Useful martingales for stochastic storage processes with Lévy-type input

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- $Z = X + Y$.

- (K-Whitt) The following are (local) martingales:

$$
\begin{aligned}
\psi(\alpha)\int_0^t e^{i\alpha Z(s)}ds \quad &+ \quad e^{i\alpha Z(0)} - e^{i\alpha Z(t)} \\
&+ \quad i\alpha \int_0^t e^{i\alpha Z(s)} dY^c(s) \\
&+ \quad \sum_{0<s\leq t} e^{iZ(s)}\left(1 - e^{-i\Delta Y(s)}\right)
\end{aligned}
$$

- When $\nu(-\infty,0)=0$:

$$\begin{aligned}\varphi(\alpha)\int_0^t e^{-\alpha Z(s)}ds &+ e^{-\alpha Z(0)} - e^{-\alpha Z(t)}\\ &- \alpha\int_0^t e^{-\alpha Z(s)}dY^c(s)\\ &+ \sum_{0<s\leq t} e^{-\alpha Z(s)}\left(1-e^{\alpha\Delta Y(s)}\right)\end{aligned}$$

(K-Boxma, 2013)
$X = (X_1, \ldots, X_K)$ - càdlàg, K-dim. Lévy proc. w.r.t. $\mathcal{F}$ with exponent

$$\psi(\alpha) = ic^T\alpha - \frac{\alpha^T\Sigma\alpha}{2} + \int_{\mathbb{R}^K} \left(e^{i\alpha^T x} - 1 - i\alpha^T x 1_{\{\|x\| \le 1\}}\right) \nu(dx)$$

$$\varphi(\alpha) = -c^T\alpha + \frac{\alpha^T\Sigma\alpha}{2} + \int_{\mathbb{R}_+^K} \left(e^{-\alpha^T x} - 1 + \alpha^T x 1_{\{\|x\| \le 1\}}\right) \nu(dx)$$

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- $Z(t) = \sum_{k=1}^{K} \int_{(0,t]} I_k(s-) dX_k(s) + Y(t)$

then the following are (local) martingales:

$$\int_0^t \psi(\alpha I(s)) e^{i\alpha Z(s)} ds + e^{i\alpha Z(0)} - e^{i\alpha Z(t)} + i\alpha \int_0^t e^{i\alpha Z(s)} dY^c(s) + \sum_{0<s\leq t} e^{i\alpha Z(s)} \left(1 - e^{-i\alpha \Delta Y(s)}\right)$$

$$\int_0^t \varphi(\alpha I(s)) e^{-\alpha Z(s)} ds + e^{-\alpha Z(0)} - e^{-\alpha Z(t)} - \alpha \int_0^t e^{-\alpha Z(s)} dY^c(s) + \sum_{0<s\leq t} e^{-\alpha Z(s)} \left(1 - e^{\alpha \Delta Y(s)}\right)$$

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and ideas from K&W(1992).

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other than $Z \geq 0$ *a.s. when* $\nu\left(\left(\mathbb{R}_+^K\right)^c\right) = 0$:

- M *is an* L^2*-martingale.*
- $M(t)/t \to 0$ *a.s. and in* L^2.

Lemma

Let X be a semimartingale and $f \in \mathcal{C}^2$. Then $f(X)$ is also a semimartingale with:

$$[f(X), f(X)](t) = \int_0^t \left(f'(X(s))\right)^2 d[X,X]^c(s) + \sum_{0 \le s \le t} (\Delta f(X(s)))^2 .$$

Corollary

X is a semimartingale, Y is VF, adapted and $Z = X + Y$, then

$$\left[e^{-\alpha Z}, e^{-\alpha Z}\right](t) = \alpha^2 \int_0^t e^{-2\alpha Z(s)} d[X,X]^c(s) + \sum_{0 \le s \le t} e^{-2\alpha Z(s-)} \left(1 - e^{-\alpha \Delta Z(s)}\right)^2.$$

Lemma

X is a semimartingale and $f, g \in \mathcal{C}^2$, then

$$[f(X), g(X)](t) = \int_0^t f'(X(s))g'(X(s))d[X,X]^c(s) + \sum_{0 \le s \le t} \Delta f(X(s))\Delta g(X(s))$$

Corollary

$$\left[e^{i\alpha Z}, e^{i\alpha Z}\right](t) = \alpha^2 \int_0^t e^{i2\alpha Z(s)} d[X,X]^c(s) + \sum_{0\le s\le t} e^{i2\alpha Z(s-)} \left(1 - e^{i\alpha \Delta Z(s)}\right)^2$$

$$\begin{aligned} M(t) &= \int_0^t \varphi(\alpha I(s)) e^{-\alpha Z(s)} ds + e^{-\alpha Z(0)} - e^{-\alpha Z(t)} \\ &\quad -\alpha \int_0^t e^{-\alpha Z(s)} dY^c(s) + \sum_{0<s\le t} e^{-\alpha Z(s)} \left(1 - e^{\alpha \Delta Y(s)}\right) \end{aligned}$$

Corollary

$$[M, M](t) = \alpha^2 \int_0^t e^{-2\alpha Z(s)} d\left[\tilde{X}, \tilde{X}\right]^c(s) + \sum_{0<s\leq t} e^{-2\alpha Z(s-)} \left(1 - e^{-\alpha \Delta \tilde{X}(s)}\right)^2$$

Lemma

$$[M, M](t) = \int_0^t e^{-2\alpha Z(s)} A(s) ds + \tilde{M}(t)$$

where

$$A(s) = \varphi(2\alpha I(s)) - 2\varphi(\alpha I(s)) \geq 0$$

is bounded and $\tilde{M}$ is a zero mean martingale having bounded jumps.

Corollary

$$E[M,M](t) = \int_0^t E e^{-2\alpha Z(s)} A(s)\,ds$$

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\end{aligned}
$$

Hence M is an L^2 martingale (with $EM^2(t) = E[M,M](t)$) and $M(t)/t \to 0$ a.s. and in L^2.

Since $e^{-\alpha Z(t)} \leq 1$ we actually have that:

$$\frac{1}{t}\int_0^t \varphi(\alpha I(s)) e^{-\alpha Z(s)} ds - \alpha \frac{1}{t}\int_0^t e^{-\alpha Z(s)} dY^c(s)$$

$$+\frac{1}{t}\sum_{0<s\leq t} e^{-\alpha Z(s)}\left(1 - e^{\alpha \Delta Y(s)}\right) \to 0$$

Or

$$\frac{1}{t}\int_0^t \varphi(\alpha I(s))e^{-\alpha Z(s)}ds - \alpha\frac{1}{t}\int_{(0,t]} e^{-\alpha Z(s)}dY(s)$$

$$-\frac{1}{t}\sum_{0<s\leq t} e^{-\alpha Z(s)}\left(e^{\alpha\Delta Y(s)} - 1 - \alpha\Delta Y(s)\right) \to 0$$

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(iiia) $L(t-) < L(s), \ \forall s > t \Rightarrow Z(t) = 0$
(**K.** 2006 - Reflecting thoughts, SPL).

Then

$$\frac{1}{t}\int_0^t \psi(\alpha I(s))e^{i\alpha Z(s)}ds + i\alpha\frac{1}{t}\int_{(0,t]} e^{i\alpha Z(s)}dL(s)$$

$$-\frac{1}{t}\sum_{0<s\leq t} e^{i\alpha Z(s)}\left(e^{-i\alpha\Delta L(s)} - 1 + i\alpha\Delta L(s)\right) \to 0$$

becomes:

$$\frac{1}{t}\int_0^t \psi(\alpha I(s))e^{i\alpha Z(s)}ds + i\alpha\frac{L(t)}{t}$$

$$-\frac{1}{t}\sum_{0<s\leq t}\left(e^{-i\alpha\Delta L(s)} - 1 + i\alpha\Delta L(s)\right) \to 0$$

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In particular if $\xi \leq 0$ then $(\xi^+, -\xi^-) = (0, -\xi)$

Consequently, if $\tilde{X}(t)/t \to \xi \leq 0$ a.s. then

$$\frac{1}{t}\int_0^t \psi(\alpha I(s))e^{i\alpha Z(s)}ds$$

$$-\frac{1}{t}\sum_{0<s\leq t}\left(e^{-i\alpha \Delta L(s)} - 1 + i\alpha \Delta L(s)\right) \to i\alpha\xi \ .$$

When $\nu\left(\left(\mathbb{R}_+^K\right)^c\right) = 0$ then $\Delta L(s) = 0$ which gives

$$\frac{1}{t}\int_0^t \varphi(\alpha I(s))e^{-\alpha Z(s)}ds \to -\alpha\xi \ .$$

Theorem

(strong law for $\tilde{X} = \sum_{k=1}^{K} \int I_k dX_k$)

Assume that $\int_{|x|>1} |x| \nu_k(dx) < \infty$ *(equivalently,* $E|X_k(1)| < \infty$*), for each* k*, and that, a.s.*

$$\frac{1}{t} \int_0^t I_k(s) ds \to \beta_k$$

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Then, a.s.,

$$\xi = \lim_{t\to\infty} \frac{\tilde{X}(t)}{t} = \sum_{k=1}^{K} \beta_k EX_k(1) = -i\sum_{k=1}^{K} \beta_k \psi_k'(0).$$

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and when $\varphi'(0) > 0$ we have on the right the famous generalized PK formula

$$\frac{1}{t}\int_0^t e^{-\alpha Z(s)}ds \to \frac{\alpha\varphi'(0)}{\varphi(\alpha)} \; .$$

The key idea for the theorem is to show, for $K = 1$ with $X = X_1$ and $I = I_1$, that if $E|X(1)| < \infty$, then a.s.,

$$\frac{\int_{(0,t]} I(s-)dX(s) - EX(1)\int_0^t I(s)ds}{t} \to 0$$

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Since also: $X(t)/t \to EX(1)$,
then, when $EX(1) \neq 0$, this is equivalent to

$$\frac{1}{X(t)}\int_{(0,t]} I(s-)dX(s) - \frac{1}{t}\int_0^t I(s)ds \to 0$$

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$$\frac{\sum_{i=1}^{n} I(T_i-)Y_i}{\sum_{i=1}^{n} Y_i} \qquad \text{or} \qquad \frac{1}{n}\sum_{i=1}^{n} I(T_i-)\frac{Y_i}{\eta} \ ,$$

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both $= \dfrac{1}{n}\displaystyle\sum_{i=1}^{n} I(T_i-)$ when $Y_i = const.$, converges a.s. iff

$$\frac{1}{t}\int_0^t I(s)ds$$

converges a.s. and the limits are equal.

Boxma & **K**. Decomposition results for stochastic storage processes and queues with alternating Lévy inputs. Under review.

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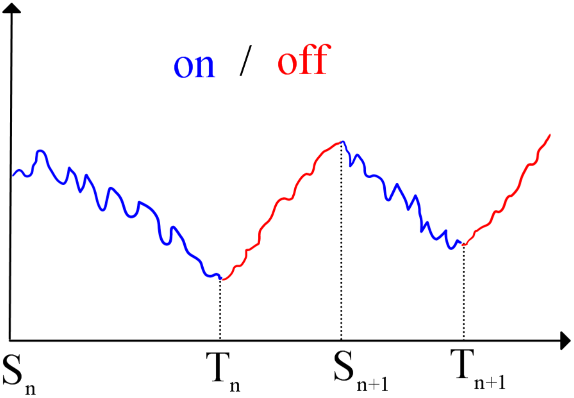
- $N(t) = \sup\{n|\ T_n \le t\}$
- $J(t) = 1_{\{S_{N(t)+1} > t\}} = 1_{\{J(t)=1\}}$ (1 if *off*)


on / off
S n
T n
S n+1
T n+1

- During *on* times the process behaves like a (reflected) Lévy process with no negative jumps with exponent

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- During *off* times it behaves like a subordinator with exponent

$$-\eta(\alpha) = -c_d\alpha - \int_{(0,\infty)} (1 - e^{-\alpha x})\nu_d(dx)$$

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- Then

$$\frac{\eta(\alpha)}{\alpha\eta'(0)} = \frac{c_d}{c_d + \bar{\nu}_d} + \frac{\bar{\nu}_d}{c_d + \bar{\nu}_d}\int_0^\infty e^{-\alpha x}\,\frac{\nu_d(x,\infty)}{\bar{\nu}_d}dx$$

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Thus

$$\frac{1}{t}\int_0^t \varphi(\alpha I(s))e^{-\alpha Z(s)}ds \to -\xi\alpha$$

where $I(s) = (J(s), 1 - J(s))$ and

$$\varphi(\alpha I(s)) = -J(s)\eta(\alpha) + (1 - J(s))\varphi(\alpha) \ .$$

Thus

$$\varphi(\alpha) E e^{-\alpha Z^*} - (\eta(\alpha) + \varphi(\alpha)) \lim_{t\to\infty} \frac{1}{t} \int_0^t J(s) e^{-\alpha Z(s)} ds = -\xi\alpha$$

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$$\frac{1}{t} \int_0^t J(s) ds \to q$$

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and thus

$$\varphi(\alpha)Ee^{-\alpha Z^*} = (\eta(\alpha) + \varphi(\alpha))qEe^{-\alpha Z_d} - \xi\alpha$$

where

$$-\xi = -q\eta'(0) + (1-q)\varphi'(0)$$

$$Ee^{-\alpha Z^*} = (1-\pi)\frac{\alpha\varphi'(0)}{\varphi(\alpha)} + \pi\left(1 - p + p\,\frac{\eta(\alpha)}{\alpha\eta'(0)}\,\frac{\alpha\varphi'(0)}{\varphi(\alpha)}\right) Ee^{-\alpha Z_d}$$

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When $\varphi(\alpha) = r\alpha - \eta(\alpha)$ then

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where $I \sim \mathrm{B}(1, \pi)$ and all variables on the right are independent. π can be one (e.g., no need for reflection) or zero (e.g., negligible *off* times).

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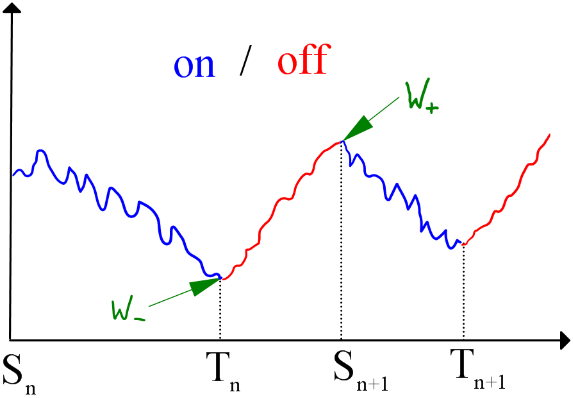
a.s. for some r.v. $W_+ \geq 0$ if and only if

$$\frac{1}{n}\sum_{k=1}^{n} e^{-\alpha W(T_{k-1})} \to Ee^{-\alpha W_-} \qquad \textit{(beginning of off period)}$$

a.s. for some r.v. $W_- \geq 0$.


on / off
W+
W-
S n
T n
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Moreover, if any two of EW_d, EW_-, EW_+ *are finite, then so is the third and:*

$$\frac{Ee^{-\alpha W_-} - Ee^{-\alpha W_+}}{\alpha(EW_+ - EW_-)} = \frac{\eta(\alpha)}{\alpha\eta'(0)} Ee^{-\alpha W_d} \quad .$$

THANK YOU!