Anticipating Linear Stochastic Differential Equations Driven by Lévy Processes

Josep Vives

Departament de Probabilitat, Lògica i Estadística
Universitat de Barcelona

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Jointly with Jorge A. León (CINVESTAV-IPN) and David Márquez-Carreras (UB)

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The problem

In this talk, we consider the equation

$$
\begin{aligned}
X_t &= X_0 + \int_0^t b_s X_s ds \\
&+ \int_0^t a_s X_s \delta W_s \\
&+ \int_0^t \int_{\{|y|>1\}} v_{s-}(y) X_{s-} dN(s,y) \\
&+ \int_0^t \int_{\{0<|y|\le 1\}} v_{s-}(y) X_{s-} d\widetilde{N}(s,y),
\end{aligned}
$$

with $t \in [0, T]$.

The problem

- The stochastic integral with respect to W is in the Skorohod sense.
- The stochastic integrals with respect to Poisson random measures (with parameter ν) N and $\widetilde{N}$ are pathwise defined.
- Recall that $d\widetilde{N}(t,y) := dN(t,y) - dt\,\nu(dy)$.
- The random variable X_0 and the random processes a, b and $v(y)$ for any $y \in \mathbb{R}$ are not necessarily adapted to the underlying filtration.

The problem

- The goal is to prove the existence of a unique solution of the previous equation.
- In the adapted case, for not necessarily linear coefficients, the problem has been analyzed by many authors. The existence of a unique solution is proved using Picard iteration procedure and Gronwall's lemma. See for example the book of Ikeda and Watanabe.

The problem

In the general case, we cannot use the Picard iteration procedure nor Gronwall's lemma to deal with the equation because the L^2-norm of the solution depends on its derivative in the Malliavin calculus sense and this derivative can be estimated only in terms of the second derivative, and so on. Therefore we do not have a closed argument. See for example the book of Nualart.

The problem

Our technique to solve the problem will be based in three steps :
First, to consider the aproximated equation without little jumps :

$$\begin{aligned} X_t^\varepsilon &= X_0 + \int_0^t b_s X_s^\varepsilon ds + \int_0^t a_s X_s^\varepsilon \delta W_s \\ &+ \int_0^t \int_{\{|y|>1\}} v_{s-}(y) X_{s-}^\varepsilon dN(s,y) \\ &+ \int_0^t \int_{\{\varepsilon<|y|\leq 1\}} v_{s-}(y) X_{s-}^\varepsilon d\widetilde{N}(s,y) \end{aligned}$$

and to prove the existence of a unique solution of this equation reducing the problem between two big jumps to a Wiener type problem.

The problem

Secondly, solving the equation between two big jumps using the anticipative Girsanov transformations on the Wiener space introduced by Buckdhan.

Finally, showing that the unique solution of the aproximated equation converges to a unique solution of the general equation.

Contents

The canonical Lévy space

- In this paper we consider all the processes defined on the canonical Lévy space on $[0, T]$,

$$(\Omega, \mathcal{F}, P) = (\Omega_W \otimes \Omega_N, \mathcal{F}_W \otimes \mathcal{F}_N, P_W \otimes P_N),$$

  where $(\Omega_W, \mathcal{F}_W, P_W)$ is the canonical Wiener space and $(\Omega_N, \mathcal{F}_N P_N)$ is the canonical Lévy space for a pure jump Lévy process with Lévy measure ν.
- We will assume $\int_{\mathbb{R}} x^2 \nu(dx) < \infty$.

The canonical Lévy space

The canonical space for a pure jump Lévy process is defined as follows :

Let $\{\varepsilon_n : n \in \mathbb{N}\}$ be a strictly decreasing sequence of positive numbers such that $\varepsilon_1 = 1$, $\lim_{n\to\infty} \varepsilon_n = 0$ and $\nu(S_n) > 0$ for any $n \geq 1$, where

$$S_1 = \{x \in \mathbb{R} : \varepsilon_1 < |x|\}$$

and

$$S_n = \{x \in \mathbb{R} : \varepsilon_n < |x| \leq \varepsilon_{n-1}\}.$$

The canonical Lévy space

With the previous notation the canonical Lévy space with measure ν is

$$(\Omega_N, \mathcal{F}_N, P_N) = \bigotimes_{n\geq 1} (\Omega^{(n)}, \mathcal{F}^{(n)}, P^{(n)}),$$

where $(\Omega^{(n)}, \mathcal{F}^{(n)}, P^{(n)})$ is the canonical space for a compound Poisson process with intensity $\lambda_n := \nu(S_n)$ and probability measure $Q_n := \nu(\cdot \cap S_n)/\lambda_n$.

The canonical Lévy space

Finally, the canonical space for a compound Poisson process, for $n \in \mathbb{N}$, is given by

$$\Omega^{(n)} := \bigcup_{k \geq 0} \left([0, T] \times S_n\right)^k ,$$

with $([0, T] \times S_n)^0 = \{\alpha\}$, where α is an arbitrary point,

$$\mathcal{F}^{(n)} := \left\{ B \subset \Omega^{(n)} : B \cap \left([0, T] \times S_n\right)^k \in \mathcal{B}\left(\left([0, T] \times S_n\right)^k\right), \forall\, k \in \mathbb{N} \right\}$$

and for any $B \in \mathcal{F}^{(n)}$,

$$P^{(n)}(B) := e^{-\lambda_n T} \sum_{k=0}^{\infty} \frac{\lambda_n^k (\ell \otimes Q_n)^{\otimes k}(B \cap ([0, T] \times S_n)^k)}{k!}.$$

The canonical Lévy space

The canonical Wiener process $W = \{W_t : t \in [0, T]\}$ is defined as $W_t(\omega) = \omega(t)$ for $\omega \in \Omega_W$, that is, ω is a continuous function on $[0, T]$ such that $\omega(0) = 0$.

The canonical Lévy space

The canonical pure jump process $J_t = \{J_t : t \in [0, T]\}$, with Lévy measure ν, is

$$J_t(\omega) = \lim_{k\to\infty} \sum_{n=2}^{k} \left(X_t^{(n)}(\omega^{(n)}) - t \int_{S_n} x\nu(dx) \right) + X_t^{(1)}(\omega^{(1)})$$

where $\omega = (\omega^{(n)})_{n\geq 1} \in \Omega_N$ and where the limit exists with probability 1 and

$$X_t^{(n)}(\omega^{(n)}) = \begin{cases} \sum_{l=1}^{m} x_l 1_{[0,t]}(t_l), & \text{if } \omega^{(n)} = ((t_1, x_1), \ldots, (t_m, x_m)), \\ 0, & \text{if } \omega^{(n)} = \alpha. \end{cases}$$

The canonical Lévy space

Finally, the canonical Lévy process with triplet (γ, σ, ν) is defined as

$$X_t(\omega) = \gamma t + \sigma W_t(\omega') + J_t(\omega''),$$

for $\omega = (\omega', \omega'') \in \Omega_W \otimes \Omega_N$.

The derivative operator

We recall some ideas of Malliavin calculus for Hilbert space valued random variables.

Consider the set $\mathcal{S}^W(L^2(\Omega_N))$, the set of all smooth $L^2(\Omega_N)$-random variables of the form

$$F = \sum_{i=1}^{n} f_i\Big(\int_0^T h_{1,i}(s)dW_s, \ldots, \int_0^T h_{n_i,i}(s)dW_s\Big)G_i.$$

where
$h_{j,i} \in L^2([0,T])$, $G_i \in L^2(\Omega_N)$ and $f_i \in \mathcal{C}_b^\infty(\mathbb{R}^{n_i})$.

The derivative operator

The derivative of the random variable F with respect to W is the $L^2(\Omega_N \times [0, T])$-valued random variable

$$D^W F = \sum_{i=1}^{n} \sum_{j=1}^{n_i} (\partial_j f_i)\Big(\int_0^T h_{1,i}(s) dW_s, \dots, \int_0^T h_{n_i,i}(s) dW_s\Big) h_{j,i} G_i.$$

The operator can be extended to the derivative operator

$$D^W : \mathbb{D}^W_{1,2}(L^2(\Omega_N)) \subset L^2(\Omega) \to L^2(\Omega \times [0, T]).$$

The derivative operator

In general, for any $k, p \geq 1$, we can consider the spaces $\mathbb{D}^W_{k,p}(L^2(\Omega_N))$ as the closure of $\mathcal{S}^W(L^2(\Omega_N))$ with respect to the norm

$$\|F\|^p_{W,k,p} := \left|\, |F|_{L^2(\Omega_N)} \,\right|^p_{L^p(\Omega_W)} + \sum_{j=1}^{k} \left| \left(\int_{[0,T]^j} |D^{W,j}_z F|^2_{L^2(\Omega_N)} dz \right)^{\frac{1}{2}} \right|^p_{L^p(\Omega_W)}$$

The derivative operator

In the case $k=1$ and $p=2$, we have $\mathbb{D}^W_{1,2}(L^2(\Omega_N))$ as the closure of $\mathcal{S}^W(L^2(\Omega_N))$ with respect to the norm

$$\|F\|^2_{W,1,2} := |F|^2_{L^2(\Omega)} + |D^W F|^2_{L^2(\Omega\times[0,T])}.$$

The divergence operator

The Skorohod integral with respect to W, denoted by δ^W, is the adjoint of the derivative operator D^W.

A process u is in *Dom* δ^W if and only if $u \in L^2(\Omega \times [0,T])$ and there exists a random variable $\delta^W(u) \in L^2(\Omega)$ satisfying the duality relation

$$\mathbb{E}\left[\int_0^T u_t D_t^W F dt\right] = \mathbb{E}\left[\delta^W(u) F\right] \quad \text{for every} \quad F \in \mathbb{D}_{1,2}^W(L^2(\Omega_N)).$$

The divergence operator

We can consider the operator D restricted to

$$\mathbb{D}^W_{1,\infty}(L^\infty(\Omega_N)) := \{F \in \mathbb{D}^W_{1,2}(L^2(\Omega_N)) \cap L^\infty(\Omega) : DF \in L^\infty(\Omega \times [0,T])\}$$

and extend δ^W to the dual operator of D on this space, that is u is in $Dom\ \delta^W$ if and only if $u \in L^1(\Omega \times [0,T])$ and there exists a random variable $\delta^W(u) \in L^1(\Omega)$ satisfying the duality relation

$$\mathbb{E}\left[\int_0^T u_t D_t^W F dt\right] = \mathbb{E}\left[\delta^W(u) F\right] \quad \text{for every} \quad F \in \mathbb{D}^W_{1,\infty}(L^\infty(\Omega_N)). \tag{1}$$

The divergence operator

For $u \in Dom\ \delta^W$, we will make use of the notation

$$\delta^W(u) = \int_0^T u_t \delta W_t$$

and for $u \mathbb{1}_{[0,t]} \in Dom\ \delta^W$, we will write

$$\delta^W(u \mathbb{1}_{[0,t]}) = \int_0^t u_s \delta W_s.$$

Properties of D and δ

Lemma

Let $F \in \mathbb{D}^W_{1,2}(L^2(\Omega_N))$ and $u \in Dom\ \delta^W \cap L^1(\Omega \times [0, T])$. Then, for almost all $\omega'' \in \Omega_N$,

$$F(\cdot, \omega'') \in \mathbb{D}^W_{1,2},$$

$$u(\cdot, \omega'') \in Dom\ \delta^W \cap L^1\left(\Omega_W \times [0, T]\right),$$

$$D^W F(\cdot, \omega'') = (D^W F)(\cdot, \omega'')$$

and

$$\delta^W(u(\cdot, \omega'')) = \delta^W(u)(\cdot, \omega'').$$

Properties of D and δ

Proof : $H \in \mathcal{S}^W$ and $G \in L^\infty(\Omega_N)$. Then,

$$\mathbb{E}\left[G \int_0^T u_t D_t^W H dt\right] = \mathbb{E}\left[G\delta^W(u)H)\right].$$

Consequently, for a.a. $\omega'' \in \Omega_N$,

$$\mathbb{E}_W\left[\int_0^T u_t(\cdot,\omega'') D_t^W H dt\right] = \mathbb{E}_W\left[\delta^W(u)(\cdot,\omega'')H(\cdot,\omega'')\right].$$

Then, using the duality relation and taking limits, we conlude the proof. $\square$

Contents

Girsanov transformations

- Consider $\omega = (\omega', \omega'') \in \Omega_W \times \Omega_N$.
- Given a process $a \in L^2(\Omega \times [0, T])$, we define the transformation $T_a : \Omega \to \Omega_W$ as the application defined by

$$T_a(\omega', \omega'') := \omega' + \int_0^{\cdot} a_s(\omega', \omega'') ds.$$

Observe that for ω'' fixed, we obtain a transformation on the Wiener space.

Girsanov transformations

- We say T_a is absolutely continuous if the measure $P_W \circ (T_a(\cdot, \omega''))^{-1}$ is absolutely continuous with respect to P_W, for almost all $\omega'' \in \Omega_N$.
- The Cameron-Martin space CM, that is, the subspace of absolutely continuous functions of Ω_W, with square-integrable derivatives, endowed with the norm

$$|\omega'|_{CM} := \left(\int_0^T \dot{\omega}'(t)^2 dt \right)^{\frac{1}{2}}.$$

Girsanov transformations

Given $a \in L^2([0, T]; \mathbb{D}^W_{1,\infty}(L^2(\Omega_N)))$ we consider

$$\{ T_t : \Omega \to \Omega_W : 0 \leq t \leq T\}$$

and

$$\{A_{s,t} : \Omega \to \Omega_W : 0 \leq s \leq t \leq T\}$$

as a solutions of the equations

$$(T_t \omega)_\cdot = \omega'_\cdot + \int_0^{t\wedge\cdot} a_s(T_s\omega, \omega'') \, ds.$$

and

$$(A_{s,t} \omega)_\cdot = \omega'_\cdot - \int_{s\wedge\cdot}^{t\wedge\cdot} a_r(A_{r,t}\omega, \omega'') \, dr,$$

respectively.

Girsanov transformations

Proposition

Let $a \in L^2([0,T]; \mathbb{D}^W_{1,\infty}(L^2(\Omega_N)))$. *Then, there exist two unique families of absolutely continuous transformations*

$$\{T_t : 0 \leq t \leq T\}$$

and

$$\{A_{s,t} : 0 \leq s \leq t \leq T\}$$

that satisfy above equations.

Moreover, for each $s < t$, $A_{s,t} = T_s A_t$, T_t *is invertible with inverse* $A_t := A_{0,t}$ *and* $a_\cdot(T_\cdot(*, \omega''), \omega'') \in L^2([0,T]; \mathbb{D}^W_{1,\infty})$, *for a.a.* $\omega'' \in \Omega_N$.

Auxiliary result

Lemma

Let $a \in L^2([0,T]; \mathbb{D}^W_{1,\infty}(L^2(\Omega_N)))$.

Then, for any $u \le s \le t$, *we have*

$$|A_{u,t}\omega - A_{u,s}\omega|^2_{CM} \le \left(\int_s^t ||a_r||^2_\infty dr \right) \exp\left\{ \int_0^T || \, |D^W a_r|_2^2 \, ||_\infty dr \right\}.$$

Auxiliary result

Proof Let $u \leq s \leq t$. Then,

$$
\begin{aligned}
&|A_{u,s}\omega - A_{u,t}\omega|^2_{CM} \\
&= \left| \int_{u\wedge\cdot}^{s\wedge\cdot} a_r(A_{r,s}\omega, \omega'')dr - \int_{u\wedge\cdot}^{t\wedge\cdot} a_r(A_{r,t}\omega, \omega'')dr \right|^2_{CM} \\
&= \int_0^T |\mathbb{1}_{(u,s]}(r) a_r(A_{r,s}\omega, \omega'') - \mathbb{1}_{(u,t]}(r) a_r(A_{r,t}\omega, \omega'')|^2 dr \\
&= \int_s^t |a_r(A_{r,t}\omega, \omega'')|^2 dr + \int_u^s |a_r(A_{r,s}\omega, \omega'') - a_r(A_{r,t}\omega, \omega'')|^2 dr \\
&\leq \int_s^t ||a_r||^2_\infty dr + \int_u^s || \, |D^W a_r|^2_2 \, ||_\infty |A_{r,s}\omega - A_{r,t}\omega|^2_{CM} dr.
\end{aligned}
$$

A Buckdahn result

$$\mathbb{E}\left[F(A_{s,t}\omega,\omega'')L_{s,t}(\omega)\right]=\mathbb{E}\left[F\right]$$

and

$$\mathbb{E}\left[F(A_{s,t}\omega,\omega'')\right]=\mathbb{E}\left[F\mathcal{L}_{s,t}\right],$$

where

$$\begin{aligned} L_{s,t}(\omega) &= \exp\left\{\int_s^t a_r(A_{r,t}\omega,\omega'')\delta W_r-\frac{1}{2}\int_s^t a_r^2(A_{r,t}\omega,\omega'')dr\right. \\ &\left. -\int_s^t\int_r^t (D_u^W a_r)(A_{r,t}\omega,\omega'')D_r^W[a_u(A_{u,t}\omega,\omega'')]dudr\right\} \end{aligned}$$

and

$$\begin{aligned} \mathcal{L}_{s,t}(\omega) &= \exp\left\{-\int_s^t a_r(T_tA_r\omega,\omega'')\delta W_r-\frac{1}{2}\int_s^t a_r^2(T_tA_r\omega,\omega'')dr\right. \\ &\left. -\int_s^t\int_s^r (D_u^W a_r)(T_tA_r\omega,\omega'')D_r^W[a_u(T_tA_u\omega,\omega'')]dudr\right\} \end{aligned}$$

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SDE on canonical Wiener space

On the canonical Wiener space, consider the linear stochastic differential equation

$$Z_t = Z_0 + \int_0^t h_s Z_s \, ds + \int_0^t a_s Z_s \, \delta W_s, \qquad t \in [0, T].$$

SDE on canonical Wiener space

The following theorem is due to Buckdhan :

Theorem

Assume $a \in L^2([0,T], \mathbb{D}^W_{1,\infty})$, $h \in L^1([0,T], L^\infty(\Omega))$ *and* $Z_0 \in L^\infty(\Omega)$. *Then, the process*

$$Z_t := Z_0(A_{0,t}) \exp\left\{ \int_0^t h_s(A_{s,t})\, ds \right\} L_{0,t} \tag{2}$$

belongs to $L^1(\Omega \times [0,T])$ *and is a solution of the previous equation. Conversely, if* $Y \in L^1(\Omega \times [0,T])$ *is a global solution of the previous equation, and if moreover,* $a, h \in L^\infty(\Omega \times [0,T])$ *and* $D^W a \in L^\infty(\Omega \times [0,T]^2)$, *then* Y *is of the form (2).*

SDE on canonical Wiener space

We need to go further and in the paper we have proved the following theorem :

Theorem

Assume $Z_0 \in \mathbb{D}^W_{1,\infty}$, $h \in L^1([0,T], \mathbb{D}^W_{1,\infty})$ *and that, for some* $p > 2$,

$$a \in L^{2p}([0,T], \mathbb{D}^W_{1,\infty}) \cap L^2([0,T], \mathbb{D}^W_{2,\infty}).$$

Then,

$$Z_t := Z_0(A_{0,t}) \exp\left\{ \int_0^t h_s(A_{s,t}) \, ds \right\} \, L_{0,t}$$

has continuous trajectories a.s.

SDE on canonical Wiener space

Idea of the proof :

$$
\begin{aligned}
&\left|\int_0^t g_r(A_{r,t})dr - \int_0^s g_r(A_{r,s})dr\right| \\
&\qquad \leq \int_s^t ||g_r||_\infty dr + \int_0^s ||\,|Dg_r|_2\,||_\infty |A_{r,t} - A_{r,s}|_{CM} dr \\
&\qquad \leq \int_s^t ||g_r||_\infty dr + \left(\int_0^T ||\,|Dg_r|_2\,||_\infty dr\right) \sup_{r\leq s} |A_{r,t} - A_{r,s}|_{CM}.
\end{aligned}
$$

Contents

Hypotheses

(H1) Assume that

- $X_0 \in \mathbb{D}^W_{1,\infty}(L^\infty(\Omega_N))$.
- b and $v_{\cdot -}(y)$, for any $y \in \mathbb{R}_0$, belong to $L^1([0,T], \mathbb{D}^W_{1,\infty}(L^\infty(\Omega_N))$.
- There exists $p > 2$ such that
$$a \in L^2([0,T], \mathbb{D}^W_{2,\infty}(L^\infty(\Omega_N)) \cap L^{2p}([0,T], \mathbb{D}^W_{1,\infty}(L^\infty(\Omega_N)).$$

Hypotheses

(H2) There exist a positive function $g \in L^2(\mathbb{R}_0, \nu) \cap L^1(\mathbb{R}_0, \nu)$ such that

$$|v_{s-}(y, \omega)| \leq g(y),$$

uniformly on ω and s, and such that

$$\lim_{|y| \to 0} g(y) = 0.$$

(H3) The function g satisfies $\int_{\mathbb{R}_0} (e^{2g(y)} - 1)\nu(dy) < \infty$.

Remark

Note that under the hypothesis (H2), the main equation and the aproximated one can be written (changing b), respectively as

$$X_t = X_0 + \int_0^t b_s X_s ds + \int_0^t a_s X_s \delta W_s + \int_0^t \int_{|y|>0} v_{s-}(y) X_{s-} d\widetilde{N}(s,y),$$

and

$$X_t^{\varepsilon} = X_0 + \int_0^t b_s X_s^{\varepsilon} ds + \int_0^t a_s X_s^{\varepsilon} \delta W_s + \int_0^t \int_{\{|y|>\varepsilon\}} v_{s-}(y) X_{s-}^{\varepsilon} d\widetilde{N}(s,y)$$

Aproximated solution

Given $\varepsilon > 0$, set

$$
\begin{aligned}
X_t^\varepsilon &= X_0(A_{0,t}) \exp\left\{ \int_0^t b_s(A_{s,t})\ ds \right\}\ L_{0,t} \\
&\times \exp\left\{ - \int_0^t \int_{\{|y|>\varepsilon\}} v_s(y, A_{s,t})\ \nu(dy)ds \right\} \\
&\times \prod_{s\le t, \varepsilon<|y|} \left[1 + v_s(y, A_{s,t}) \Delta N(s,y) \right]
\end{aligned}
$$

Aproximated solution

It is easy to see that it can also be written

$$X_t^\varepsilon = X_0(A_{0,t}) \exp\left\{ \int_0^t b_s^\varepsilon(A_{s,t})\, ds \right\} L_{0,t} \prod_{i=1}^{N_t^\varepsilon} \left[1 + v_{\tau_i^\varepsilon}(y_i^\varepsilon, A_{\tau_i^\varepsilon,t}) \right]$$

where $b_s^\varepsilon(\omega) := b_s(\omega) - \int_{\{|y|>\varepsilon\}} v_s(y,\omega)\nu(dy)$, $\{\tau_i^\varepsilon, i \geq 1\}$ are the jump times which jump size is greater than ε, y_i^ε denotes the amplitude of jump τ_i^ε, and N_t^ε is the number of jumps before t, with size bigger than ε.

Solution candidate

Proposition

Assume (H1) and (H2) hold. For each $t \in [0, T]$, the process X_t^ε converges almost surely to

$$
\begin{aligned}
X_t &= X_0(A_{0,t}) \exp\left\{ \int_0^t b_s(A_{s,t})\, ds \right\} L_{0,t} \\
&\times \exp\left\{ -\int_0^t \int_{\mathbb{R}_0} v_s(y, A_{s,t})\, \nu(dy)\, ds \right\} \\
&\times \prod_{s \le t, y \in \mathbb{R}_0} \Big[1 + v_s(y, A_{s,t}) \Delta N(s,y) \Big].
\end{aligned}
$$

Idea of the proof

The main fact is to see that

$$
\begin{aligned}
&\int_0^t \int_{\mathbb{R}_0} |v_s(y, A_{s,t})| \; dN(s,y) + \int_0^t \int_{\mathbb{R}_0} |v_s(y, A_{s,t})| \; \nu(dy)ds \\
\leq \; & \int_0^t \int_{\mathbb{R}_0} g(y) \; d\widetilde{N}(s,y) + 2 \int_0^t \int_{\mathbb{R}_0} g(y) \; \nu(dy)ds.
\end{aligned}
$$

and that this quantity is finite a.s. because (H2) and the fact that

$$
\mathbb{E}\left[\left(\int_0^t \int_{\mathbb{R}_0} g(y) \; d\tilde{N}(s,y) \right)^2 \right] = \int_0^t \int_{\mathbb{R}_0} g(y)^2 \; \nu(dy)ds.
$$

Solution candidate

Proposition

Assume (H1), (H2) and (H3). Then, the processes X^ε and X belong to $L^1(\Omega \times [0, T])$ and

$$X_t^\varepsilon \to X_t,$$

in $L^1(\Omega \times [0, T])$, as ε goes to zero.

Proof

The idea is to note that, thanks to (H3),

$$
\begin{aligned}
|X_t^\varepsilon| &\leq CL_{0,t} \exp\left\{ \sum_{s\leq t, \varepsilon<|y|} g(y)\Delta N(s,y) \right\} \\
&\leq CL_{0,t} \exp\left\{ \int_0^T \int_{\mathbb{R}_0} g(y) dN(s,y) \right\} \in L^1(\Omega),
\end{aligned}
$$

and to apply the dominated convergence theorem.

Existence and uniqueness of the solution of the aproximated equation

Theorem

Let $\varepsilon > 0$. Then, the process

$$
\begin{aligned}
X_t^{\varepsilon} &= X_0(A_{0,t})L_{0,t}\exp\left\{\int_0^t b_s^{\varepsilon}(A_{s,t})\ ds\right\} \\
&\times \prod_{i=1}^{N_t^{\varepsilon}}\left[1 + v_{\tau_i{}^{\varepsilon}-}(y_i{}^{\varepsilon}, A_{\tau_i{}^{\varepsilon},t})\right]
\end{aligned}
$$

is the unique solution in $L^1(\Omega \times [0,T])$ of

$$X_t^{\varepsilon} = X_0 + \int_0^t b_s X_s^{\varepsilon} ds + \int_0^t a_s X_s^{\varepsilon} \delta W_s + \int_0^t \int_{\{|y|>\varepsilon\}} v_{s-}(y) X_{s-}^{\varepsilon}\ d\tilde{N}(s,y),$$

Proof : Step 1

The process X^{ε} is a.s. càdlàg because as a consequence of (H1) and the previous results is easy to see that

$$X_0(A_{0,t})L_{0,t}\exp\left\{\int_0^t b_s^{\varepsilon}(A_{s,t})\ ds\right\}$$

is continuous on t and

$$\prod_{i=1}^{N_t{}^{\varepsilon}}\left[1+v_{\tau_i{}^{\varepsilon}-}(y_i{}^{\varepsilon},A_{\tau_i{}^{\varepsilon},t})\right]$$

is a finite product with all terms well defined and continuous.

Proof : Step 2

Second step is to check that X^ε is a solution of the aproximated equation. Consider a smooth random variable G and define

$$\Phi_s^\varepsilon(\omega) = \prod_{r\le s, \varepsilon<|y|} \Big[1 + v_{r-}(y, T_r)\Delta N(r, y) \Big].$$

Due to Girsanov's theorem and that $A_{r,t} = T_r \circ A_t$, we have

$$\begin{aligned}
&\mathbb{E}\left[\int_0^t a_s X_s^\varepsilon D_s^W G ds \right] \\
&= \mathbb{E}\left[\int_0^t a_s X_0(A_{0,s}) \exp\left\{ \int_0^s b_r^\varepsilon(A_{r,s}) dr \right\} L_{0,s} \Phi_s^\varepsilon(A_s) D_s^W G ds \right] \\
&= \mathbb{E}\left[\int_0^t a_s(T_s) X_0 \exp\left\{ \int_0^s b_r^\varepsilon(T_r) dr \right\} \Phi_s^\varepsilon \, (D_s^W G)(T_s) ds \right].
\end{aligned}$$

Proof : Step 3

Since

$$\frac{d}{ds}G(T_s) = a_s(T_s)(D_s^W G)(T_s),$$

we get

$$\begin{aligned} &\mathbb{E}\left[\int_0^t a_s X_s^\varepsilon D_s^W G ds\right] \\ &= \mathbb{E}\left[\int_0^t \left(\frac{d}{ds}G(T_s)\right) X_0 \exp\left\{\int_0^s b_r^\varepsilon(T_r)dr\right\}\Phi_s^\varepsilon ds\right]. \end{aligned}$$

Proof : Step 4

Using

$$\int_0^t \int_{\{|y|>\varepsilon\}} dN(s,y) < \infty, \; a.s.,$$

we have

$$\mathbb{E}\left[\int_0^t a_s X_s^\varepsilon D_s^W G ds\right] = \sum_{i=1}^{\infty} \mathbb{E}\left[\int_{\tau_{i-1}^\varepsilon \wedge t}^{\tau_i^\varepsilon \wedge t} \left(\frac{d}{ds} G(T_s)\right) X_0 \exp\left\{\int_0^s b_r^\varepsilon(T_r) dr\right\} \Phi^\varepsilon_{\tau_{i-1}^\varepsilon \wedge t} ds\right].$$

Proof : Step 5

Finally, integration by parts and the fact that

$$\Phi^{\varepsilon}_{\tau^{\varepsilon}_i} = \Phi^{\varepsilon}_{\tau^{\varepsilon}_{i-1}}\left(1 + v_{\tau^{\varepsilon}_i -}(y^{\varepsilon}_i, T_{\tau^{\varepsilon}_i})\right)$$

imply

$$\begin{aligned}\mathbb{E}\left[\int_0^t a_s X^{\varepsilon}_s D^W_s G ds\right] &= \mathbb{E}\Bigg[G\Bigg(X^{\varepsilon}_t - X_0 - \int_0^t b_s X^{\varepsilon}_s ds \\ &\quad - \int_0^t \int_{\{|y|>\varepsilon\}} v_{s-}(y) X^{\varepsilon}_{s-} \, d\tilde{N}(s,y)\Bigg)\Bigg].\end{aligned}$$

Proof : Step 6

To show the uniqueness, let Y^ε be a solution of

$$X_t^\varepsilon = X_0 + \int_0^t b_s X_s^\varepsilon ds + \int_0^t a_s X_s^\varepsilon \delta W_s + \int_0^t \int_{\{|y|>\varepsilon\}} v_{s-}(y) X_{s-}^\varepsilon \, d\widetilde{N}(s,y).$$

Then, for $t \in [\tau_1^\varepsilon, \tau_2^\varepsilon)$,

$$\mathbb{E}\left[Y_t^\varepsilon G(A_{\tau_1^\varepsilon,t})\right] = \mathbb{E}\left[X_{\tau_1^\varepsilon}^\varepsilon G\right] + \mathbb{E}\left[\int_{\tau_1^\varepsilon}^t b_s^\varepsilon Y_s^\varepsilon G(A_{\tau_1^\varepsilon,s}) ds\right].$$

Proof : Step 7

Thus,

$$\mathbb{E}\left[Y_t^{\varepsilon}(T_{\tau_1^{\varepsilon},t})\mathcal{L}_{\tau_1^{\varepsilon},t}G\right] = \mathbb{E}\left[X_{\tau_1^{\varepsilon}}^{\varepsilon}G\right] + \mathbb{E}\left[\int_{\tau_1^{\varepsilon}}^{t} b_s^{\varepsilon}(T_{\tau_1^{\varepsilon},s})Y_s^{\varepsilon}(T_{\tau_1^{\varepsilon},s})\mathcal{L}_{\tau_1^{\varepsilon},s}Gds\right].$$

And so,

$$Y_t^{\varepsilon}(T_{\tau_1^{\varepsilon},t})\mathcal{L}_{\tau_1^{\varepsilon},t} = X_{\tau_1^{\varepsilon}}^{\varepsilon} + \int_{\tau_1^{\varepsilon}}^{t} b_s^{\varepsilon}(T_{\tau_1^{\varepsilon},s})Y_s^{\varepsilon}(T_{\tau_1^{\varepsilon},s})\mathcal{L}_{\tau_1^{\varepsilon},s}ds.$$

Proof : Step 8

Solving the equation for $t \in [\tau_1^\epsilon, \tau_2^\epsilon)$, so in the Gaussian setting, we have

$$Y_t^\varepsilon(T_{\tau_1^\varepsilon,t})\mathcal{L}_{\tau_1^\varepsilon,t} = X_{\tau_1^\varepsilon}^\varepsilon \exp\{\int_{\tau_1^\varepsilon}^t b_s^\varepsilon(T_{\tau_1^\varepsilon,s})ds\}.$$

And using Girsanov transformations and the precise expression of $X_{\tau_1^\varepsilon}^\varepsilon$ we obtain exactly X_t^ε.

The rest of the cases can be treate similarly.

Contents

Main result

Theorem

$$
\begin{aligned}
X_t &= X_0(A_{0,t}) L_{0,t} \prod_{s \le t, y \in \mathbb{R}_0} \Big[1 + v_{s-}(y, A_{s,t}) \Delta N(s,y) \Big] \\
&\quad \times \exp\left\{ \int_0^t b_s(A_{s,t})\, ds - \int_0^t \int_{\mathbb{R}_0} v_{s-}(y, A_{s,t})\, \nu(dy)\, ds \right\}
\end{aligned}
$$

is the unique solution in $L^1(\Omega \times [0,T])$ *of*

$$
X_t = X_0 + \int_0^t b_s X_s ds + \int_0^t a_s X_s \delta W_s + \int_0^t \int_{\mathbb{R}_0} v_{s-}(y) X_{s-}\, d\widetilde{N}(s,y),
$$

such that $\int_0^T \int_{\mathbb{R}_0} |v_s(y) X_{s-}| dN(s,y) \in L^1(\Omega)$

Proof : Step 1

We have

$$X^\varepsilon \to X \quad \text{in } L^1(\Omega \times [0, T])$$

and we know

$$\begin{aligned} \mathbb{E}\left[\int_0^t a_s X_s^\varepsilon D_s^W G ds\right] &= \mathbb{E}\Big[G\Big(X_t^\varepsilon - X_0 - \int_0^t b_s X_s^\varepsilon ds \\ &\quad - \int_0^t \int_{\{|y|>\varepsilon\}} v_{s-}(y) X_{s-}^\varepsilon \, d\widetilde{N}(s,y)\Big)\Big]. \end{aligned}$$

Then,

$$\begin{aligned} \mathbb{E}\left[\int_0^t a_s X_s D_s^W G ds\right] &= \mathbb{E}\Big[G\Big(X_t - X_0 - \int_0^t b_s X_s ds \\ &\quad - \int_0^t \int_{\{|y|>0\}} v_{s-}(y) X_{s-} d\widetilde{N}(s,y)\Big)\Big]. \end{aligned}$$

Proof : Step 2

For the uniqueness we have, given a solution Y, and erasing omegas for simplicity,

$$
\begin{aligned}
&\mathbb{E}_W\left[Y_t G(A_t)\right] = \mathbb{E}_W\left[X_0 G\right] \\
&+\mathbb{E}_W\left[\int_0^t G(A_s) b_s Y_s ds\right] \\
&+\mathbb{E}_W\left[\int_0^t \int_{\mathbb{R}_0} G(A_s) v_s(y) Y_{s-} d\widetilde{N}(s,y)\right].
\end{aligned}
$$

Proof : Step 3

Thus, Girsanov theorem implies, for any ω'' a.s.,

$$\mathbb{E}_W\left[\mathcal{L}_t Y_t(T_t) G\right] = \mathbb{E}_W\Bigg[G\Bigg(X_0 + \int_0^t b_s(T_s) Y_s(T_s)\mathcal{L}_s ds$$

$$+ \int_0^t \int_{\mathbb{R}_0} v_s(y, T_s) Y_{s-}(T_s)\mathcal{L}_s d\tilde{N}(s,y)\Bigg)\Bigg],$$

Proof : Step 4

Consequently, by Fubini's theorem, we also have that, for a. a. ω',

$$
\begin{aligned}
Y_t(T_t)\mathcal{L}_t = X_0 &+ \int_0^t b_s(T_s)Y_s(T_s)\mathcal{L}_s ds \\
&+ \int_0^t \int_{\mathbb{R}_0} v_s(y, T_s)Y_{s-}(T_s)\mathcal{L}_s d\tilde{N}(s,y), \qquad \omega'' \text{ a.s.}
\end{aligned}
$$

Finally, solving the equation, we have

$$
\begin{aligned}
Y_t(T_t)\mathcal{L}_t &= X_0 \exp\left\{\int_0^t b_s(T_s)ds + \int_0^t \int_{\mathbb{R}_0} v_s(y, T_s)d\tilde{N}(s,y)\right\} \\
&\times \prod_{0\le s\le t} \left[1 + v_s(y, T_s)\Delta N(s,y)\right],
\end{aligned}
$$

wich means that $Y = X$.

Thank you for your attention