such that

$$\sup_y |f(y) - s^{-1} \phi(\frac{y - m}{s})| \leq AL + d, \quad (7)$$

where $L = \sum_1^n \nu_r / s^4$ and, with $l = (4Ls^2)^{-1}$,

$$d = 2 \int_l^{\infty} |\Psi(u)| du.$$

We apply the previous estimate with

$W \stackrel{\text{Law}}{=} Y_t = \sum_{r=1}^n Y_{\frac{rt}{[x\rho_t]}} - Y_{\frac{(r-1)t}{[x\rho_t]}}$, $n = [x\rho_t]$, and prove that

$$\sqrt{t}\sigma(\rho_t)L \lesssim \frac{1}{(t\rho_t^2\sigma^2(\rho_t))^{1/2}} \left(6\frac{Q_{\Pi}(1/\rho_t)}{\rho_t^2\sigma^2(\rho_t)} + 2 \right) \rightarrow 0;$$

with

$$Q_{\Pi}(z) = 2z^{-2} \int_0^z y\bar{\Pi}(y)dy, \quad z > 0.$$

$\exists \beta_0 > 0$, s.t.

$$\gamma := \sqrt{t}\sigma(\rho_t) \int_l^{\infty} e^{-t\Re(\psi_{\rho_t}(i\theta))} d\theta \leq \frac{1}{(tH(\rho_t))^{\beta_0}} \rightarrow 0.$$

uniformly in x .

The same method work in general but we should consider $W = Y_t + U_h + \Delta_a = \sum_1^n Z_r$, where $n = [x\rho t] + 2$,
 $Z_r := Y_{\frac{rt}{[x\rho t]}} - Y_{\frac{(r-1)t}{[x\rho t]}} \stackrel{\text{Law}}{=} Y_{\frac{t}{[x\rho t]}}$ for $1 \leq r < n - 1$, and the two independent random variables U_h and Δ_a are independent of Y_t , U_h has a uniform distribution on $[-h, h]$, and $\Delta_a \stackrel{\text{Law}}{=} U_a + \tilde{U}_a$, with U_a and \tilde{U}_a independent. The density of W satisfies

$$n_t(z) = (2h)^{-1} \mathbb{P}(Y_t + \Delta_a \in (z - h, z + h)),$$

Theorem

In the settings $(SC_0-(I-II))$, (SC_∞) and (G) the following estimates

$$\begin{aligned} & \mathbb{P}(X_t \in (x - u, x]) \\ &= \frac{e^{-tH(\rho_t)}}{\sqrt{t}\sigma(\rho_t)} \left(\int_0^u e^{-\rho_t v} \phi\left(\frac{v}{\sqrt{t}\sigma(\rho_t)}\right) dv + o(1) \frac{1 - e^{-u\rho_t}}{\rho_t} \right), \end{aligned}$$

$$\mathbb{P}(X_t \leq x) \sim \frac{e^{-tH(\rho_t)}}{\sqrt{2\pi t}\sigma(\rho_t)\rho_t},$$

hold uniformly in $u < x$ and uniformly in x .

We plug the former estimates in

$$\frac{\mathbb{P}(T_x \in dt, X_{T_x-} < x)}{dt} = \int_0^x \mathbb{P}(X_t \in dy) \bar{\Pi}(x-y), \quad x > 0, t > 0.$$

If $b > 0$,

$$\mathbb{P}(T_x \in (t, t+h], X_{T_x} = x) = b \partial_z \int_t^{t+\Delta} ds \mathbb{P}(X_s \leq z) |_{z=x}, \quad x > 0, t \geq 0.$$

If X_s has a density $f_s(\cdot)$, $s \geq 0$, then

$$\mathbb{P}(T_x \in dt, X_{T_x} = x) = b f_t(x), \quad x > 0, t \geq 0.$$

Theorem

Let $\Delta_0 > 0$ fixed. In the settings $(SC_0\text{-}(I\text{-}II))$, (SC_∞) and (G) , the following estimates

$$h_x^J(t) \sim \frac{e^{-tH(\rho_t)}}{\sqrt{2\pi t\sigma(\rho_t)}} \left(\frac{\psi(\rho_t)}{\rho_t} - b \right), \quad (8)$$

$$h_x^C(t, \Delta) \sim b \int_t^{t+\Delta} \frac{e^{-sH(\rho(\frac{x}{s}))}}{\sqrt{2\pi s\sigma(\rho(\frac{x}{s}))}} ds, \quad (9)$$

hold uniformly in $0 < \Delta < \Delta_0$ and in x . Furthermore, under the settings $(SC_0\text{-}(I\text{-}II))$ the more precise estimate

$$\mathbb{P}(T_x \in dt, X_{T_x} = x) = \frac{b}{\sqrt{2\pi t\sigma(\rho_t)}} (1 + o(1)) e^{-tH(\rho_t)} dt, \quad (10)$$

holds uniformly in x .

Corollary

In the settings (SC₀-(I-II)), the hazard rate of T_x on the event of discontinuous crossing is

$$\frac{h_x^J(t)}{\mathbb{P}(T_x > t)} \sim \int_0^\infty (1 - e^{-\rho t y}) \Pi(dy) \rightarrow 0,$$

and on the event of creeping

$$\frac{h_x^C(t)}{\mathbb{P}(T_x > t)} \sim b \rho t \rightarrow \infty.$$

$$\frac{h_x^C(t)}{h_x^J(t)} \sim \frac{b}{\int_0^\infty e^{-y\rho t} \Pi(y, \infty) dy} \rightarrow b \times \infty.$$

$$\mathbb{P}(T_x \in (t, t + \Delta]) \sim \Delta h_x^C(t)$$

Corollary

Let $\Delta_0 > 0$ fixed and assume $\Pi(y, \infty) = y^{-\alpha} \ell(y)$ at ∞ . Then the following estimates,

$$h_x^J(t) \sim \frac{e^{-tH(\rho_t)}}{\sqrt{2\pi t\sigma(\rho_t)}} \cdot \frac{\bar{\Pi}(1/\rho_t)}{\rho_t},$$

$$h_x^C(t, \Delta) \sim \frac{b\Delta e^{-tH(\rho_t)}}{\sqrt{2\pi t\sigma(\rho_t)}}, \quad (11)$$

hold uniformly in $0 < \Delta < \Delta_0$ and in x such that $x/t \rightarrow \mathbb{E}(X_1)$ and $(x - bt)/c(t) \rightarrow 0$, with c taken as $t\Pi(c(t), \infty) = 1$. We have also

$$\frac{h_x^C(t)}{h_x^J(t)} \sim \frac{b}{\int_0^y e^{-\rho_t y} \Pi(y, \infty) dy} \rightarrow \frac{b}{\int_0^\infty \Pi(y, \infty) dy}$$

Corollary

In the settings $(SC_0\text{-}(I-II))$, (SC_∞) and (G) the estimate

$$\mathbb{P}(\rho_t \times (x - X_t) \in [0, u] | x - X_t > 0) = (1 - e^{-u})(1 + o(1))$$

holds uniformly in $u < x$ and uniformly in x .

Corollary

In the settings $(SC_0\text{-}(I\text{-}II))$, (SC_∞) and (G) the estimate

$$\mathbb{P}(\rho_t \times (x - X_t) \in [0, u] | x - X_t > 0) = (1 - e^{-u})(1 + o(1))$$

holds uniformly in $u < x$ and uniformly in x .

$$\begin{aligned} & \mathbb{P}(X_t \in (x - (u/\rho_t), x] | X_t \leq x) \\ &= \sqrt{2\pi} \int_0^u e^{-v} \phi\left(\frac{v}{\sqrt{t\rho_t}\sigma(\rho_t)}\right) dv + o(1)(1 - e^{-u}), \end{aligned}$$

$$t\rho_t^2\sigma^2(\rho_t) \asymp tH(\rho_t) \rightarrow \infty.$$

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Our estimates are not valid in this setting. Under further assumptions we can use [Stone's Local Limit Theorem](#) to estimate the density of X .

Assume that X is in the domain of attraction of a stable law at infinity, $\Pi(x, \infty) = x^{-\alpha}\ell(x)$ with ℓ slowly varying at infinity, $\alpha \in (0, 1)$. Then uniformly in Δ and $y \in \mathbb{R}$,

$$c(t)\mathbb{P}(X_t \in (y, y + \Delta]) = \Delta(\tilde{g}_1(\frac{y}{c(t)}) + o(1)) \text{ as } t \rightarrow \infty,$$

with \tilde{g}_1 the α -stable density, and $t\Pi(c(t), \infty) = 1$.

Theorem

Suppose now that X is a non-lattice subordinator which has drift $b \geq 0$ and $\bar{\Pi}(\cdot) \in RV(-\alpha)$ at ∞ with $\alpha \in (0, 1)$. Define c by $t\bar{\Pi}(c(t)) = 1$, so that $(X(t\cdot)/c(t))$ converges weakly to S , a stable subordinator of index α . Let $\tilde{g}_t(\cdot)$ and $\tilde{h}_x(\cdot)$ denote the density functions of S_t and $T_x^S := \inf\{t : S_t > x\}$ respectively. Then for any fixed $D > 1$, uniformly for $y_t := x/c(t) \in (D^{-1}, D)$

$$th_x^J(t) = \tilde{h}_{y_t}(1) + o(1) \text{ as } t \rightarrow \infty.$$

and, if $b > 0$, uniformly for $y_t \in (D^{-1}, D)$ and $0 < \Delta < \Delta_0$,

$$c(t)h_x^C(t, \Delta) = \frac{\alpha b \Delta}{1 - \alpha} (\tilde{g}_1(y_t) + o(1)) \text{ as } t \rightarrow \infty.$$

$$\frac{h_x^C(t, \Delta)}{h_x^J(t, \Delta)} \asymp \frac{t}{c(t)} \rightarrow 0, \quad \text{as } t \rightarrow \infty.$$