Some spectral problems associated to non-self-adjoint self-similar semi-groups

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Definition

$P = (P_t)_{t\geq 0}$ is a *H*-invariant Lamperti-Feller semi-group on $\mathbb{R}^+$ (for short H-iLF) if P is a Feller semi-group on $\mathbb{R}^+$ and for any $f \in C_0(\mathbb{R}^+)$,

1. $\exists\, \nu$ a non-degenerate probability measure on $\mathbb{R}^+$ such that

$$\nu P_t f = \nu f.$$

2. $\exists H > 0$ such that the deterministic space-time transform

$$(K_t)_{t\geq 0} = \left(P_{\log(1+t)} \circ d_{(1+t)^H}\right)_{t\geq 0}$$

defines a Feller semi-group on $\mathbb{R}^+$, where $d_c f(x) = f(cx)$.

3. $\lim_{x\to 0} P_t f(x) = P_t f(0),\ \forall t > 0.$

We aim to characterize this class of semi-groups and study their discrete spectrum and eigenvalues expansion.

Some motivation/remarks

- From a result of Lamperti (62), we know that K is the semi-group of a positive H-self-similar Markov processes (pssMp) X, with infinite lifetime and 0 as an entrance boundary point. With $H = 1$, we have

  $$P_t f(x) = K_{e^t-1} \circ d_{e^{-t}} f(x) = \mathbb{E}_x \left[f(e^{-t} X_{e^t-1}) \right] = \mathbb{E}_{e^{-t}x} \left[f(X_{1-e^{-t}}) \right].$$

  Thus,

  $$\lim_{t\to\infty} P_t f(x) = \lim_{x\downarrow 0} \mathbb{E}_x \left[f(X_1) \right]. \tag{1}$$

- The class of infinitesimal generators associated to K or P includes well-known operators such as several type of fractional derivatives and differential-difference operators.

- Beside one specific instance, these semi-groups are non self-adjoint.

  Unlike for self-adjoint operators, there is no spectral theorem available.

  The spectral analysis of non-self-adjoint operator is *Terra incognita* (Pavlov, 83).

  Davies (06): *Studying non-self-adjoint operators is like being a vet rather than a doctor: one has to acquire a much wider range of knowledge, and to accept that one cannot expect to have as high a rate of success when confronted with particular cases.*

- McKean(56), Getoor(58), Karlin and McGregor(66), Ogura (70).

Characterization of H-iLF semi-groups

We denote by $\mathcal{N}$ the set of functions Ψ of the form, for any $z \in i\mathbb{R}$,

$$\Psi(z) = mz + \frac{\sigma^2}{2} z^2 + \int_{\mathbb{R}} (e^{zy} - 1 - zy)\Pi(dy), \tag{2}$$

where $m, \sigma \geq 0$, Π is a σ-finite measure satisfying $\int_{-\infty}^{\infty} (y^2 \wedge |y|)\, \Pi(dy) < \infty$.

Theorem

There is a bijection between $\mathcal{N}$ and the set of H-iLF semi-groups.
For each $\Psi \in \mathcal{N}$, the infinitesimal generator of P^{Ψ}, when $H = 1$, is given by

$$\mathbf{L}^{\Psi} f(x) = \frac{\sigma^2}{2} x f''(x) + \left(m + \frac{\sigma^2}{2} - x \right) f'(x) + \frac{1}{x} \int_{\mathbb{R}} (f(e^y x) - f(x) - yxf'(x))\, \Pi(dy).$$

There exists an absolutely continuous probability measure ν^{Ψ} such that

$$\nu^{\Psi} P_t^{\Psi} f = \lim_{x \downarrow 0} \mathbb{E}_x \left[f(X_1^{\Psi}) \right].$$

$P_t^{\Psi} : L^2(\nu^{\Psi}) \to L^2(\nu^{\Psi})$ is a contraction.

The program

If $\Pi \neq 0$, we show that P is non-self-adjoint w.r.t. to ν, i.e. there exists a Feller semi-group $\widehat{P} \neq P$ such that

$$\langle P_t f, g\rangle_\nu = \langle f, \widehat{P}_t g\rangle_\nu.$$

1. Discrete spectrum S_d: Find $\lambda_n \in \mathbb{C}$ s.t. $\exists\, \mathcal{P}_n \in L^2(\nu)$ with

$$P_t \mathcal{P}_n(x) = e^{-\lambda_n t} \mathcal{P}_n(x).$$

2. Properties of $(\mathcal{P}_n)_{\lambda_n \in S_d}$ in $L^2(\nu)$ (completeness, Bessel property, ...)
3. Existence and properties of co-eigenfunctions ν_n of P_t, i.e. for $\lambda_n \in S_d$,

$$\langle P_t f, \nu_n\rangle_\nu = e^{-\lambda_n t}\langle f, \nu_n\rangle_\nu.$$

4. If $S = S_d$, deduce for $t > T$, expansion on $\mathcal{H} \subset L^2(\nu)$, i.e. for $f \in \mathcal{H}$

$$P_t f(x) = \sum_{n=0}^{\infty} e^{-\lambda_n t}\langle f, \nu_n\rangle_\nu \mathcal{P}_n(x).$$

A special instance:

Let $\Psi(z) = z^2$, i.e. $\sigma = \sqrt{2}$, $m = 0$, $\Pi = 0$ in (2).

- Its semi-group $Q = P^{\Psi}$ defines a diffusion which is self-adjoint w.r.t. $\nu^{\Psi}(x) = \gamma(x) = e^{-x}$ and

$$Lf(x) = xf''(x) + (1-x)f'(x).$$

- Moreover, $\forall f \in L^2(\gamma)$, $t \geq 0$,

$$Q_t f(x) = \sum_{n \geq 0} e^{-nt} \langle f, \mathcal{L}_n \rangle_{\gamma} \, \mathcal{L}_n(x)$$

  where $\mathcal{L}_n$ are the Laguerre polynomials

$$\mathcal{L}_n(x) = \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{k!} x^k$$

  or also

$$\mathcal{L}_n(x) = e^x \frac{1}{n!} \frac{d^n}{dx^n} \left(x^n e^{-x} \right) = \frac{1}{\gamma(x)} \mathcal{R}^{(n)} \left(\gamma(x) \right).$$

- This is our starting point ...

Intertwining of semi-groups

The semi-groups P and Q acting on $L^2(\nu_P)$ and $L^2(\nu_Q)$ respectively are said to intertwine if there exists

$$\Lambda : L^2(\nu_Q) \mapsto L^2(\nu_P)$$

a Markov kernel such that

$$\Lambda Q_t = P_t \Lambda.$$

Dynkin (65), Diaconis-Fill (90), Carmona et al. (98).

- Assume that $\mathcal{L}_n$ is an eigenfunction of Q associated to $e^{-\lambda_n t}$, then

$$P_t \Lambda \mathcal{L}_n = \Lambda Q_t \mathcal{L}_n = e^{-\lambda_n t} \Lambda \mathcal{L}_n.$$

- Assume that ν_n is a co-eigenfunction of P associated to $e^{-\lambda_n t}$, then $\Lambda^* \nu_n = \mathcal{L}_n$

$$\begin{aligned} \langle Q_t f, \Lambda^* \nu_n \rangle_{\nu_Q} &= \langle \Lambda Q_t f, \nu_n \rangle_{\nu_P} = \langle P_t \Lambda f, \nu_n \rangle_{\nu_P} = e^{-\lambda_n t} \langle \Lambda f, \nu_n \rangle_{\nu_P} \\ &= e^{-\lambda_n t} \langle f, \Lambda^* \nu_n \rangle_{\nu_Q}. \end{aligned}$$

Intertwining of iHFL semi-groups

For a positive r.v. V, we define its multiplicative kernel by

$$\mathcal{V}f(t) = \mathbb{E}\left[f(tV)\right] = \int_{\mathbb{R}^+} f(ty)\nu(y)dy.$$

- From (1), we have, with $\Psi \in \mathcal{N}$,

$$\mathbb{E}_0\left[f(X_t^{\Psi})\right] = \mathbb{E}_0\left[f(tX_1^{\Psi})\right] = \mathcal{V}^{\Psi}f(t).$$

  where V_{Ψ} is a positive r.v. with density distribution ν^{Ψ}.

- If $\mathcal{V}^{\Psi}$ is injective on $C_0(\mathbb{R}^+)$ and for some positive r.v. M

$$V_{\Psi} \stackrel{d}{=} V_{\Psi_1} \times M$$

  with $\Psi_1 \in \mathcal{N}$ then using a criteria from Carmona et al. (01), one can show that

$$\mathcal{M}P_t^{\Psi} = P_t^{\Psi_1}\mathcal{M}.$$

We recall the Wiener-Hopf factorization, for $z \in i\mathbb{R}$,

$$\Psi(z) = -\phi_+(-z)\phi_-(z),$$

where the Bernstein functions $\phi_\pm$

$$\phi_\pm(z) = q_- + \delta_\pm z + \int_0^\infty \left(1 - e^{-zy}\right) \mu_\pm(dy).$$

where $q_-, \delta_\pm \geq 0$ and $\int_0^\infty (1 \wedge y)\mu_\pm(dy) < \infty$.

$\phi_\pm$ correspond to the Laplace exponents of increasing Lévy processes $H^\pm$ associated to the Lévy process linked to Ψ.

Let us denote

$$\mathcal{N}_- = \{\psi \in \mathcal{N};\ \Pi(dy)\mathbb{I}_{\{y>0\}} = 0\},$$

$$\mathcal{N}_\mathcal{P} = \{\Psi \in \mathcal{N} \setminus \mathcal{N}_-;\ \Pi(dy)\mathbb{I}_{\{y>0\}} = \pi(y)dy \text{ and } \pi \text{ non-increasing}\}.$$

$\mathcal{N}_-$ corresponds to processes with downwards jumps only.

Theorem (Intertwining)

1. *For any $\psi \in \mathcal{N}_-$, we have $\psi(z) = z\phi_-(z)$ and*

$$P_t^\psi \mathcal{I}_{\phi_-} = \mathcal{I}_{\phi_-} Q_t,$$

where $I_{\phi_-} = \int_0^\infty e^{-H_s^-} ds$ and $\mathcal{I}_{\phi_-} : L^2(\gamma) \mapsto L^2(\nu^\psi)$ is a contraction.

2. *For any $\Psi \in \mathcal{N}_\mathcal{P}$, we have*

$$\mathcal{V}_{\phi_+} P_t^\Psi = P_t^\psi \mathcal{V}_{\phi_+},$$

where $\Psi(z) = -\phi_+(-z)\phi_-(z)$, $\psi(z) = z\phi_-(z) \in \mathcal{N}_-$ and ν^{ϕ_+} is the stationary measure of P^{ϕ_+}, $\mathcal{V}_{\phi_+} : L^2(\nu^\Psi) \mapsto L^2(\nu^\psi)$ is a contraction.

Ideas of the proof:

1- With $\mathbf{e} \sim Exp(1)$ the stationary law of Q_t and V_ψ the stationary law of P^ψ we prove that

$$\mathbf{e} \stackrel{d}{=} V_\psi \times I_{\phi_-}, \tag{3}$$

through moment identification, due to Bertoin and Yor (02) and P.(10),

$$\mathbb{E}\left[\left(V_\psi \times I_{\phi_-}\right)^k\right] = \prod_{j=1}^{k} \phi_-(j) \times \frac{k!}{\prod_{j=1}^{k} \phi_-(j)} = k! = \mathbb{E}\left[\mathbf{e}^k\right], \quad \forall k \in \mathbb{N},$$

and (3) follows.
For the intertwining to hold, it remains to prove the injectivity of $\mathcal{V}_\psi f(t) = \mathbb{E}\left[f(tV_\psi)\right]$ or equivalently

$$\mathbb{E}\left[V_\psi^{iu-1}\right] \neq 0, \ \forall u \in \mathbb{R}. \tag{4}$$

Then (3) and (4) imply that

$$P_t^\psi \mathcal{I}_{\phi_-} = \mathcal{I}_{\phi_-} Q_t.$$

2- Similarly,

$$\mathcal{V}_{\phi_+} P_t^{\Psi} = P_t^{\psi} \mathcal{V}_{\phi_+},$$

would follow from

$$V_{\Psi} \stackrel{d}{=} V_{\phi_+} \times V_{\psi}.$$

With $\mathcal{M}_{V_\Psi}(z) = \mathbb{E}\left[V_{\Psi}^{z-1}\right]$ we have from Maulik-Zwart (06) that, for $z \in i\mathbb{R}$,

$$\mathcal{M}_{V_\Psi}(z+1) = \frac{\Psi(z)}{z}\mathcal{M}_{V_\Psi}(z) = \frac{-\phi_-(z)\phi_+(-z)}{z}\mathcal{M}_{V_\Psi}(z). \tag{5}$$

Next, if $\Psi \in \mathcal{N}_{\mathcal{P}}$, then $\psi(z) = z\phi_-(z) \in \mathcal{N}_-$. Split formally to get

$$\mathcal{M}_{V_\psi}(z+1) = \frac{\psi(z)}{z}\mathcal{M}_{V_\psi}(z),\ \Re(z) > 0,$$

$$\mathcal{M}_{V_{\phi_+}}(z+1) = \frac{-\phi_+(-z)}{z}\mathcal{M}_{V_{\phi_+}}(z),\ \Re(z) \le 1.$$

Based on result a Webster (97) on $\mathbb{R}^+$, we show that both equations have an unique solution which can be expressed in terms of generalized Weierstrass products and thus **assuming that uniqueness of** (5) **holds**

$$\mathcal{M}_{V_\Psi}(z) = \mathcal{M}_{V_\psi}(z)\mathcal{M}_{V_{\phi_+}}(z) = \frac{1}{\phi'_+(0+)}\frac{\Gamma(1-z)}{W_{\phi_+}(1-z)}W_{\phi_-}(z),$$

where

$$W_\phi(z) = \frac{e^{-\gamma_\phi z}}{\phi(z)}\prod_{j=1}^{\infty}\frac{\phi(j)}{\phi(j+z)}e^{\frac{\phi'(k)}{\phi(k)}z}$$

with $|\gamma_\phi| = \left|\lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{\phi'(k)}{\phi(k)} - \log\phi(n)\right)\right| < \infty$.

The uniqueness follows from an argument due to Kuznetsov and Pardo (12) which requires estimates of the rate of exponential decay of $f(b) = |W_\phi(a+ib)|$.

Eigenfunctions and discrete spectrum

Proposition (Eigenfunctions 1-)

Let $\psi \in \mathcal{N}_-$, then for any $n \in \mathbb{N}$,

$$P_t^{\psi}\mathcal{P}_n^{\psi}(x) = e^{-nt}\mathcal{P}_n^{\psi}(x)$$

where

$$\mathcal{P}_n^{\psi}(x) = \mathcal{I}_{\phi_-}\mathcal{L}_n(x) = \sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^k}{k!}\frac{k!}{\prod_{j=1}^{k}\phi_-(k)}x^k.$$

$\left(\mathcal{P}_n^{\psi}\right)_{n\geq 0}$ *is a complete Bessel sequence in $L^2(\nu^{\psi})$ but it is not a Riesz basis.*

$\left(\mathcal{P}_n^{\psi}\right)_{n\geq 0}$ is a Bessel sequence in $L^2(\nu^{\psi})$ if there exists $C > 0$ such that

$$\sum_{n=0}^{\infty}|\langle f, \mathcal{P}_n^{\psi}\rangle_{\nu^{\psi}}|^2 \leq C||f||^2_{\nu^{\psi}} \quad \forall f \in L^2(\nu^{\psi}). \tag{6}$$

For any $(c_n)_{n\geq 0} \in \ell^2$, the series $\sum_{n=0}^{\infty} c_n\mathcal{P}_n^{\psi}(x)$ converges in $L^2(\nu^{\psi})$.

Proposition (Eigenfunctions 2-)

For $\Psi \in \mathcal{N}_{\mathcal{P}}$ we have $S_d^{\Psi} = \{0, 1, 2, ..., n\}$ iff Ψ is analytic at least on the strip $0 < \mathfrak{Re}(z) \leq n + \epsilon,\ \epsilon > 0$. Moreover,

$$\mathcal{P}_n(x) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{k!}{\prod_{j=1}^{k} \Psi(j)} x^k.$$

If $S_d^{\Psi} = \mathbb{N}$, then $\left(\mathcal{P}_n^{\Psi}\right)_{n \geq 0}$ is not a Bessel sequence in $L^2(\nu^{\Psi})$.

It is conjectured (see Bertoin and Yor (04)) that ν^{Ψ} is moment indeterminate. This would imply that even if $S_d = \mathbb{N}$ then $\left(\mathcal{P}_n^{\Psi}\right)_{n \geq 0}$ is not complete in $L^2(\nu^{\Psi})$.

Throughout, we assume that $\psi \in \mathcal{N}_{-}$.

Eigenvalues expansions

From the intertwining relationship and the expansion of Q_t, we deduce

$$P_t \mathcal{I}_{\phi_-} f(x) = \mathcal{I}_{\phi_-} \sum_{n=0}^{\infty} e^{-nt} \langle f, \mathcal{L}_n \rangle_\gamma \mathcal{L}_n(x) = \sum_{n=0}^{\infty} e^{-nt} \langle f, \mathcal{L}_n \rangle_\gamma \mathcal{P}_n(x).$$

which holds whenever

$$\sum_{n=0}^{\infty} e^{-2nt} \langle f, \mathcal{L}_n \rangle_\gamma^2 < \infty.$$

Theorem

For any $\psi \in \mathcal{N}_-$ and any $f \in \mathbf{P}$, the space of polynomials, we have

$$P_t f(x) = \sum_{n=0}^{\infty} e^{-nt} \langle \mathcal{I}_{\phi_-}^{-1} f, \mathcal{L}_n \rangle_\gamma \mathcal{P}_n(x), \quad t \geq 0.$$

A. Existence of co-eigenfunctions: If for any $n \geq 0$, the equation $\mathcal{I}^*_{\phi_-} g(x) = \mathcal{L}_n(x)$ has a solution $g = \nu_n$ in $L^2(\nu)$ then clearly

$$P_t f(x) = \sum_{n=0}^{\infty} e^{-nt} \langle f, \nu_n \rangle_\nu \mathcal{P}_n(x).$$

Since $(\mathcal{P}_n)_{n \geq 0}$ is a Bessel sequence, the expansion holds whenever

$$\sum_{n=0}^{\infty} e^{-2nt} \langle f, \nu_n \rangle_\nu^2 < \infty. \tag{7}$$

B. When does (7) holds? We have $\mathcal{I}_{\phi_-}$ is dense but not onto since

$$||\mathcal{I}_{\phi_-} f||_{L^2(\nu)} \geq c||f||_{L^2(\gamma)}$$

fails even on monomials. Since for any $f \in L^2(\mathbb{R}^+)$, we have

$$\langle f, \nu_n \rangle_\nu \leq ||f||_{L^2(\mathbb{R}^+)} ||\nu_n \nu||_{L^2(\mathbb{R}^+)},$$

the expansion holds for any $t > B$, whenever

$$||\nu_n \nu||_{L^2(\mathbb{R}^+)} \leq e^{Bn}.$$

A- What is the form/regularity of the co-eigenfunctions?

By Mellin inversion, the identity $\mathcal{I}^*_{\phi_-}\nu_n(x) = \mathcal{L}_n(x)$ gives, in the sense of distributions,

$$\nu_n(x) = \frac{\mathcal{R}^{(n)}(\nu(x))}{\nu(x)} = \frac{1}{\nu(x)n!}\frac{d^n}{dx^n}\left(x^n\nu(x)\right).$$

Problems :

1. $\nu \in C^\infty(\mathbb{R}^+)$?
2. $\nu_n \in L^2(\nu)$?

For (1) to hold, we show that we must consider processes of infinite variation:

$$\sigma > 0 \quad \text{or} \quad \int_{-1}^{0} |y|\Pi(dy) = \infty.$$

Otherwise, $\nu \in C^{[K]}(\mathbb{R}^+)$ where $K = \frac{\overline{\Pi}(0^+)}{\overline{\overline{\Pi}}(0^+)+m} < \infty$. Note also that in this latter case $supp(\nu) = (0, \overline{\overline{\Pi}}(0^+) + m)$.

For (2), i.e. $\nu_n \in L^2(\nu)$?

- $\nu(x) = m^{-1}x^{-1}\nu^*(x^{-1})$
  ($m = 0$ *is treated by means of an orbit argument*,)
- ν^* is a positive self-decomposable density distribution.

Proposition

For almost all $\psi \in \mathcal{N}_-^\infty = \{\psi \in \mathcal{N}_-; \sigma > 0 \; or \; \int_{-1}^0 |y|\Pi(dy) = \infty\}$*, we have, for any* $n \in \mathbb{N}$*, as* $x \to 0$

$$(\nu^*)^{(n)}(x) \sim Cmx^{-n-1}\varphi^n(x^{-1})\sqrt{\varphi'(x^{-1})}e^{-\int_{\phi_-(0)}^{x^{-1}} \frac{\varphi(s)}{s}ds}, \qquad (8)$$

where $\varphi : [\phi_-(0), \infty) \mapsto [0, \infty)$ *is the inverse function of* $\phi_-(z) = \frac{\psi(z)}{z}$ *and* $C > 0$.

Sketch of the proof.

- Denote by $f_k(y) = e^{-ky}(\nu^*)^{(k)}(e^{-y})$. Then, for $z \in i\mathbb{R}$,

$$\int_{\mathbb{R}} e^{zy} f_k(y)dy = \widehat{f_k}(z) = m\,(z)_k \mathcal{M}_{V_\psi}(z)$$

  where $(z)_k = z(z-1)\dots(z-k+1)$.

- Since for $x > 0$

$$\mathcal{M}_{V_\psi}(x+1) = \phi_-(x)\mathcal{M}_{V_\psi}(x)$$

  and $\phi_-(x)$ is log-concave, we have, from Webster (97), as $x \to \infty$,

$$\widehat{f_k}(x) = m(x)_k \mathcal{M}_{V_\psi}(x) \sim Cm(x)_k \sqrt{\phi_-(x)} e^{\int \log \phi_-(y)dy}.$$

- Also $\int \log \phi_-(y)dy$ is asymptotically parabolic and a Tauberian theorem due to Balkema et al. (95) gives the proof **provided** $(\nu^*)^{(k)}(e^{-y})$ **is log-convex in a neighbourhood of infinity**.
- This is stronger than the fact due to Sato and Yamazato (78): $(\nu^*)^{(k)}(y)$ *is log-concave at zero*.

Then, we have:

- Asymptotics of $(\nu^*)^{(k)}$ at zero leads to asymptotics for $(x^k\nu(x))^{(k)}$ at ∞. Using this asymptotics

$$\int_1^\infty (\nu_n(x))^2\nu(x)dx = \frac{1}{(n!)^2}\int_1^\infty \frac{\left((x^n\nu(x))^{(n)}\right)^2}{\nu(x)}dx < \infty.$$

- To show that

$$\int_0^1 (\nu_n(x))^2\nu(x)dx < \infty,$$

  we prove a lower bound for ν and uniform bounds for $(x^n\nu(x))^{(n)}$ through Mellin inversion.

Hence $\nu_n \in L^2(\nu)$.

We have $\langle \mathcal{P}_m, \nu_n\rangle_\nu = \delta_{nm}$ and $(\mathcal{P}_n, \nu_n)_{n\geq 0}$ is a biorthogonal sequence in $L^2(\nu)$.

B- Spectral Expansions via uniform bounds for $||\nu_n \nu||_{L^p(\mathbb{R}^+)}$

Theorem

For almost all $\psi \in \mathcal{N}_-^\infty$, we have, for any $f \in L^p(\mathbb{R}^+)$, $1 \leq p \leq \infty$ and $t > T_\psi$, $T_\psi \geq 0$,

$$P_t f(x) = \sum_{n=0}^{\infty} e^{-nt} \langle f, \nu_n \rangle_\nu \mathcal{P}_n(x).$$

In particular, for any $t > T_\psi$,

$$p_t(x,y) = \sum_{n=0}^{\infty} e^{-nt} \nu_n(y) \mathcal{P}_n(x) \nu(y); \quad k_u(x,y) = p_{\ln \frac{1}{1-u}} \left(\frac{x}{1-u}, y \right).$$

In both cases, we also have absolute convergence. Note that if $\sigma > 0$ then $T_\psi = 0$.

Sketch of the proof.

Recall that $\mathcal{M}_{\nu_n\nu}(z) = (-1)^n \frac{\Gamma(z)}{\Gamma(n+1)\Gamma(z-n)} \mathcal{M}_{V_\psi}(z)$, for $\mathfrak{Re}(z) > 0$ and

$$\mathcal{M}_{V_\psi}(z) = \frac{e^{-\gamma_{\phi_-} z}}{\phi_-(z)} \prod_{k=1}^{\infty} \frac{\phi_-(k)}{\phi_-(k+z)} e^{\frac{\phi'_-(k)}{\phi_-(k)} z}.$$

Next, we have

- Along lines $a + ib$

$$\left|\mathcal{M}_{V_\psi}(a+ib)\right| \asymp C(a) e^{-b \int_{ab^{-1}}^{\infty} \ln\left|\frac{\phi_-(by+ib)}{\phi_-(by)}\right| dy}.$$

- If $\limsup_{x\to 0} \overline{\overline{\Pi}}_-(\lambda x)/\overline{\overline{\Pi}}_-(x) < 1$, for some $\lambda > 1$, then with some $\varepsilon \in (0, \pi/2]$

$$\left|\mathcal{M}_{V_\psi}(a+ib)\right| \lesssim e^{-\varepsilon|b|}.$$

The important observation is the dependence of the asymptotics on

$$H_{\phi_-}(a,b) = \int_{ab^{-1}}^{\infty} \ln\left|\frac{\phi_-(by+ib)}{\phi_-(by)}\right| dy \in \left(0, \frac{\pi}{2}\right).$$

$\sigma > 0$ or $\limsup_{x\to 0} \overline{\overline{\Pi}}_-(\lambda x)/\overline{\overline{\Pi}}_-(x) < 1$ ensures that $\liminf_{b\to\infty} H_{\phi_-}(a,b) > 0$.

- Mellin inversion gives

$$\nu_n(x)\nu(x) = C \int_{-\infty}^{\infty} x^{-a-ib} \frac{(-1)^n \Gamma(z)}{\Gamma(z-n)\Gamma(n+1)} \mathcal{M}_{V_\psi}(a+ib)db$$

- If $\left|\mathcal{M}_{V_\psi}(a+ib)\right| \leq e^{-\varepsilon|b|}$ then

$$||\nu_n \nu||_{L^p(\mathbb{R}^+)} \leq Ce^{B_\varepsilon n}$$

- Henceforth our expansion holds for $t > B_\varepsilon$.
- When $\sigma > 0$, i.e.

$$\Psi(z) = \frac{\sigma^2 z^2}{2} + \dots$$

then $B_\epsilon = 0$.

Thank you !