A variation of the Canadisation algorithm for the pricing of American options driven by Lévy processes [3]

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Motivation

We consider the following classic optimal stopping problem with expiry date $T \in [0, \infty)$:

$$v(T, x) = \sup_{\tau \in \mathcal{T}} \mathbb{E}_x \left[e^{-r(\tau \wedge T)} f(X_{\tau \wedge T}) \right], \tag{1}$$

where f is a payoff function and $r \geq 0$ the discount rate. A classic example is the American put option with $f(x) = (K - e^x)^+$. Let X be a Lévy process defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$ with characteristic exponent Ψ. For $x \in \mathbb{R}$ we denote by $\mathbb{P}_x$ ($\mathbb{E}_x$) the law (associated expectation operator) of X when it is started at x. We denote by $\mathcal{T}$ the set of $\mathbf{F}$-stopping times.

Problem: No closed form formula exists for (1)! Therefore, one typically has to settle for approximating v rather.

Our set-up

Our setting can be seen as a modification of Carr's 'Canadisation' method [2] (see also [1]).

For any $n \in \mathbb{N}$, we enlarge the above probability space to contain a Poisson process $N^{(n)}$ with intensity n, independent of X. We denote the k-th jump time of $N^{(n)}$ by $T_k^{(n)}$, i.e. $T_k^{(n)} := \inf\{t \geq 0 \,|\, N_t^{(n)} \geq k\}$ with $T_0^{(n)} := 0$. Let $\widetilde{\mathbf{F}}^{(n)}$ denote the naturally enlarged filtration generated by the pair $(X, N^{(n)})$. Note that for any $T \geq 0$, if $(k(n))_{n \geq 1}$ is a sequence of natural numbers such that $k(n)/n \to T$ as $n \to \infty$ then $T_{k(n)}^{(n)} \to T$ a.s. by the law of large numbers. Furthermore, denote by $\widetilde{\mathcal{T}}^{(n)}$ the set of $\widetilde{\mathbf{F}}^{(n)}$-stopping times on this enlarged probability space by

$$\widetilde{\mathcal{T}}^{(n)} := \left\{ \tau \;\middle|\; \tau \text{ is an } \widetilde{\mathbf{F}}^{(n)}\text{-stopping time } \& \ \tau \in \{0 = T_0^{(n)}, T_1^{(n)}, \ldots\} \right\}.$$

Now consider modifying the original optimal stopping problem (1) in three steps. Replace the deterministic expiry date T by the random variable $T_k^{(n)}$ (for suitably chosen k), the set of stopping times $\mathcal{T}$ over which is optimised by $\widetilde{\mathcal{T}}^{(n)}$ and the discount factor $e^{-r\tau}$ by some $D^{(n)}(\tau)$. Together this amounts to defining for each $n \in \mathbb{N}$ and $k \in \mathbb{N}$:

$$V_k^{(n)}(x) := \sup_{\tau \in \widetilde{\mathcal{T}}^{(n)}} \mathbb{E}_x \left[D^{(n)}(\tau \wedge T_k^{(n)}) f(X_{\tau \wedge T_k^{(n)}}) \right]. \tag{2}$$

Main result

Theorem 0.1 *Let f be a bounded and continuous function. Let the value function v be given by (1). For each $n \geq 1$ and $k \geq 0$ let the function $V_k^{(n)}$ be given by (2). We have the following:*

(i) For any $n \geq 1$, the sequence of functions $(V_k^{(n)})_{k \geq 0}$ satisfies the following recursion:

$$V_0^{(n)}(x) = f(x), \; V_k^{(n)}(x) = \max\left\{ f(x), e^{-r/n} \mathbb{E}_x \left[V_{k-1}^{(n)}(X_{\xi^{(n)}}) \right] \right\} \text{ for } k \geq 1, \tag{3}$$

where $\xi^{(n)}$ is independent of X and follows an exponential distribution with mean $1/n$.

(ii) For any $T \in [0, \infty)$, if $(k(n))_{n \geq 1}$ is a sequence such that $k(n)/n \to T$ as $n \to \infty$ then:

$$T_{k(n)}^{(n)} \longrightarrow T \text{ a.s.} \quad \text{and} \quad V_{k(n)}^{(n)}(x) \longrightarrow v(T, x) \text{ for all } x \in \mathbb{R}.$$

Meromorphic Lévy processes and the American Put option

Now we work out the algorithm in detail for the classic example of the American put, e.g. $f(x) = (K - e^x)^+$. The key we need to make our algorithm (3) explicit are expressions for the law of X evaluated at an independent exponentially distributed random variable $\xi^{(q)}$ with mean $1/q$. A very rich class of those Lévy processes X was introduced in [4] with the name **'Meromorphic Lévy processes'** is fulfilling this property.

Proposition 0.2 *Suppose that $\mathbb{P}(X_1 < 0) > 0$, i.e. X is not a subordinator. Let $n \geq 1$. There exists a decreasing sequence of points $(\bar{x}_k^{(n)})_{k \geq 0}$, with $\bar{x}_0^{(n)} := \log K$, such that for each $k \geq 1$ we have:*

(i) the point $\bar{x}_k^{(n)}$ is the unique solution on $(-\infty, \bar{x}_{k-1}^{(n)})$ of the equation in z

$$e^{-r/n} \mathbb{E}_z \left[V_{k-1}^{(n)}(X_{\xi^{(n)}}) \right] - f(z) = 0$$

(ii) $V_k^{(n)}$ is given by

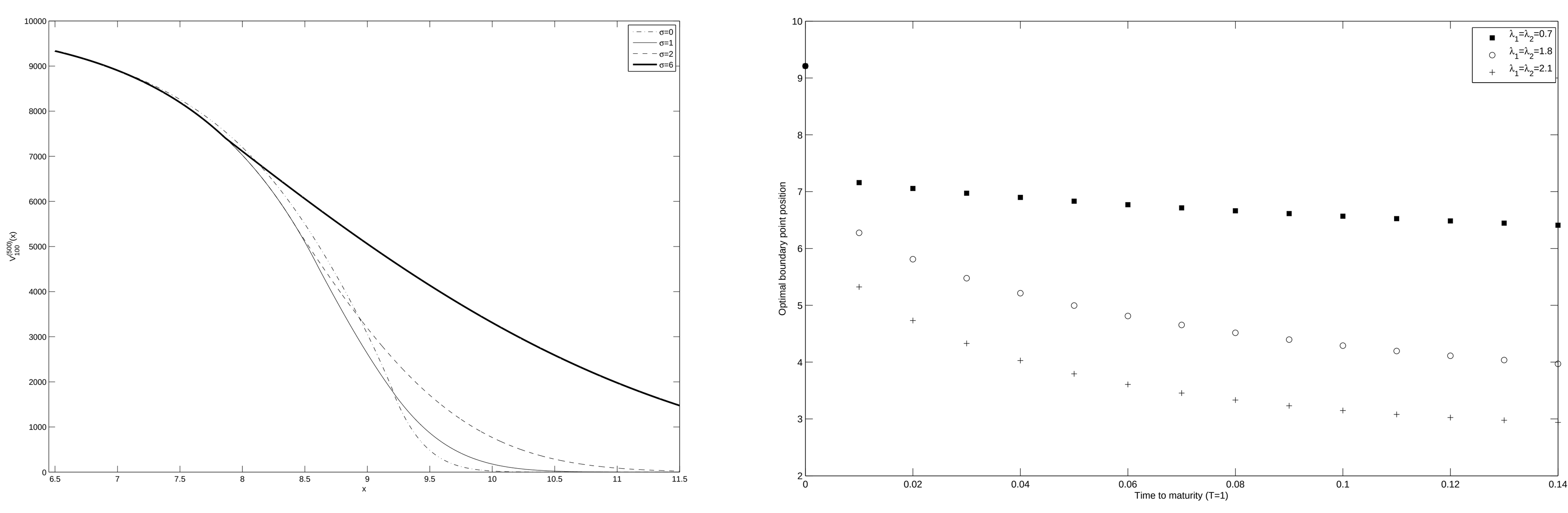
$$V_k^{(n)}(x) = \begin{cases} f(x) & \text{if } x \leq \bar{x}_k^{(n)} \\ e^{-r/n} \mathbb{E}_x \left[V_{k-1}^{(n)}(X_{\xi^{(n)}}) \right] & \text{if } x > \bar{x}_k^{(n)} \end{cases}$$

(iii) $V_k^{(n)} > V_{k-1}^{(n)}$ on $(\bar{x}_k^{(n)}, \infty)$.

Proposition 0.3 *Suppose that X is a meromorphic Lévy process and is not compound Poisson. Fix some $n \geq 1$. For any $k \geq 0$ the function $V_k^{(n)}$ can be expressed in piecewise form as follows:*

$$\begin{aligned} V_k^{(n)}(x) = \mathbf{1}_{\{x \geq \bar{x}_0^{(n)}\}} \sum_{j=0}^{N_+} e^{-\zeta_+^{(n)}(j)x} \sum_{i=0}^{k-1} A_+^{(n)}(i,j,0,k) x^i \\ + \sum_{m=1}^{k} \mathbf{1}_{\{x \in [\bar{x}_m^{(n)}, \bar{x}_{m-1}^{(n)})\}} \Bigg(\sum_{j=0}^{N_-} e^{-\zeta_-^{(n)}(j)x} \sum_{i=0}^{k-m} A_-^{(n)}(i,j,m,k) x^i \\ + \sum_{j=0}^{N_+} e^{-\zeta_+^{(n)}(j)x} \sum_{i=0}^{k-m} A_+^{(n)}(i,j,m,k) x^i - B^{(n)}(m,k) e^x \\ + C^{(n)}(m,k) \Bigg) + \mathbf{1}_{\{x < \bar{x}_k^{(n)}\}} (K - e^x), \end{aligned}$$

where $\bar{x}_0^{(n)} = \log K$. Expressions for $\bar{x}_k^{(n)}$ and all the coefficients involved can be found in Appendix of our paper.



In the left figure plots of the approximating value function $V_k^{(n)}$ for different values of σ for a β-class process are presented and the right figure shows three sets of boundary points $\bar{x}_0^{(n)}$ for different values of $\lambda_1 = \lambda_2$ for a β-class process.

References

[1] Bouchard, B., El Karoui, N. and Touzi, N. (2005) Maturity randomization for stochastic control problems. *Ann. Appl. Probab.* **15**, 2575–2605.

[2] Carr, P. (1998) Randomization and the American put. *Rev. Fin. Studies* **11**, 597–626.

[3] Kleinert, F. and van Schaik, K. (2013) A variation of the Canadisation algorithm for the pricing of American options driven by Lévy processes. `arXiv:1304.4534v1 [math.PR]`

[4] Kuznetsov, A. and Kyprianou, A. E. and Pardo, J. C. (2012) Meromorphic Lévy processes and their fluctuation identities. *Ann. Appl. Probab.* **22**, 1101–1135.