Self-decomposability of the Fréchet distribution

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Abstract

Let $X \sim \mathrm{Exp}(1)$, it is shown that for all $\xi \in (-1,0)$, X^ξ is self-decomposable.

Problem

Let $F_\xi(x) = \begin{cases} e^{-e^{-x}} & \text{if } \xi = 0 \\ e^{-x^{1/\xi}}\mathbf{1}_{x>0} & \text{if } \xi < 0 \\ (1-e^{-x^{1/\xi}})\mathbf{1}_{x>0} & \text{if } \xi > 0 \end{cases}$. When $\xi < 0$, F_ξ is the Fréchet distribution function of parameter $-\xi$, if $\xi = 0$ F_ξ is the Gumbel distribution function and if $\xi > 0$ F_ξ is the Weibull distribution function of parameter ξ. Let $X \sim \mathrm{Exp}(1)$ and $\xi \neq 0$, then $X^\xi \sim F_\xi$ and $-\log(X) \sim F_0$.

Write for $\alpha \in (0,1)$, $-\log(X) \stackrel{d}{=} -\alpha\log(X) + \alpha\log(S_\alpha)$ where S_α is positive α-stable. This means that Gumbel distribution is self-decomposable.

If $\xi = 1$, $X \sim \mathrm{Exp}(1)$ which has a infinite divisible law. When $\xi \geq 1$, X^ξ has a completely monotone density, so its distribution is an exponential mixture which entails infinite divisibility. When $|\xi| \geq 1$, X^ξ has a density which is hyperbolically completely monotone in the sense of Thorin and Bondesson [4], so that its distribution is infinitely divisible. However it is not infinitely divisible if $\xi \in (0,1)$ because of its surexponential tails.

The natural question whether X^ξ is infinitely divisible for $\xi \in (-1,0)$ is raised in Steutel (1973), Bondesson (1992) and Steutel and Van Harn (2003).

Theorem : $\forall \xi \in (-1,0)$, *the distribution of* X^ξ *is self-decomposable.*
More precisely, let

$$\Gamma_t \sim \frac{1}{\Gamma(t)} x^{t-1} e^{-x} dx,$$

then the distribution of Γ_t^ξ *is self-decomposable for every* $\xi \in (-1,0)$.

Corollary : *The distribution of* Γ_t^ξ *is infinitely divisible if and only if* $\xi \notin (0,1)$. *In particular the only extreme value distribution which is not infinitely divisible is the Weibull with parameter in* $(0,1)$.

Sketch of the Proof

Set $\alpha = -\xi \in (0,1)$. The entire moments of Γ_t^α are given for every $n \geq 1$ by

$$\mathbb{E}[\Gamma_t^{\alpha n}] = \frac{\Gamma(t+\alpha n)}{\Gamma(t)} = m\,\frac{\Psi(1)\ldots\Psi(n-1)}{(n-1)!}$$

where Ψ is the Lévy-Khintchine exponent of a certain spectrally negative Lévy process Z with infinite variation and positive mean m. By Proposition 2 in [3], this entails

$$\mathbb{E}[\Gamma_t^{\alpha n}] \;=\; \mathbb{E}[I^{-n}]$$

for every $n \geq 1$, where I is the exponential functional of ξ:

$$I \;=\; \int_0^\infty e^{-Z_s}\, ds.$$

Since Z has no positive jumps, Proposition 2 in [3] shows that the random variable I^{-1} is moment-determinate, hence

$$\Gamma_t^{-\alpha} \;\stackrel{d}{=}\; I.$$

The self-decomposability of I is now a direct consequence of the strong Markov property at $T_y = \inf\{s > 0,\ Z_s = y\}$ for every $y > 0$.

The case $\xi = -1$

The case $\xi = -1$ is classical and can be obtained in four manners.

Exponential functional of Brownian motion : The SD property follows directly from Dufresne's identity [5] which reads

$$\frac{2}{\Gamma_\nu} \stackrel{d}{=} \int_0^\infty \exp\left(B_t - \frac{\nu t}{2}\right) dt.$$

First passage time : It's well known that the distribution of $\Gamma_{1/2}^{-1}$ is an inverse gaussian. So its distribution is infinitely divisible. If $\nu \neq 1/2$ the distribution of Γ_ν^{-1} is a genenral inverse gaussian, in [2] O. Barndorff-Nielsen, P. Blæsild and C. Halgreen proved that a GIG is infinite divisibile as first-passage time of a diffusion.

Last passage time : We just saw that Γ_ν^{-1} can be write as the first-passage time of a diffusion. Getoor in [6] proved that this distribution can also be seen as the last-passage time of some other diffusion. More precisely, let R (respectively $\hat{R}$) be a Bessel process starting from 1 of dimension $2(1-\nu)$ (respectively starting from 0 of dimension $2(1+\nu)$), then

$$\frac{1}{\Gamma_\nu} \stackrel{d}{=} 2T_0(R) \stackrel{d}{=} 2L_1(\hat{R})$$

with the notations $T_0(R) = \inf\{t/R_t = 0\}$ and $L_1(\hat{R}) = \sup\{t/\hat{R}_t = 1\}$ (recall that $\hat{R}_t \to \infty$ as $t \to \infty$ almost surely since $2(1+\nu) > 2$).

Computation on Laplace transform : Let $\Phi_t(\lambda) = \mathbb{E}\left(e^{-\lambda\Gamma_t^{-1}}\right)$, a simple computation gives $-\frac{\Phi_t'(\lambda)}{\Phi_t(\lambda)} = \frac{K_{t-1}(2\sqrt{\lambda})}{\sqrt{\lambda}K_t(2\sqrt{\lambda})}$.

Grosswald has shown in [7] that $\lambda \mapsto \frac{K_{t-1}(2\sqrt{\lambda})}{\sqrt{\lambda}K_t(2\sqrt{\lambda})}$ is a Stieltjes transform. It follows that the distribution of Γ_t^{-1} is a GGC.

The case $\xi \in (-1,0)$

As in the case $\xi = -1$, the self-decomposability of Γ_t^ξ can be interpreted in several way.

Exponential functional of a Lévy process : Our theorem is a consequence of the identification

$$\Gamma_t^\xi \stackrel{d}{=} \int_0^\infty e^{-Z_s}\, ds.$$

First passage time : By using Lamperti transformation we obtain

$$\Gamma_t^\xi \stackrel{d}{=} \inf\{u/Y_u = 0\}$$

where Y is a spectrally positive 1-self-similar Markov process starting from 1.

Last passage time : The latter first passage time interpretation also makes it possible to view Γ_t^ξ as a last passage time :

$$\Gamma_t^\xi \stackrel{d}{=} \sup\{u/X_u = 1\}$$

where X is a positive, spectrally negative, 1-self-similar Markov process starting from 0.

Consequence on Laplace transform : Our theorem gives an analytical interpretation in terms of generalized Bessel functions. Setting $\alpha = -\xi \in (0,1)$ and writing down

$$\mathbb{E}\left(e^{-\lambda\Gamma_t^{-\alpha}}\right) = \frac{\mathcal{Z}_{1/\alpha}^{t/\alpha}(\lambda)}{\alpha\Gamma(t)}$$

with the notation of [9]. The infinite divisibility of $\Gamma_t^{-\alpha}$ entails that the function

$$\lambda \mapsto -\frac{\mathcal{Z}_{1/\alpha}^{t/\alpha}(\lambda)}{\alpha\Gamma(t)}$$

is completely monotone. However, at the present time unlike to the case $\xi = -1$ there isn't a direct proof and it would be interesting to see if $\lambda \mapsto -\frac{\mathcal{Z}_{1/\alpha}^{t/\alpha}(\lambda)}{\alpha\Gamma(t)}$ is a Stieltjes transform.

References

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