Asymptotic behaviour of the first passage time distribution for subordinators

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[1] Joint work with Ron Doney (Univ. Manchester, UK)

Let X be a real valued Lévy process with characteristic exponent Ψ, associated to the characteristics (a, σ^2, Π),

$$\mathbb{E}(\exp\{i\lambda X_t\}) = \exp\{-t\Psi(\lambda)\},$$

$$\Psi(\lambda) = -ia\lambda + \frac{\lambda^2\sigma^2}{2} + \int_{\mathbb{R}\setminus\{0\}} \left(1 - e^{i\lambda x} + i\lambda x 1_{\{|x|<1\}}\right) \Pi(dx), \qquad \lambda \in \mathbb{R}$$

$a \in \mathbb{R}$, $\sigma \in \mathbb{R}$, and $\int_{\mathbb{R}\setminus\{0\}} 1 \wedge x^2 \Pi(dx) < \infty$.
If X is a subordinator, i.e. has non-decreasing paths, then it is characterized via its Laplace exponent ψ

$$\mathbb{E}(\exp\{-\lambda X_t\}) = \exp\{-t\psi(\lambda)\},$$

$$\psi(\lambda) = b\lambda + \int_0^\infty (1 - e^{-\lambda x})\Pi(dx), \qquad \lambda \geq 0.$$

For $x > 0$, let $T_x = \inf\{t \geq 0 : X_t > x\}$, be the first passage time above level x for X.

Aim: under some assumptions on X, describe the local behaviour of the first passage time distribution for X, i.e. obtain estimates for

$$\mathbb{P}(T_x \in (t, t + \Delta]),$$

uniform in $0 < \Delta \leq \Delta_0$, and in x in different regimes, when either $t \to \infty$ or $t \to 0$.

More precisely, we will describe the contributions

$$h_x^C(t, \Delta) = \mathbb{P}(T_x \in (t, t + \Delta], X_{T_x} = x), \qquad \text{creeping,}$$

and

$$h_x^J(t, \Delta) = \mathbb{P}(T_x \in (t, t+\Delta], X_{T_x -} < x), \qquad \text{discontinuous crossing.}$$

There are just a few cases were the law of T_x is known explicitly.

Example

If X is a stable subordinator with index α, $\psi(\lambda) = \lambda^\alpha$,

$$T_x \overset{\text{Law}}{=} x^\alpha X_1^{-\alpha} \sim x^\alpha \text{Mittag-Lefler}(\alpha).$$

In a work with R. Doney, we described the case of a non-monotone process when $\exists c : [0, \infty) \to [0, \infty)$ s.t. $\dfrac{X_t}{c(t)} \xrightarrow[t\to\infty]{\text{Law}} Y_1$, with Y an α-stable process.

- We obtained estimates for $h_x^C(t, \Delta) = \mathbb{P}(T_x \in (t, t+\Delta], X_{T_x} = x)$ and $h_x^J(t, \Delta) = \mathbb{P}(T_x \in (t, t+\Delta], X_{T_x -} < x)$, when $t \to \infty$ and either $x/c(t) \to 0$ or $x \geq Dc(t)$, In a work with R. Doney, we described the case of a non-monotone process when $\exists c : [0,\infty) \to [0,\infty)$ s.t. $\frac{X_t}{c(t)} \xrightarrow[t\to\infty]{\text{Law}} Y_1$, with Y an α-stable process.

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- We proved that uniformly in x

$$\lim_{t\to\infty} \frac{h^C_x(t,\Delta)}{h^J_x(t,\Delta)} = 0,$$

unless Y is spectrally negative and the upward ladder height subordinator of X has finite mean, in which case both contributions are equivalent.

We say that a subordinator is *stochastically compact or belongs to the Feller class* at infinity, respectively at 0,

[SC] if

$$\limsup \frac{y^2\overline{\Pi}(y)}{\int_0^y z^2\Pi(dz)} < \infty$$

as $y \to \infty$, respectively as $y \to 0+$, where

$$\overline{\Pi}(y) := \Pi(y, \infty), \qquad y > 0.$$

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[SC'] equivalently, if $\exists \alpha \in (0,2]$ and $c \geq 1$ such that $\limsup \frac{\int_0^{\lambda z} y\overline{\Pi}(y)dy}{\int_0^z y\overline{\Pi}(y)dy} \leq c\lambda^{2-\alpha}$ for $\lambda > 1$, as $z \to \infty$, respectively as $z \to 0+$;

Example:

- $\Pi(x,\infty) = x^{-\alpha}\ell(x)$, where ℓ is slowly varying at infinity, resp. at 0, and $\alpha \in (0,1)$. Hence X is in the domain of attraction of an α-stable law. Taking $c(t)$ s.t.

$$\int_0^\infty (1 - e^{-y/c(t)})\Pi(dy) = \frac{1}{t},$$

$$\frac{X_t - bt}{c(t)} \xrightarrow[t\to\infty(t\to 0)]{\text{Law}} \mathcal{S}_\alpha \sim \alpha - \text{stable}$$

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- $\Pi(x,\infty) = x^{-\beta}\ell(x)$, where ℓ is a function that is slowly varying at infinity and $\beta \in [1,2)$. There exists a function $c : [0,\infty) \to (0,\infty)$ s.t.

$$\frac{X_t - t\mu}{c(t)} \xrightarrow[t\to\infty]{} \widetilde{\mathcal{S}}_\beta \sim \text{SP} - \beta - \text{stable},$$

with $\mu = \mathbb{E}(X_1)$,

- $\Pi(dx) = \pi(x)dx$, with $\pi(x) \asymp x^{-\alpha-1}\ell(x)$, for $\alpha \in (0,1)$ for large (small) values of x

Theorem (Maller and Mason (2009, 2010))

If X is stochastically compact at infinity (respectively at zero), $\exists \mathrm{c} : [0,\infty) \to (0,\infty)$ and $\mathbf{b} : [0,\infty) \to [0,\infty)$ s.t. for any sequence $(t_k, k \geq 0)$ tending towards infinity (respectively, towards 0) there is a subsequence $(t'_k, k \geq 0)$ such that

$$\frac{X_{t'_k} - \mathbf{b}(t'_k)}{\mathrm{c}(t'_k)} \xrightarrow[k\to\infty]{\text{Law}} Y', \tag{1}$$

where Y' is a real valued non-degenerate random variable, whose law may depend on the subsequence taken. The functions may be taken to solve, for all $t > 0$

$$\frac{1}{t} = 2\,(c(t))^{-2}\int_0^{c(t)} y\overline{\Pi}(y)dy, \quad \mathbf{b}(t) = t\left(b + \int_0^{\mathrm{c}(t)} y\Pi(dy)\right)$$

Theorem (Jain and Pruitt (1987))

Let $x > 0$, $b < x_t := x/t < \mu =: \mathbb{E}(X_1)$. *Assume either*

(SC_0-I) *the Lévy measure* Π *satisfies the condition* SC_0, $t \to \infty$, $x_t \to b$;

(SC_0-II) *the Lévy measure* Π *satisfies the condition* SC_0, $t \to 0$, $x_t \to b$, *and "a technical condition on* x*".*

(SC_∞-I) *the Lévy measure* Π *satisfies the condition* SC_∞, $t \to \infty$, $x_t \to \mu$, *and "a technical condition on* x*".*

(G) $t \to \infty$ *and* $b < \liminf_{t\to\infty} \frac{x}{t} \leq \limsup_{t\to\infty} \frac{x}{t} < \mu$, *and* X *is non-lattice.*

With $\psi'(\rho_t) = x/t$, $H(\rho_t) = \psi(\rho_t) - \rho_t\psi'(\rho_t)$, $\sigma^2(\rho_t) = \int_0^\infty y^2 e^{-\rho_t y}\Pi(dy)$, *we have* $tH(\rho_t) \to \infty$ *and*

$$\mathbb{P}(X_t \leq x) = \mathbb{P}(T_x \geq t) \sim \frac{e^{-tH(\rho_t)}}{\sqrt{2\pi t}\sigma(\rho_t)\rho_t}.$$

The "technical condition on x" is that if

$$\limsup_{y\to 0 (y\to\infty)} \frac{y(b+\int_0^y z\Pi(dz))}{\int_0^y z^2\Pi(dz)} < \infty, \tag{2}$$

equivalently

$$\limsup_{t\to 0 (t\to\infty)} \frac{\mathbf{b}(t)}{c(t)} < \infty. \tag{3}$$

then

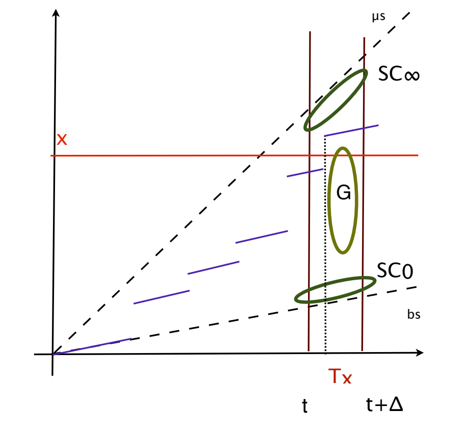
$$\frac{x-bt}{c(t)} \to 0 \quad \text{as } t\to 0 (t\to\infty)$$

If (2) does not hold, then

$$\frac{x-\mathbf{b}(t)}{c(t)} \to -\infty \quad \text{as } t\to 0 (t\to\infty)$$

We will say that we are in the situation $x-bt=o(c(t))$

Barriers and hypotheses



$$\mathbb{P}(T_x \in (t, t+\Delta], X_{T_x} = x) \sim ? \qquad \mathbb{P}(T_x \in (t, t+\Delta], X_{T_x-} < x) \sim ?$$

Lemma (First passage time density, M. Meerschaert, H. Scheffler 2008, Doney & R. 2013, Kesten 1969, Winkel 2005, Griffin & Maller 2011)

$$\frac{\mathbb{P}(T_x \in dt, X_{T_x-} < x)}{dt} = \int_0^x \mathbb{P}(X_t \in dy)\overline{\Pi}(x-y), \quad x > 0, t > 0.$$

If $b > 0$,

$$\int_0^\infty dt\, \mathbb{P}(X_t \in dx) = u(x)dx, \quad \mathbb{P}(X_{T_x} = x) = bu(x),$$

$$\mathbb{P}(T_x \le t, X_{T_x} = x) = b\partial_z \int_0^t ds \mathbb{P}(X_s \le z)|_{z=x}, \; x > 0, \; t \ge 0.$$

If X_s *has a density* $f_s(\cdot)$, $s \ge 0$, *then* $\mathbb{P}(T_x \in dt, X_{T_x} = x) = bf_t(x)$, $x > 0$, $t \ge 0$.

Theorem

Suppose that X is a subordinator which has drift $b \geq 0$ and Lévy measure Π. For $b < x/t < \mu := \mathbb{E}(X_1)$, define $x_t := x/t$ and $\rho_t := \rho(x/t)$, that is $\psi'(\rho_t) = x/t$, $H(u) = \psi(u) - u\psi'(u)$, and $\sigma^2(u) = \int_0^\infty y^2 e^{-uy}\Pi(dy)$.

(i) *If X is stochastically compact at 0, the unidimensional law of X admits a density, say $\mathbb{P}(X_t \in dy) = f_t(y)dy$, $y \geq 0$, and such that $f_t \in C^\infty(\mathbb{R})$.*

(ii) *In the settings (SC_0-(I-II)) we have the estimate*

$$\sqrt{t}\sigma(\rho_t)f_t(z) = \left(\phi((z-x)/\sqrt{t}\sigma(\rho_t)) + o(1)\right) e^{-tH(\rho_t)} e^{\rho_t(z-x)}, \tag{4}$$

uniformly in $z > 0$ and x, with ϕ the standard normal density.

To prove the first claim we establish that there is $0 < \beta < 2$ and a constant c_1 s.t.

$$\Re(\psi(-i\theta)) = \int_0^\infty (1 - \cos(\theta y))\Pi(dy) \geq c_1|\theta|^\beta.$$

$$\int_{\mathbb{R}} |\theta|^n e^{-\Re(\psi(-i\theta))} d\theta < \infty, \qquad \forall n \geq 1.$$

To prove the second claim we introduce a change of measure. Let $(Y_s \quad, s \geq 0)$, a subordinator whose Laplace exponent is given by ψ_{ρ_t},

$$\psi_{\rho_t}(\lambda) = \psi(\rho_t+\lambda)-\psi(\rho_t) = b\lambda+\int_{(0,\infty)}(1-e^{-\lambda y})e^{-\rho_t y}\Pi(dy), \qquad \lambda \geq 0. \tag{5}$$

In particular we have the following key relation:

$$\mathbb{P}(Y_t \quad \in dy) = e^{tH(\rho_t)}e^{-\rho_t(y-tx_t)}\mathbb{P}(X_t \in dy), \qquad y \in \mathbb{R}^+. \tag{6}$$

Y has moment of all orders, we use a Normal approximation

To prove the second claim we introduce a change of measure. Let $(Y_s^{(x_t)}, s \geq 0)$, a subordinator whose Laplace exponent is given by ψ_{ρ_t},

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In particular we have the following key relation:

$$\mathbb{P}(Y_t^{(x_t)} \in dy) = e^{tH(\rho_t)} e^{-\rho_t(y-tx_t)} \mathbb{P}(X_t \in dy), \qquad y \in \mathbb{R}^+. \tag{6}$$

$Y^{(x_t)}$ has moment of all orders, we use a Normal approximation

Lemma (Petrov 1975)

Let $Z_1, Z_2, \cdots Z_n$ be independent rvs having finite 3rd moments, write $\mathbb{E}Z_r = \mu_r, Var(Z_r) = \sigma_r^2$, and $\mathbb{E}|Z_r - \mu_r|^3 = \nu_r$, and put $W = \sum_1^n Z_r, m = \mathbb{E}(W) = \sum_1^n \mu_r$, and $s^2 = Var(W) = \sum_1^n \sigma_r^2$. Assume further that $\int_{-\infty}^{\infty} |\widetilde{\Psi}(u)|du < \infty$, where $\widetilde{\Psi}(u) = \mathbb{E}(e^{iuW})$, and denote by f and ϕ the pdf of W and the standard Normal pdf. Then there is an absolute constant A such that

$$\sup_y |f(y) - s^{-1}\phi(\frac{y-m}{s})| \leq AL + d, \tag{7}$$

where $L = \sum_1^n \nu_r/s^4$ and, with $l = (4Ls^2)^{-1}$,

$$d = 2\int_l^{\infty} |\Psi(u)|du.$$

We apply the previous estimate with
$W \stackrel{\text{Law}}{=} Y_t = \sum_{r=1}^{n} Y_{\frac{rt}{[x\rho_t]}} - Y_{\frac{(r-1)t}{[x\rho_t]}}$, $n = [x\rho_t]$, and prove that

$$\sqrt{t}\sigma(\rho_t)L \lesssim \frac{1}{\left(t\rho_t^2\sigma^2(\rho_t)\right)^{1/2}}\left(6\frac{Q_\Pi(1/\rho_t)}{\rho_t^2\sigma^2(\rho_t)} + 2\right) \to 0;$$

with

$$Q_\Pi(z) = 2z^{-2}\int_0^z y\overline{\Pi}(y)dy, \qquad z > 0.$$

$\exists \beta_0 > 0$, s.t.

$$\gamma := \sqrt{t}\sigma(\rho_t)\int_l^\infty e^{-t\Re(\psi_{\rho_t}(i\theta))}d\theta \leq \frac{1}{(tH(\rho_t))^{\beta_0}} \to 0.$$

uniformly in x.

The same method work in general but we should consider $W = Y_t + U_h + \Delta_a = \sum_1^n Z_r$, where $n = [x\rho_t] + 2$, $Z_r := Y_{\frac{rt}{[x\rho_t]}} - Y_{\frac{(r-1)t}{[x\rho_t]}} \overset{\text{Law}}{=} Y_{\frac{t}{[x\rho_t]}}$ for $1 \leq r < n-1$, and the two independent random variables U_h and Δ_a are independent of Y_t, U_h has a uniform distribution on $[-h, h]$, and $\Delta_a \overset{\text{Law}}{=} U_a + \tilde{U}_a$, with U_a and $\widetilde{U}_a$ independent. The density of W satisfies

$$n_t(z) = (2h)^{-1}\mathbb{P}(Y_t + \Delta_a \in (z-h, z+h)),$$

Theorem

In the settings (SC_0-(I-II)), (SC_∞) and (G) the following estimates

$$\mathbb{P}(X_t \in (x-u, x]) = \frac{e^{-tH(\rho_t)}}{\sqrt{t}\sigma(\rho_t)} \left(\int_0^u e^{-\rho_t v} \phi\left(\frac{v}{\sqrt{t}\sigma(\rho_t)} \right) dv + o(1) \frac{1-e^{-u\rho_t}}{\rho_t} \right),$$

$$\mathbb{P}(X_t \leq x) \sim \frac{e^{-tH(\rho_t)}}{\sqrt{2\pi t}\sigma(\rho_t)\rho_t},$$

hold uniformly in $u < x$ and uniformly in x.

We plug the former estimates in

$$\frac{\mathbb{P}(T_x \in dt, X_{T_x -} < x)}{dt} = \int_0^x \mathbb{P}(X_t \in dy)\overline{\Pi}(x-y), \quad x > 0, t > 0.$$

If $b > 0$,

$$\mathbb{P}(T_x \in (t, t+h], X_{T_x} = x) = b\partial_z \int_t^{t+\Delta} ds \mathbb{P}(X_s \leq z)|_{z=x}, \; x > 0, \; t \geq 0.$$

If X_s has a density $f_s(\cdot)$, $s \geq 0$, then
$\mathbb{P}(T_x \in dt, X_{T_x} = x) = bf_t(x)$, $x > 0$, $t \geq 0$.

Theorem

Let $\Delta_0 > 0$ *fixed. In the settings (*SC_0*-(I-II)),* (SC_∞) *and* (G)*, the following estimates*

$$h_x^J(t) \sim \frac{e^{-tH(\rho_t)}}{\sqrt{2\pi t}\sigma(\rho_t)} \left(\frac{\psi(\rho_t)}{\rho_t} - b \right), \tag{8}$$

$$h_x^C(t,\Delta) \sim b \int_t^{t+\Delta} \frac{e^{-sH\left(\rho\left(\frac{x}{s}\right)\right)}}{\sqrt{2\pi s}\sigma\left(\rho\left(\frac{x}{s}\right)\right)} ds, \tag{9}$$

hold uniformly in $0 < \Delta < \Delta_0$ *and in* x*. Furthermore, under the settings (*SC_0*-(I-II)) the more precise estimate*

$$\mathbb{P}(T_x \in dt, X_{T_x} = x) = \frac{b}{\sqrt{2\pi t}\sigma(\rho_t)}(1 + o(1))e^{-tH(\rho_t)}dt, \tag{10}$$

holds uniformly in x*.*

Corollary

In the settings (SC_0-(I-II)), the hazard rate of T_x on the event of discontinuous crossing is

$$\frac{h_x^J(t)}{\mathbb{P}(T_x > t)} \sim \int_0^\infty (1 - e^{-\rho_t y})\Pi(dy) \to 0,$$

and on the event of creeping

$$\frac{h_x^C(t)}{\mathbb{P}(T_x > t)} \sim b\rho_t \to \infty.$$

$$\frac{h_x^C(t)}{h_x^J(t)} \sim \frac{b}{\int_0^\infty e^{-y\rho_t}\Pi(y,\infty)dy} \to b \times \infty.$$

$$\mathbb{P}(T_x \in (t, t+\Delta]) \sim \Delta h_x^C(t)$$

Corollary

Let $\Delta_0 > 0$ *fixed and assume* $\Pi(y,\infty) = y^{-\alpha}\ell(y)$ **at** ∞. *Then the following estimates,*

$$h_x^J(t) \sim \frac{e^{-tH(\rho_t)}}{\sqrt{2\pi t}\sigma(\rho_t)} \cdot \frac{\overline{\Pi}(1/\rho_t)}{\rho_t},$$

$$h_x^C(t,\Delta) \sim \frac{b\Delta e^{-tH(\rho_t)}}{\sqrt{2\pi t}\sigma(\rho_t)}, \tag{11}$$

hold uniformly in $0 < \Delta < \Delta_0$ *and in* x *such that* $x/t \to \mathbb{E}(X_1)$ *and* $(x - bt)/c(t) \to 0$, *with* c *taken as* $t\Pi(c(t),\infty) = 1$. *We have also*

$$\frac{h_x^C(t)}{h_x^J(t)} \sim \frac{b}{\int_0^y e^{-\rho_t y}\Pi(y,\infty)dy} \to \frac{b}{\int_0^\infty \Pi(y,\infty)dy}$$

Corollary

In the settings (SC_0-(I-II)), (SC_∞) and (G) the estimate

$$\mathbb{P}\left(\rho_t \times (x - X_t) \in [0, u) | x - X_t > 0\right) = (1 - e^{-u})(1 + o(1))$$

holds uniformly in $u < x$ and uniformly in x.

Corollary

In the settings (SC_0-(I-II)), (SC_∞) and (G) the estimate

$$\mathbb{P}\left(\rho_t \times (x - X_t) \in [0,u) | x - X_t > 0\right) = (1 - e^{-u})(1 + o(1))$$

holds uniformly in $u < x$ and uniformly in x.

$$\begin{aligned} &\mathbb{P}(X_t \in (x - (u/\rho_t), x] | X_t \leq x) \\ &= \sqrt{2\pi} \int_0^u e^{-v} \phi\left(\frac{v}{\sqrt{t}\rho_t \sigma(\rho_t)}\right) dv + o(1)(1 - e^{-u}), \end{aligned}$$

$$t\rho_t^2 \sigma^2(\rho_t) \asymp tH(\rho_t) \to \infty.$$

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Our estimates are not valid in this setting. Under further assumptions we can use Stone's Local Limit Theorem to estimate the density of X.

Assume that X is in the domain of attraction of a stable law at infinity, $\Pi(x, \infty) = x^{-\alpha}\ell(x)$ with ℓ slowly varying at infinity, $\alpha \in (0, 1)$. Then uniformly in Δ and $y \in \mathbb{R}$,

$$c(t)\mathbb{P}(X_t \in (y, y + \Delta]) = \Delta(\widetilde{g}_1(\frac{y}{c(t)}) + o(1)) \text{ as } t \to \infty,$$

with $\widetilde{g}_1$ the α-stable density, and $t\Pi(c(t), \infty) = 1$.

Theorem

Suppose now that X *is a non-lattice subordinator which has drift* $b \geq 0$ *and* $\overline{\Pi}(\cdot) \in RV(-\alpha)$ ***at*** ∞ *with* $\alpha \in (0,1)$. *Define* c *by* $t\overline{\Pi}(c(t)) = 1$, *so that* $(X(t\cdot)/c(t))$ *converges weakly to* S, *a stable subordinator of index* α. *Let* $\tilde{g}_t(\cdot)$ *and* $\tilde{h}_x(\cdot)$ *denote the density functions of* S_t *and* $T_x^S := \inf\{t : S_t > x\}$ *respectively. Then for any fixed* $D > 1$, *uniformly for* $y_t := x/c(t) \in (D^{-1}, D)$

$$t h_x^J(t) = \tilde{h}_{y_t}(1) + o(1) \text{ as } t \to \infty.$$

and, if $b > 0$, *uniformly for* $y_t \in (D^{-1}, D)$ *and* $0 < \Delta < \Delta_0$,

$$c(t) h_x^C(t, \Delta) = \frac{\alpha b \Delta}{1-\alpha}\left(\tilde{g}_1\left(y_t\right) + o(1)\right) \text{ as } t \to \infty.$$

$$\frac{h_x^C(t,\Delta)}{h_x^J(t,\Delta)} \asymp \frac{t}{c(t)} \to 0, \qquad \text{as } t \to \infty.$$