

Useful martingales for stochastic  
storage processes with **Lévy-type**  
input

Offer Kella

*The Hebrew University of Jerusalem*

# 7th International Conference on Lévy Processes

July 15-19, 2013, Wrocław

- ▶ **K & Whitt.** (1992). Useful martingales for stochastic storage processes with Lévy input. *J. Appl. Prob.* **29**, 396-403.
- ▶ **Asmussen & K.** (2000). A multi-dimensional martingale for Markov additive processes and its applications. *Adv. Appl. Prob.* **32**, 376-393.
- ▶ **Asmussen & K.** (2001). On optional stopping of some exponential martingales for Lévy processes with or without reflection. *Stoch. Proc. Appl.* **91**, 47-55.
- ▶ **K & Boxma.** (2013). Useful martingales for stochastic storage processes with **Lévy-type** input. *J. Appl. Prob.* **50**, 439-449.

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$$\psi(\alpha) = i\alpha a - \frac{\sigma^2}{2}\alpha^2 + \int_{\mathbb{R}} (e^{i\alpha x} - 1 - i\alpha x \mathbf{1}_{\{|x| \leq 1\}}) \nu(dx)$$

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$$\varphi(\alpha) = -c \alpha + \frac{\sigma^2}{2} \alpha^2 + \int_{(0, \infty)} (e^{-\alpha x} - 1 + \alpha x \mathbf{1}_{\{|x| \leq 1\}}) \nu(dx)$$

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- ▶  $Y = \{Y(t) \mid t \geq 0\}$  - càdlàg, FV, adapted.
- ▶  $Z = X + Y$ .



- ▶ (K-Whitt) The following are (local) martingales:

$$\begin{aligned} \psi(\alpha) \int_0^t e^{i\alpha Z(s)} ds &+ e^{i\alpha Z(0)} - e^{i\alpha Z(t)} \\ &+ i\alpha \int_0^t e^{i\alpha Z(s)} dY^c(s) \\ &+ \sum_{0 < s \leq t} e^{iZ(s)} \left( 1 - e^{-i\Delta Y(s)} \right) \end{aligned}$$

- ▶ When  $\nu(-\infty, 0) = 0$ :

$$\begin{aligned} \varphi(\alpha) \int_0^t e^{-\alpha Z(s)} ds &+ e^{-\alpha Z(0)} - e^{-\alpha Z(t)} \\ &- \alpha \int_0^t e^{-\alpha Z(s)} dY^c(s) \\ &+ \sum_{0 < s \leq t} e^{-\alpha Z(s)} \left( 1 - e^{\alpha \Delta Y(s)} \right) \end{aligned}$$

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(K-Boxma, 2013)

$X = (X_1, \dots, X_K)$  - càdlàg,  $K$ -dim. Lévy proc. w.r.t.  $\mathcal{F}$  with exponent

$$\psi(\alpha) = ic^T \alpha - \frac{\alpha^T \Sigma \alpha}{2} + \int_{\mathbb{R}^K} \left( e^{i\alpha^T x} - 1 - i\alpha^T x 1_{\{\|x\| \leq 1\}} \right) \nu(dx)$$

$$\varphi(\alpha) = -c^T \alpha + \frac{\alpha^T \Sigma \alpha}{2} + \int_{\mathbb{R}_+^K} \left( e^{-\alpha^T x} - 1 + \alpha^T x 1_{\{\|x\| \leq 1\}} \right) \nu(dx)$$

- ▶  $I = (I_1, \dots, I_K)$  - bounded, adapted, càdlàg.

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▶  $I = (I_1, \dots, I_K)$  - bounded, adapted, càdlàg.

▶ 
$$Z(t) = \sum_{k=1}^K \int_{(0,t]} I_k(s-) dX_k(s) + Y(t)$$

then the following are (local) martingales:

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$$\int_0^t \psi(\alpha l(s)) e^{i\alpha Z(s)} ds + e^{i\alpha Z(0)} - e^{i\alpha Z(t)} + i\alpha \int_0^t e^{i\alpha Z(s)} dY^c(s) \\ + \sum_{0 < s \leq t} e^{i\alpha Z(s)} (1 - e^{-i\alpha \Delta Y(s)})$$

$$\int_0^t \varphi(\alpha l(s)) e^{-\alpha Z(s)} ds + e^{-\alpha Z(0)} - e^{-\alpha Z(t)} - \alpha \int_0^t e^{-\alpha Z(s)} dY^c(s) \\ + \sum_{0 < s \leq t} e^{-\alpha Z(s)} (1 - e^{\alpha \Delta Y(s)})$$

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$$e^{i\alpha \sum_{k=1}^K \int_{(0,t]} I_k(s-) dX_k(s) - \int_0^t \psi(\alpha I(s)) ds}$$

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2nd Ed., Cambridge Univ. Press.

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and ideas from K&W(1992).

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- ▶  *$M$  is an  $L^2$ -martingale.*

## Theorem

*WITHOUT ADDITIONAL CONDITIONS*

*other than  $Z \geq 0$  a.s. when  $\nu((\mathbb{R}_+^K)^c) = 0$ :*

- ▶  *$M$  is an  $L^2$ -martingale.*
- ▶  *$M(t)/t \rightarrow 0$  a.s. and in  $L^2$ .*

### Lemma

Let  $X$  be a semimartingale and  $f \in \mathcal{C}^2$ . Then  $f(X)$  is also a semimartingale with:

$$[f(X), f(X)](t) = \int_0^t (f'(X(s)))^2 d[X, X]^c(s) + \sum_{0 \leq s \leq t} (\Delta f(X(s)))^2.$$

### Corollary

$X$  is a semimartingale,  $Y$  is VF, adapted and  $Z = X + Y$ , then

$$\begin{aligned} [e^{-\alpha Z}, e^{-\alpha Z}](t) &= \alpha^2 \int_0^t e^{-2\alpha Z(s)} d[X, X]^c(s) \\ &+ \sum_{0 \leq s \leq t} e^{-2\alpha Z(s-)} \left(1 - e^{-\alpha \Delta Z(s)}\right)^2. \end{aligned}$$



### Lemma

$X$  is a semimartingale and  $f, g \in \mathcal{C}^2$ , then

$$\begin{aligned} [f(X), g(X)](t) &= \int_0^t f'(X(s))g'(X(s))d[X, X]^c(s) \\ &\quad + \sum_{0 \leq s \leq t} \Delta f(X(s))\Delta g(X(s)) \end{aligned}$$

## Corollary

$$\begin{aligned} \left[ e^{i\alpha Z}, e^{i\alpha Z} \right] (t) &= \alpha^2 \int_0^t e^{i2\alpha Z(s)} d[X, X]^c(s) \\ &+ \sum_{0 \leq s \leq t} e^{i2\alpha Z(s-)} \left( 1 - e^{i\alpha \Delta Z(s)} \right)^2 \end{aligned}$$

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$$M(t) = \int_0^t \varphi(\alpha I(s)) e^{-\alpha Z(s)} ds + e^{-\alpha Z(0)} - e^{-\alpha Z(t)} \\ - \alpha \int_0^t e^{-\alpha Z(s)} dY^c(s) + \sum_{0 < s \leq t} e^{-\alpha Z(s)} (1 - e^{\alpha \Delta Y(s)})$$

## Corollary

$$\begin{aligned} [M, M](t) &= \alpha^2 \int_0^t e^{-2\alpha Z(s)} d[\tilde{X}, \tilde{X}]^c(s) \\ &\quad + \sum_{0 < s \leq t} e^{-2\alpha Z(s-)} \left(1 - e^{-\alpha \Delta \tilde{X}(s)}\right)^2 \end{aligned}$$

## Lemma

$$[M, M](t) = \int_0^t e^{-2\alpha Z(s)} A(s) ds + \tilde{M}(t)$$

where

$$A(s) = \varphi(2\alpha I(s)) - 2\varphi(\alpha I(s)) \geq 0$$

*is bounded and  $\tilde{M}$  is a zero mean martingale having bounded jumps.*

Corollary

$$E[M, M](t) = \int_0^t Ee^{-2\alpha Z(s)} A(s) ds$$

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$$\int_{[0, \infty)} \frac{d[M, M](s)}{(1+s)^2} = \int_0^\infty \frac{Ee^{-2\alpha Z(s)} A(s)}{(1+s)^2} ds$$



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Hence  $M$  is an  $L^2$  martingale (with  $EM^2(t) = E[M, M](t)$ ) and  $M(t)/t \rightarrow 0$  a.s. and in  $L^2$ .

Since  $e^{-\alpha Z(t)} \leq 1$  we actually have that:

$$\begin{aligned} & \frac{1}{t} \int_0^t \varphi(\alpha I(s)) e^{-\alpha Z(s)} ds - \alpha \frac{1}{t} \int_0^t e^{-\alpha Z(s)} dY^c(s) \\ & + \frac{1}{t} \sum_{0 < s \leq t} e^{-\alpha Z(s)} \left( 1 - e^{\alpha \Delta Y(s)} \right) \rightarrow 0 \end{aligned}$$

Or

$$\frac{1}{t} \int_0^t \varphi(\alpha I(s)) e^{-\alpha Z(s)} ds - \alpha \frac{1}{t} \int_{(0,t]} e^{-\alpha Z(s)} dY(s)$$

$$- \frac{1}{t} \sum_{0 < s \leq t} e^{-\alpha Z(s)} \left( e^{\alpha \Delta Y(s)} - 1 - \alpha \Delta Y(s) \right) \rightarrow 0$$

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- (iiib)  $L$  is minimal.
- (iiia)  $L(t-) < L(s)$ ,  $\forall s > t \Rightarrow Z(t) = 0$   
(K. 2006 - Reflecting thoughts, SPL).

Then

$$\frac{1}{t} \int_0^t \psi(\alpha I(s)) e^{i\alpha Z(s)} ds + i\alpha \frac{1}{t} \int_{(0,t]} e^{i\alpha Z(s)} dL(s)$$
$$- \frac{1}{t} \sum_{0 < s \leq t} e^{i\alpha Z(s)} \left( e^{-i\alpha \Delta L(s)} - 1 + i\alpha \Delta L(s) \right) \rightarrow 0$$

becomes:

$$\frac{1}{t} \int_0^t \psi(\alpha l(s)) e^{i\alpha Z(s)} ds + i\alpha \frac{L(t)}{t}$$

$$-\frac{1}{t} \sum_{0 < s \leq t} \left( e^{-i\alpha \Delta L(s)} - 1 + i\alpha \Delta L(s) \right) \rightarrow 0$$

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$$\frac{x(t)}{t} \rightarrow \xi \iff \left( \frac{z(t)}{t}, \frac{\ell(t)}{t} \right) \rightarrow (\xi^+, -\xi^-)$$

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In particular if  $\xi \leq 0$  then  $(\xi^+, -\xi^-) = (0, -\xi)$

Consequently, if  $\tilde{X}(t)/t \rightarrow \xi \leq 0$  a.s. then

$$\frac{1}{t} \int_0^t \psi(\alpha I(s)) e^{i\alpha Z(s)} ds$$
$$-\frac{1}{t} \sum_{0 < s \leq t} \left( e^{-i\alpha \Delta L(s)} - 1 + i\alpha \Delta L(s) \right) \rightarrow i\alpha \xi .$$

When  $\nu((\mathbb{R}_+^K)^c) = 0$  then  $\Delta L(s) = 0$  which gives

$$\frac{1}{t} \int_0^t \varphi(\alpha I(s)) e^{-\alpha Z(s)} ds \rightarrow -\alpha \xi .$$

### Theorem

(strong law for  $\tilde{X} = \sum_{k=1}^K \int I_k dX_k$ )

Assume that  $\int_{|x|>1} |x| \nu_k(dx) < \infty$  (equivalently,  $E|X_k(1)| < \infty$ ),  
for each  $k$ , and that, a.s.

$$\frac{1}{t} \int_0^t I_k(s) ds \rightarrow \beta_k$$

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Then, a.s.,

$$\xi = \lim_{t \rightarrow \infty} \frac{\tilde{X}(t)}{t} = \sum_{k=1}^K \beta_k EX_k(1) = -i \sum_{k=1}^K \beta_k \psi'_k(0).$$

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When  $K = 1$  and  $I(t) = 1$ , one has

$$\varphi(\alpha) \frac{1}{t} \int_0^t e^{-\alpha Z(s)} ds \rightarrow \alpha(\varphi'(0))^+$$



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When  $K = 1$  and  $I(t) = 1$ , one has

$$\varphi(\alpha) \frac{1}{t} \int_0^t e^{-\alpha Z(s)} ds \rightarrow \alpha(\varphi'(0))^+$$

and when  $\varphi'(0) > 0$  we have on the right the famous generalized PK formula

$$\frac{1}{t} \int_0^t e^{-\alpha Z(s)} ds \rightarrow \frac{\alpha \varphi'(0)}{\varphi(\alpha)} .$$

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The key idea for the theorem is to show, for  $K = 1$  with  $X = X_1$  and  $I = I_1$ , that if  $E|X(1)| < \infty$ , then a.s.,

$$\frac{\int_{(0,t]} I(s-) dX(s) - EX(1) \int_0^t I(s) ds}{t} \rightarrow 0$$

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Since also:  $X(t)/t \rightarrow EX(1)$ ,  
then, when  $EX(1) \neq 0$ , this is equivalent to

$$\frac{1}{X(t)} \int_{(0,t]} I(s-)dX(s) - \frac{1}{t} \int_0^t I(s)ds \rightarrow 0$$

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$$\text{and } EX(1) \neq 0 \Leftrightarrow -i\psi'(0) = c + \int_{[-1,1]^c} x\nu(dx) \neq 0,$$



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$$\frac{1}{X(t)} \int_{(0,t]} I(s-) dX(s)$$

converges a.s. iff

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E.g.  $X(t) = \sum_{i=1}^{\infty} Y_i 1_{\{T_i \leq t\}}$  - compound Poisson

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E.g.  $X(t) = \sum_{i=1}^{\infty} Y_i 1_{\{T_i \leq t\}}$  - compound Poisson  
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both  $= \frac{1}{n} \sum_{i=1}^n I(T_{i-})$  when  $Y_i = \text{const.}$ , converges a.s. iff

$$\frac{1}{t} \int_0^t I(s) ds$$

converges a.s. and the limits are equal.

**Boxma & K.** Decomposition results for stochastic storage processes and queues with alternating Lévy inputs. Under review.

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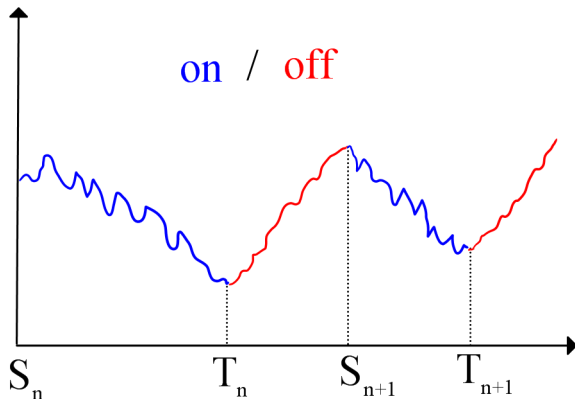
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- ▶  $J(t) = 1_{\{S_{N(t)+1} > t\}} = 1_{\{J(t)=1\}}$  (1 if *off*)

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- ▶ During *on* times the process behaves like a (reflected) Lévy process with no negative jumps with exponent

$$\varphi(\alpha) = -c_u \alpha + \frac{\sigma_u^2 \alpha^2}{2} + \int_{(0, \infty)} (e^{-\alpha x} - 1 + \alpha x \mathbf{1}_{\{x \leq 1\}}) \nu_u(dx)$$

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- ▶ During *off* times it behaves like a subordinator with exponent

$$-\eta(\alpha) = -c_d \alpha - \int_{(0, \infty)} (1 - e^{-\alpha x}) \nu_d(dx)$$

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Assume  $\varphi'(0) > 0$  (always finite) and  $\eta'(0) < \infty$  and note

- ▶  $\frac{\alpha\varphi'(0)}{\varphi(\alpha)}$  - stationary LST of RLP.

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- ▶ Then

$$\frac{\eta(\alpha)}{\alpha\eta'(0)} = \frac{c_d}{c_d + \bar{\nu}_d} + \frac{\bar{\nu}_d}{c_d + \bar{\nu}_d} \int_0^\infty e^{-\alpha x} \frac{\nu_d(x, \infty)}{\bar{\nu}_d} dx$$

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Hence:

$$\frac{\tilde{X}(t)}{t} \rightarrow qEX_d(1) + (1 - q)EX_u(1) = \xi$$

Thus

$$\frac{1}{t} \int_0^t \varphi(\alpha I(s)) e^{-\alpha Z(s)} ds \rightarrow -\xi \alpha$$

where  $I(s) = (J(s), 1 - J(s))$  and

$$\varphi(\alpha I(s)) = -J(s)\eta(\alpha) + (1 - J(s))\varphi(\alpha) .$$

Thus

$$\varphi(\alpha)Ee^{-\alpha Z^*} - (\eta(\alpha) + \varphi(\alpha)) \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t J(s)e^{-\alpha Z(s)} ds = -\xi\alpha$$

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$$\frac{\int_0^t e^{-\alpha Z(s)} J(s) ds}{\int_0^t J(s) ds} \rightarrow E e^{-\alpha Z_d}$$

and thus

$$\varphi(\alpha) E e^{-\alpha Z^*} = (\eta(\alpha) + \varphi(\alpha)) q E e^{-\alpha Z_d} - \xi \alpha$$

where

$$-\xi = -q\eta'(0) + (1 - q)\varphi'(0)$$

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$$Ee^{-\alpha Z^*} = (1 - \pi) \frac{\alpha \varphi'(0)}{\varphi(\alpha)} + \pi \left( 1 - p + p \frac{\eta(\alpha)}{\alpha \eta'(0)} \frac{\alpha \varphi'(0)}{\varphi(\alpha)} \right) Ee^{-\alpha Z_d}$$

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When  $\varphi(\alpha) = r\alpha - \eta(\alpha)$  then

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where  $I \sim B(1, \pi)$  and all variables on the right are independent.  
 $\pi$  can be one (e.g., no need for reflection) or zero (e.g., negligible *off* times).

## Theorem

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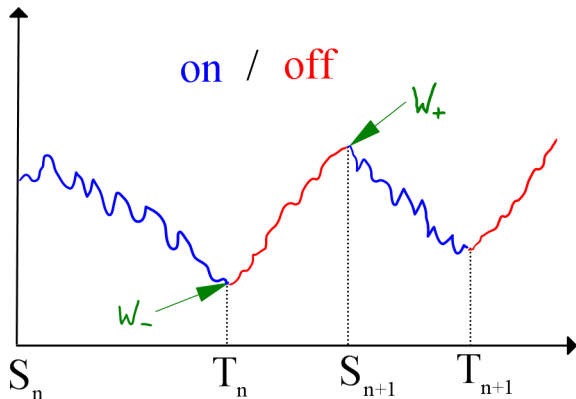
a.s. for some r.v.  $W_+ \geq 0$  if and only if

$$\frac{1}{n} \sum_{k=1}^n e^{-\alpha W(T_{k-1})} \rightarrow Ee^{-\alpha W_-} \quad (\text{beginning of off period})$$

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Moreover, if any two of  $EW_d$ ,  $EW_-$ ,  $EW_+$  are finite, then so is the third and:

$$\frac{Ee^{-\alpha W_-} - Ee^{-\alpha W_+}}{\alpha(EW_+ - EW_-)} = \frac{\eta(\alpha)}{\alpha\eta'(0)} Ee^{-\alpha W_d} .$$



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THANK YOU!