

Meta-stability of some reaction-diffusion equations with Lévy noise

A. Debussche M. Högele P. Imkeller I. Pavlyukevich

ENS Cachan (A Bretagne), HU Berlin, U Jena

Wroclaw, July 16, 2013

A. Debussche, M. Högele, P. Imkeller: The Dynamics of Nonlinear Reaction-Diffusion Equations with Small Lévy Noise. Lecture Notes in Mathematics 2085. Springer: Berlin 2013

1. Dynamical model: reaction-diffusion equation with Lévy noise

Mathematical model: conceptual EBM; energy balance: reaction term: balance of incoming vs. outgoing radiative energy; diffusion term: heat transport from equator to poles; noise: Ditlevsen's observation

Goal: study meta-stability properties of reaction-diffusion systems with α -stable Lévy noise of small intensity

state of the system: heat distribution $X^\varepsilon(t, \zeta)$: temperature at time t observed at $\zeta \in [0, 1]$ (interval of global latitudes), subject to noise, intensity ε ;

idealized version: reaction term $U(u) = \lambda(\frac{u^4}{4} - \frac{u^2}{2})$ ($\lambda > 0$), diffusion operator $\frac{\partial^2}{\partial \zeta^2}$, noise: L Lévy process in $H = H_0^1([0, 1])$

$$dX_t^\varepsilon(\zeta) = \left[\frac{\partial^2}{\partial \zeta^2} X_t^\varepsilon(\zeta) - U'(X_t^\varepsilon(\zeta)) \right] dt + \varepsilon dL_t(\zeta),$$

$$X_t^\varepsilon(0) = X_t^\varepsilon(1) = 0, \quad t > 0,$$

$$X_0^\varepsilon(\zeta) = x(\zeta), \quad \text{with } x \in H, \zeta \in [0, 1].$$

2. Deterministic part: Chafee-Infante equation

For $\pi^2 < \lambda \neq (k\pi)^2, k \in \mathbb{N}$, consider

$$du_t = \frac{\partial^2}{\partial \zeta^2} u_t - U'(u_t), \quad t \in [0, T],$$

$$u_t(0) = u_t(1) = 0, \quad t \in [0, T],$$

$$u_0(\zeta) = x(\zeta), \quad \zeta \in [0, 1].$$

For initial values $x \in H_0^1([0, 1])$ there is a **unique continuous solution in $H = H_0^1([0, 1])$** , e.g. Chafee-Infante '74, Henry '83, Carr, Pego '89

There are two **stable states** $\{\varphi^+, \varphi^-\}$ with resp. **domains of attraction** D^+ and D^- and a smooth **separatrix** $\mathcal{S} = H \setminus (D^+ \cup D^-)$.

3. Dynamics of solutions of the Chafee-Infante equation

Thm 6 (Temam '92):

All trajectories enter a large ball in $H_0^1([0, 1])$ and thus $L^\infty([0, 1])$ in **uniform relaxation time** T_{rec} .

Thm 7

For $\gamma \in (0, 1)$ one can define a **reduced domain of attraction** $D^+(\varepsilon^\gamma)$, $\varepsilon > 0$, such that all trajectories starting in $D^+(\varepsilon^\gamma)$ enter $B(\varphi^+, \varepsilon^\gamma)$ before time $T_{rec} + T_\varepsilon = T_{rec} + c_\gamma |\ln \varepsilon|$.

one-dimensional case, $\varphi^+ = 0$, $x \in B_1(0)$: $-U'(u) \sim Mu$ for u near 0. Hence

$$T_\varepsilon = \int_{\varepsilon^\gamma}^x \frac{1}{|U'(u)|} du \approx \int_{\varepsilon^\gamma}^x \frac{1}{Mu} du \approx \text{Const} + \frac{\gamma}{M} |\ln \varepsilon|.$$

4. α -stable processes in Hilbert space $(H, \langle \cdot, \cdot \rangle, \|\cdot\|)$

$(L_t)_{t \geq 0}$ càdlàg version of **symmetric α -stable process in H** , Lévy measure

$$\nu(dy) = \sigma(ds) \frac{dr}{r^{1+\alpha}}, \quad r = \|y\|, \quad s = y/\|y\|, \quad y \neq 0, \quad \sigma \text{ finite measure on } \partial B_1(0)$$

Lévy-Chinchine representation of characteristic function

$$\mathbb{E} \left[e^{i\langle u, L_t \rangle} \right] = \exp \left(t \int_H \left(e^{i\langle u, y \rangle} - 1 - i\langle u, y \rangle \mathbf{1}_{\{\|y\| \leq 1\}} \right) \nu(dy) \right), \quad u \in H.$$

for $\nu : \mathcal{B}(H) \rightarrow [0, \infty]$ satisfying $\int_H (1 \wedge \|y\|^2) \nu(dy) < \infty$,
 $\nu(A) = \nu(-A)$, $A \in \mathcal{B}(H)$.

In contrast to (Q-)Brownian motion in H : **no scalar decomposition**

$$L_t \neq \sum_{k=1}^{\infty} \lambda_k L_t^k e_k \quad \text{for ONB } (e_n) \subset H, \quad (L_t^k)_{t \geq 0} \text{ independent}$$

L is **heavy-tailed** and **need not have any moments**.

5. Stochastic Chafee-Infante equation

For $\varepsilon > 0$ and $U'(y) = \lambda(y^3 - y)$

$$dX_t^\varepsilon(\zeta) = \left[\frac{\partial^2}{\partial \zeta^2} X_t^\varepsilon(\zeta) - U'(X_t^\varepsilon(\zeta)) \right] dt + \varepsilon dL_t(\zeta),$$

$$X_t^\varepsilon(0) = X_t^\varepsilon(1) = 0, \quad t > 0,$$

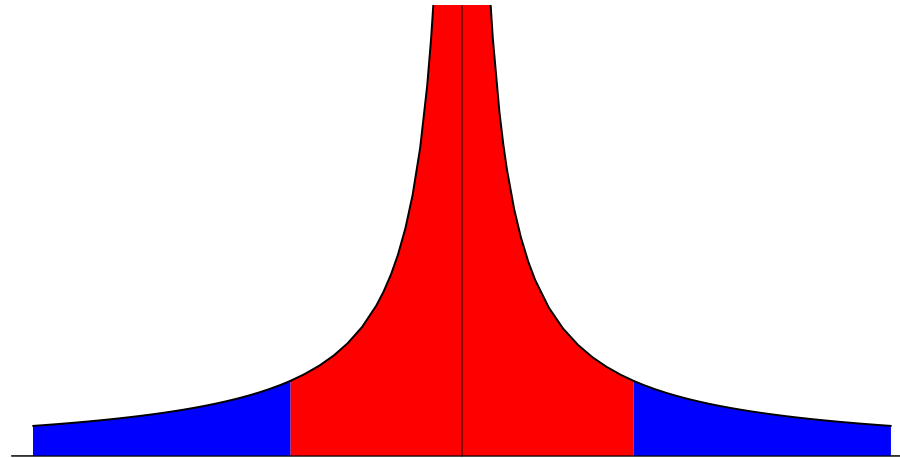
$$X_0^\varepsilon(\zeta) = x(\zeta), \quad \text{with } x \in H, \zeta \in [0, 1].$$

There is a unique **càdlàg mild solution** in H (Peszat, Zabczyk '07, Marinelli, Röckner '09), satisfying the **strong Markov property**

$$X_t^\varepsilon = S(t)x - \int_0^t S(t-s)f(X_s^\varepsilon) ds + \varepsilon \int_0^t S(t-s) dL_s.$$

Pb: Let $\tau(\varepsilon, x) = \tau(\varepsilon)$ be **exit time of X^ε starting in x from D^+** . Describe the law of $\tau(\varepsilon)$ asymptotically as $\varepsilon \rightarrow 0$.

6. First exit times



Decompose noise into **small** ($\leq \varepsilon^{-\frac{1}{2}}$) and **large** ($> \varepsilon^{-\frac{1}{2}}$) jumps.

$$L_t = \xi_t^\varepsilon + \eta_t^\varepsilon$$

$(\eta_t^\varepsilon)_{t \geq 0}$ compound Poisson process with jump measure ν^ε

$$\nu^\varepsilon(A) = \frac{\nu\left(A \cap \varepsilon^{-\frac{1}{2}} B_1^c(0)\right)}{\beta_\varepsilon}, \quad A \in \mathcal{B}(H), \quad \beta_\varepsilon = \nu\left(\varepsilon^{-\frac{1}{2}} B_1^c(0)\right) \sim \varepsilon^{\frac{\alpha}{2}}.$$

6. First exit times

$$\eta_t^\varepsilon = \sum_{T_k \leq t} W_k, \quad \mathcal{L}(W_k) = \nu^\varepsilon$$

$$T_k = \sum_{i=1}^k \tau_i \quad \mathcal{L}(\tau_i) = \text{EXP}(\beta_\varepsilon).$$

$(\xi_t^\varepsilon)_{t \geq 0}$ is a martingale with bounded jumps. Denote the stochastic convolution

$$\xi_t^* = \int_0^t S(t-s) d\xi_s^\varepsilon,$$

and the small jump part

$$Y_t^\varepsilon = S(t)x - \int_0^t S(t-s) f(Y_s^\varepsilon) ds + \varepsilon \xi_t^*.$$

7. Heuristics: why $\varepsilon^{-\frac{1}{2}}$?

Now

$$\int_0^{T_1} S(T_1 - s) dL_t = \int_0^{T_1^-} S(T_1 - s) dL_s + \Delta_{T_1} L = \xi^*(T_1) + W_1.$$

Decompose

$$Y_t^\varepsilon(x) - u_t(x) = R_t^\varepsilon(x) + \varepsilon \xi_t^*.$$

Using properties of u prove that $\|R_t^\varepsilon(x)\|_\infty \rightarrow 0+$ if $\|\xi_t^*\| \rightarrow 0+$. Control small jumps convolution by

$$\mathbb{P} \left(\sup_{t \in [0, T]} \|\varepsilon \xi_t^*\| \geq c\varepsilon^\gamma \right) \leq c_\xi T \varepsilon^{1-2\gamma+\frac{\alpha}{2}}.$$

Use this to estimate small jumps deviation: there is $\vartheta > \frac{\alpha}{2}, \gamma > 0$ such that

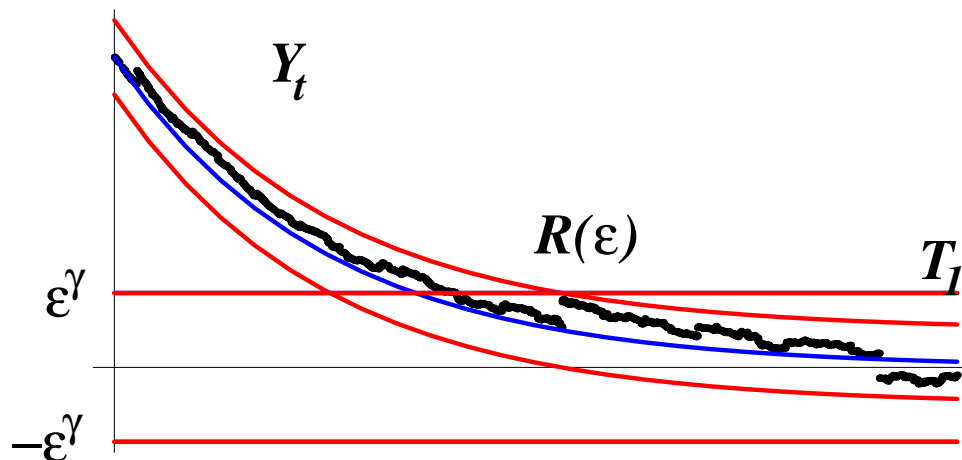
$$\mathbb{P} \left(\exists x \in D^+(\varepsilon^\gamma) : \sup_{t \in [0, T_1]} |Y_t^\varepsilon(x) - u_t(x)|_\infty \geq (1/2)\varepsilon^\gamma \right) \leq \varepsilon^\vartheta.$$

7. Heuristics: why $\varepsilon^{-\frac{1}{2}}$?

For small $\varepsilon > 0$

$$\mathbb{E}T_1 = 1/\beta_\varepsilon \sim \varepsilon^{-\frac{\alpha}{2}} > T_{rec} + c_\gamma |\ln \varepsilon|.$$

We can expect to **make large jump from close to stable state!**



Thus **large jumps** $\varepsilon\eta^\varepsilon$ should dominate **exit behavior for small $\varepsilon > 0$, exit times of polynomial order.**

Denote $D_0^+(\varepsilon^\gamma) = D^+(\varepsilon^\gamma) - \varphi^+$ the shifted reduced domain. Consider the mass the reshifted domain of attraction

$$\lambda^+(\varepsilon) = \nu \left((1/\varepsilon) \left(D_0^+(\varepsilon^\gamma) \right)^c \right).$$

8. Asymptotic first exit time from a domain of attraction

Thm 8

For any $x \in D^+$ and $\theta > -1$

$$\lim_{\varepsilon \rightarrow 0^+} \mathbb{E}_x \left[e^{-\theta \lambda^+(\varepsilon) \tau(\varepsilon)} \right] = \frac{1}{1 + \theta} = \widehat{EXP}(1).$$

Cor 1:

For all $x \in D^+$

$$\mathcal{L}(\lambda^+(\varepsilon) \tau(\varepsilon, x)) \longrightarrow EXP(1), \quad \varepsilon \rightarrow 0^+.$$

Moreover

$$\lim_{\varepsilon \rightarrow 0^+} \lambda^+(\varepsilon) \mathbb{E}_x [\tau(\varepsilon)] = 1.$$

For small $\varepsilon > 0$

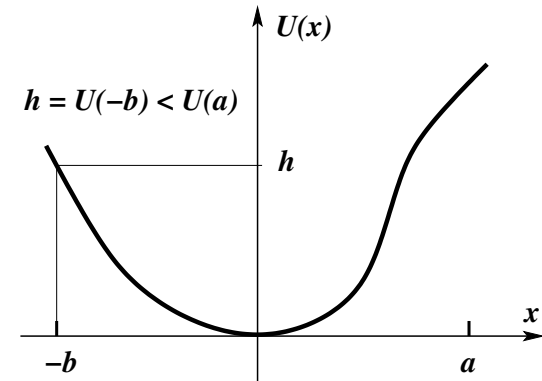
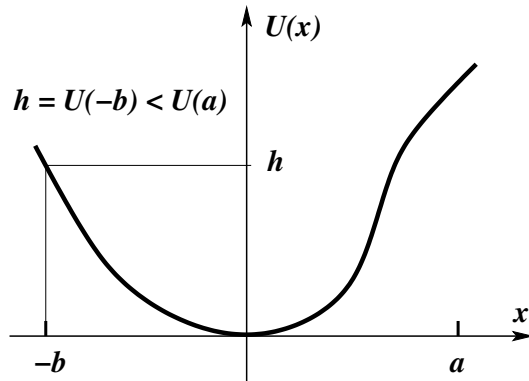
$$\mathbb{E}_x [\tau(\varepsilon)] = 1/\lambda^+(\varepsilon) \sim c/\varepsilon^\alpha.$$

Thus **exit times have polynomial growth asymptotically!**

9. comparison: Gaussian and Lévy dynamics, dim 1

$$\hat{\sigma} = \inf\{t \geq 0 : \hat{X}^\varepsilon(t) \notin [-b, a]\}$$

$$\sigma = \inf\{t \geq 0 : X^\varepsilon(t) \notin [-b, a]\}$$



$$\hat{X}^\varepsilon(t) = x - \int_0^t U'(\hat{X}^\varepsilon(s)) ds + \varepsilon W(t)$$

$$X^\varepsilon(t) = x - \int_0^t U'(X^\varepsilon(s-)) ds + \varepsilon L(t)$$

Thm 9 (Freidlin-Wentzell):

$$\mathbf{P}_x(e^{(2h-\delta)/\varepsilon^2} < \hat{\sigma} < e^{(2h+\delta)/\varepsilon^2}) \rightarrow 1$$

Thm 10

$$\mathbf{P}_x\left(\frac{1}{\varepsilon^{\alpha-\delta}} < \sigma < \frac{1}{\varepsilon^{\alpha+\delta}}\right) \rightarrow 1$$

Kramers' law ('40, Williams, Bovier et al.):

$$\mathbf{E}_x \hat{\sigma} \approx \frac{\varepsilon \sqrt{\pi}}{|U'(-b)| \sqrt{U''(0)}} e^{2h/\varepsilon^2}$$

$$\mathbf{E}_x \sigma \approx \frac{1}{\varepsilon^\alpha} \left(\int_{\mathbb{R} \setminus [-b, a]} \frac{dy}{|y|^{1+\alpha}} \right)^{-1}$$

Exponential law (Day, Bovier et al.)

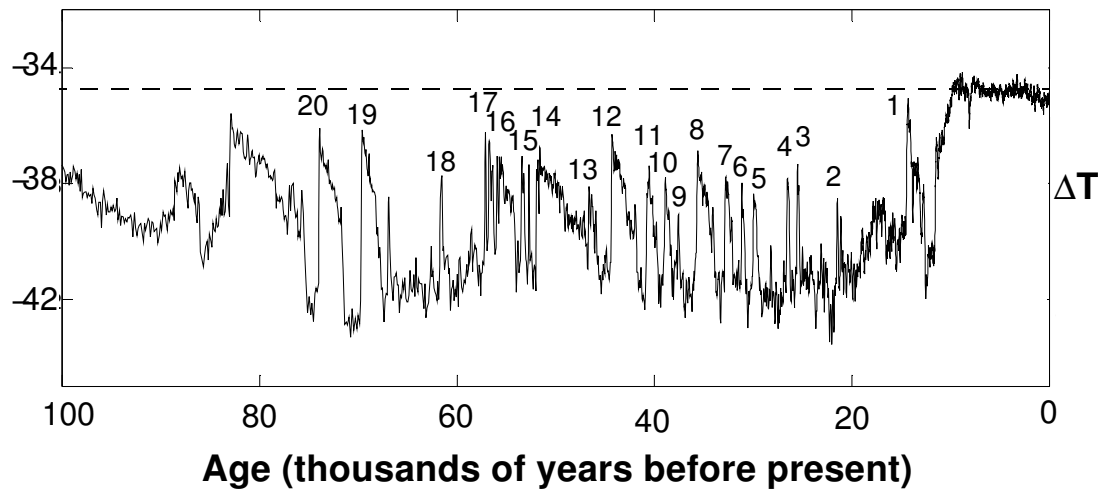
$$\mathbf{P}_x\left(\frac{\hat{\sigma}}{\mathbf{E}_x \hat{\sigma}} > u\right) \sim \exp(-u)$$

$$\mathbf{P}_x\left(\frac{\sigma}{\mathbf{E}_x \sigma} > u\right) \sim \exp(-u)$$

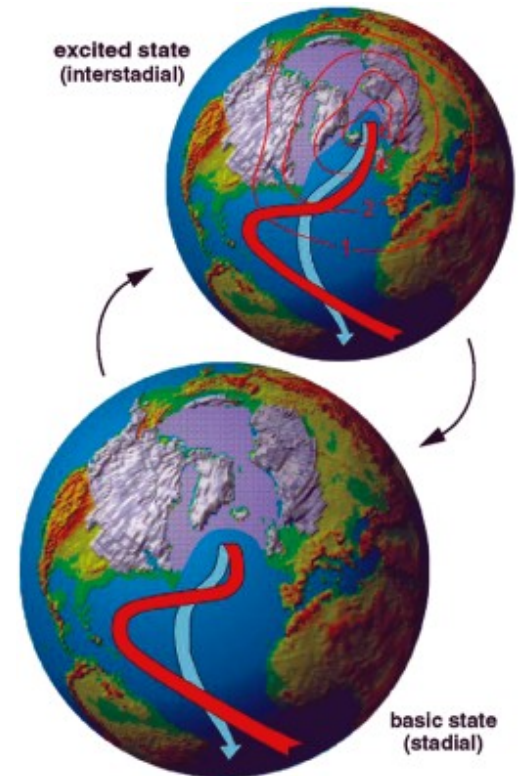
10. Paleo-climatic time series

temperature indicators: ^{18}O , ^{16}O , methane, calcium etc.

GRIP ice core data: 20 abrupt changes in climate of Greenland during last ice age (-91 000 to -11 000 y) (D/O events).



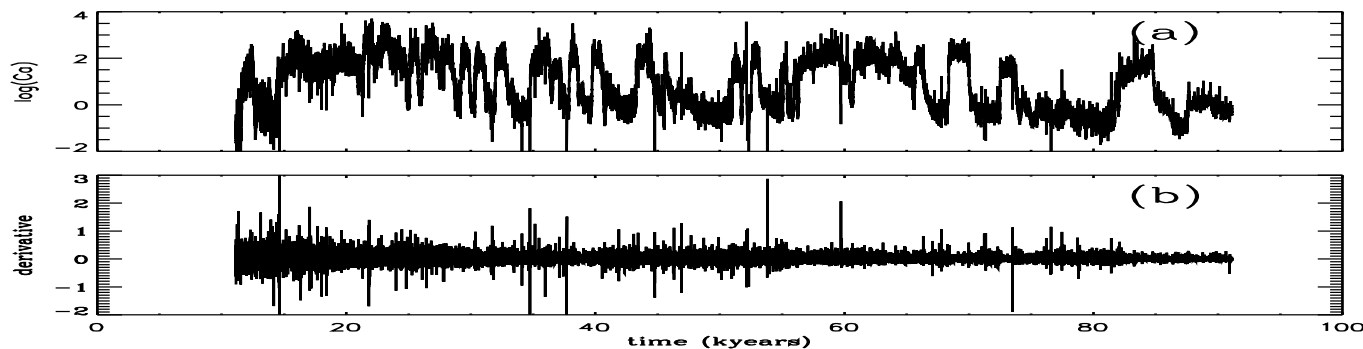
- rapid warming by 5-10°C within one decade
- subsequent slower cooling within a few centuries
- fast return to stable cold ground state



simulations: Ganopolsky/Rahmstorf,
Potsdam Institute for Climate Impact
Research

11. Dansgaard-Oeschger events. Statistical analysis

Calcium signal from GRIP: about 80 000 samples for 80 000 y



waiting times between D/O events: multiples of ~ 1470 years.

What triggers the transitions?

modeling by Langevin equation (reaction):

$$dX(t) = -U'(t, X(t))dt + \text{NOISE}$$

U — multi well potential, depicts energy balance

P. Ditlevsen (*Geophys. Res. Lett.* 1999): power spectrum analysis of time series:

NOISE contains strong α -stable component with $\alpha \approx 1.75$.

12. p -Variation as test statistic

Which **model of noise** fits best with time series: **estimate, test parameter**

Ditlevsen's analysis: **power spectrum of residua** of time series

Problem: **Stationarity?**

Aim: **better test statistics** than **peaks of power spectrum**.

Model assumption: with some U interpret data as

$$X^\varepsilon(t) = x - \int_0^t U'(X^\varepsilon(s-)) ds + \varepsilon L(t) = Y^\varepsilon(t) + L^\varepsilon(t)$$

L **Lévy process** containing **α -stable** component with unknown α , Y^ε of **bounded variation; estimate, test α**

Idea: **p -variation** characteristic for fluctuation behavior of noise processes.

$$V_t^{p,n}(X) = \sum_{i=1}^{[nt]} \left| X\left(\frac{i}{n}\right) - X\left(\frac{i-1}{n}\right) \right|^p, \quad V_t^p = \lim_{n \rightarrow \infty} V_t^{p,n}$$

13. α -stable Lévy Processes

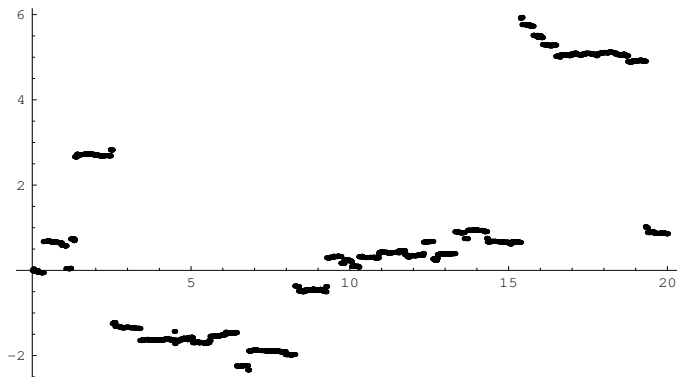
L Lévy process with characteristics (d, γ, ν) iff

$$E(e^{iuL(t)}) = e^{t(-\frac{1}{2}du^2 + i\gamma u + \int_{\mathbf{R}} [e^{iuy} - 1 - iuy1_{\{|y|\leq 1\}}] \nu(dy))}, \quad u \in \mathbf{R}, t \geq 0,$$

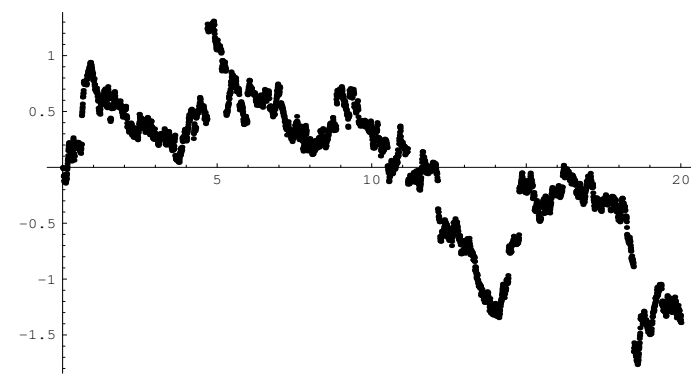
ν measure on Borel sets in \mathbf{R} with $\nu(\{0\}) = 0$, $\int_{\mathbf{R}} [|y|^2 \wedge 1] \nu(dy) < \infty$.

L α -stable symmetric Lévy process if

$$E(e^{iuL(t)}) = e^{-c(\alpha)t|u|^\alpha}, \quad \nu(dy) = \frac{1}{|y|^{\alpha+1}} dy, \quad u, y \in \mathbf{R}.$$



$\alpha = 0.75$



$\alpha = 1.75$

14. p -Variation and the Blumenthal-Gettoor Index

L α -stable process with jump measure ν ; then p -variation identified by

Blumenthal-Gettoor index

$$\beta_L = \inf\{s \geq 0 : \int_{\{|y| \leq 1\}} |y|^s \nu(dy) < \infty\}$$

$$\gamma_L = \inf\{p > 0 : V_1^p(L) < \infty\}$$

Thm 1

L symmetric α -stable. Then

$$\gamma_L = \beta_L = \alpha.$$

Problem: How to read $\gamma_L = \alpha$ off the sequence $(V_t^{p,n}(L))_{n \in \mathbb{N}}$?

Calls for results about the asymptotic behavior of the sequence.

15. The case $\alpha = 2$: Brownian motion

For $n \in \mathbf{N}$ $V_1^{p,n}(W)$ consists of n independent increments and

$$E(V_1^{p,n}(W)) = n^{1-\frac{p}{2}} E(|W(1)|^p)$$

Thm 2 (LLN type)

$$n^{-1+\frac{p}{2}} V_t^{p,n}(W) \rightarrow t E(|W(1)|^p) \quad \text{in probability,}$$

Y of bounded variation. Then also

$$n^{-1+\frac{p}{2}} V_t^{p,n}(W + Y) \rightarrow t E(|W(1)|^p) \quad \text{in probability.}$$

Thm 3 (CLT type)

$$\begin{aligned} (n^{\frac{1}{2}} [n^{-1+\frac{p}{2}} V_t^{p,n}(W) - t E(|W(1)|^p)])_{t \geq 0} &= (n^{-\frac{1}{2}+\frac{p}{2}} V_t^{p,n}(W) - n^{\frac{1}{2}} t E(|W(1)|^p))_{t \geq 0} \\ &\rightarrow ((\text{var}(|W(1)|^p))^{\frac{1}{2}} \tilde{W}(t))_{t \geq 0} \end{aligned}$$

weakly with respect to the Skorokhod metric, and an independent Brownian motion \tilde{W} .

16. The case $\alpha < 2$

(Lit: Corcuera, Nualart, Wörner '07; case $p < \alpha$ for LLN type, $p < \frac{\alpha}{2}$ for CLT type)

Problem: $p < \frac{\alpha}{2} < 1$ not satisfactory for **paleo-climatic data**! Beyond $\frac{\alpha}{2}$ no CLT type result available, no asymptotic normality, but **asymptotically of different type**.

Thm 4 (LT type)

L **α -stable** with $\alpha \in]0, 2[$. Then

$$(V_t^{p,n}(L) - B_t^n(\alpha, p))_{t \geq 0} \rightarrow \tilde{L}$$

weakly with respect to the Skorokhod metric, and an **independent $\frac{\alpha}{p}$ -stable process \tilde{L}** . Here

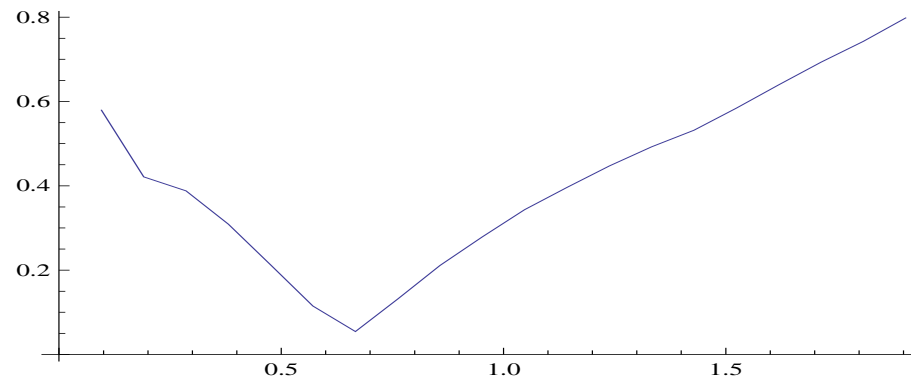
$$B_t^n(\alpha, p) = \begin{cases} n^{1-\frac{p}{\alpha}} t E(|L(1)|^p), & \frac{\alpha}{2} < p < \alpha, \\ nt^2 E(\sin((nt)^{-1} |L(1)|^p)), & p = \alpha, \\ 0, & \alpha < p. \end{cases}$$

Same result with $L + Y$ instead of L if Y is of finite p -variation and $\frac{\alpha}{2} < p < 1$ or $p > \alpha$.

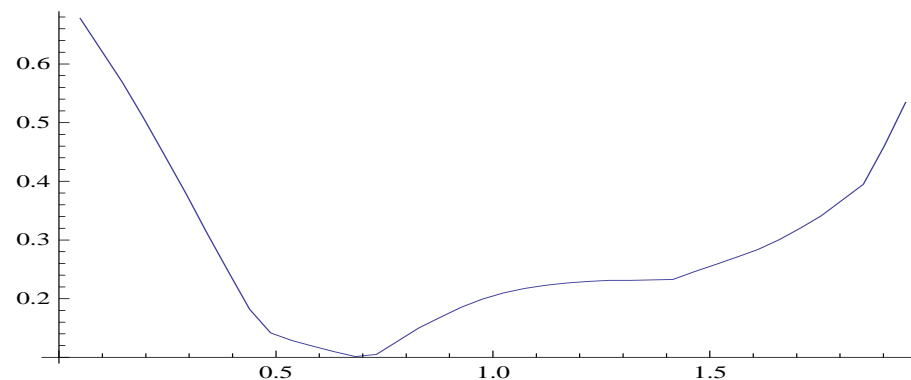
17. Test for α with real and simulated data

Thm 5: law of $V^{2\alpha,n}(X)$ converges to $\frac{1}{2}$ -stable law if data of time series X have α -stable residuals

Kolmogorov-Smirnov statistics: distance between empirical law of $V^{2p,n}(X)$ and $\frac{1}{2}$ -stable law, as a function of p ; minimum of curve: right α



simulated time series of a 0.6-stable Levy process, $n = 200$



real time series from the Greenland ice, $n = 200$

18. Gaussian and Lévy dynamics: one dimension

W Wiener process

L symmetric α -stable Lévy process

Tail behavior

$$P(|W(1)| \geq x) \sim \exp(-cx^2)$$

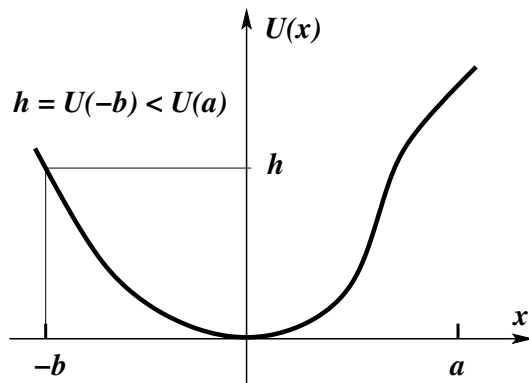
$$P(|L(1)| \geq x) \sim c \frac{1}{x^\alpha}, \quad x \rightarrow \infty$$

$$\hat{X}^\varepsilon(t) = x - \int_0^t U'(\hat{X}^\varepsilon(s)) ds + \varepsilon W(t)$$

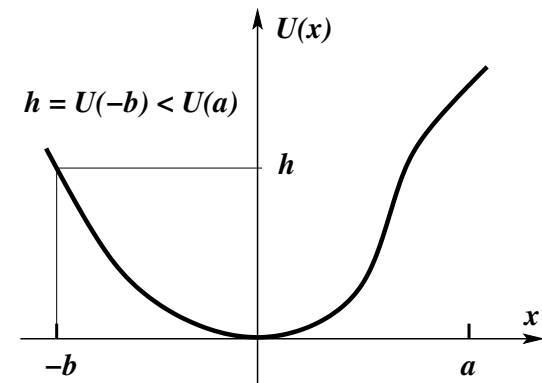
$$X^\varepsilon(t) = x - \int_0^t U'(X^\varepsilon(s-)) ds + \varepsilon L(t)$$

$$\hat{\tau}(\varepsilon) = \inf\{t \geq 0 : \hat{X}^\varepsilon(t) \notin [-b, a]\}$$

$$\tau(\varepsilon) = \inf\{t \geq 0 : X^\varepsilon(t) \notin [-b, a]\}$$



$$\mathbf{E}_x \hat{\tau}(\varepsilon) \approx \frac{\varepsilon \sqrt{\pi}}{|U'(-b)| \sqrt{U''(0)}} \exp\left(\frac{2h}{\varepsilon^2}\right)$$



$$\mathbf{E}_x \tau(\varepsilon) \approx \frac{1}{\varepsilon^\alpha} \left(\int_{\mathbb{R} \setminus [-b, a]} \frac{dy}{|y|^{1+\alpha}} \right)^{-1}$$

19. Gaussian and Lévy dynamics: one dimension

Conjecture: make **tails** of Lévy process **exponentially light** to recover Gaussian exit behavior.

Tail behavior

L Lévy process with jump measure having tails

$$P(|L(1)| \geq x) \sim \exp(-cx^\alpha), \quad x \rightarrow \infty$$

sub-exponential tails: $\alpha < 1$ super-exponential tails: $\alpha > 1$

Consider

$$X^\varepsilon(t) = x - \int_0^t U'(X^\varepsilon(s-)) ds + \varepsilon L(t)$$

$$\tau(\varepsilon) = \inf\{t \geq 0 : X^\varepsilon(t) \notin [-1, 1]\}$$

Conjecture:

$$E_x(\tau(\varepsilon)) \sim_{\varepsilon \rightarrow 0} \exp\left(\frac{c}{\varepsilon^2}\right) \quad \text{as } \alpha \uparrow 2.$$

20. The phase transition at $\alpha = 1$

Thm 10 [sub-exponential tails] For $\delta > 0$ there is $\varepsilon_0 > 0$ such that for all $0 < \varepsilon \leq \varepsilon_0, t \geq 0$:

$$(1 - \delta) \exp(-C_\varepsilon^{1-\delta} t) \leq \sup_{|x| \leq 1} \mathbf{P}_x(\tau(\varepsilon) > t) \leq \exp(-\frac{1}{2} C_\varepsilon t),$$

with $C_\varepsilon := 2\nu([\frac{1}{\varepsilon}, \infty))$. Hence for $|x| < 1$

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^\alpha \ln \mathbf{E}_x \tau(\varepsilon) = 1.$$

Thm 11 [super-exponential tails] q_ε ε -quantile of jump measure ν . Then for $\delta > 0$ there is $\varepsilon_0 > 0$ such that for all $0 < \varepsilon \leq \varepsilon_0, t \geq 0$:

$$(1 - \delta) \exp(-D_\varepsilon^{1-\delta} t) \leq \sup_{|x| \leq 1} \mathbf{P}_x(\tau(\varepsilon) > t) \leq (1 + \delta) \exp(-D_\varepsilon^{1+\delta} t),$$

where $D_\varepsilon = \exp(-d_\alpha \frac{|\ln \varepsilon|}{\varepsilon q_\varepsilon})$ and $d_\alpha = \alpha(\alpha - 1)^{\frac{1}{\alpha}-1}$. Hence for $|x| < 1$

$$d_\alpha^{-1} \lim_{\varepsilon \rightarrow 0} \varepsilon |\ln \varepsilon|^{\frac{1}{\alpha}-1} \ln \mathbf{E}_x \tau(\varepsilon) = 1.$$

21. α -stable towards Gaussian. The phase transition at $\alpha = 1$

Comparison of regimes for mean exit time

Power tails jump tails $\nu([x, \infty)) = x^{-r}$, $x \geq 1$ for some $r > 0$. Then

$$2 \lim_{\varepsilon \rightarrow 0} \varepsilon^r \mathbf{E}_x \tau(\varepsilon) = 1.$$

Sub-exponential tails jump tails $\nu([u, \infty)) = \exp(-u^\alpha)$, $u \geq 1$, $\alpha < 1$. Then

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^\alpha \ln \mathbf{E}_x \tau(\varepsilon) = 1.$$

Super-exponential tails jump tails $\nu([u, \infty)) = \exp(-u^\alpha)$, $u \geq 1$, $\alpha > 1$. Then

$$d_\alpha^{-1} \lim_{\varepsilon \rightarrow 0} \varepsilon |\ln \varepsilon|^{\frac{1}{\alpha}-1} \ln \mathbf{E}_x \tau(\varepsilon) = 1.$$

Gaussian diffusion no jumps, L one-dimensional Brownian motion. Then

$$\frac{1}{2}(U(-1) \wedge U(1))^{-1} \lim_{\varepsilon \rightarrow 0} \varepsilon^2 \ln \mathbf{E}_x \tau(\varepsilon) = 1.$$

22. Heuristics of exits: climbing versus jumping

The Brownian case

LD theory: diffusion has to climb potential in order to exit at lowest saddle point

The power tail case

for $\varepsilon > 0$ split $L = \eta^\varepsilon + \xi^\varepsilon$, compound Poisson pure jump part η^ε with jumps of height larger than $\frac{1}{\sqrt{\varepsilon}}$; small jump and Gaussian part ξ^ε with jumps not exceeding $\frac{1}{\sqrt{\varepsilon}}$; exit asymptotically due to one big jump, as shown in first talk

The case of exponential tails

for $\varepsilon > 0$ split $L = \eta^\varepsilon + \xi^\varepsilon$, compound Poisson pure jump part η^ε with jumps of height larger than g_ε ; small jump and Gaussian part ξ^ε with jumps not exceeding this bound;

choose g_ε individually according to sub- and super-exponential tails

show that exit before time T while not returning to an interval around stable fixed point 0 of radius $\delta > 0$ requires that either increments of ξ^ε exceed certain bounds (for which probability is small enough), or sum of large jumps before time T exceeds bound $1 - \delta$

in any case large jumps responsible for exits

22. Heuristics of exits: climbing versus jumping

N_T random number of large jumps before time T

W_i jump n^o i , $i \in \mathbf{N}$.

N_T Poisson with expectation $\beta_\varepsilon T$, where $\beta_\varepsilon = \nu([-g_\varepsilon, g_\varepsilon]^c) = 2 \exp(-x^\alpha)$

For n fixed, probability that sum of large jumps exceeds bound $1 - \delta$ estimated by

$$P(N_T > n) + \sum_{k=1}^n P(N_T = k) P\left(\sum_{i=1}^k |\varepsilon W_i| > 1 - \delta\right)$$

Idea for estimation:

for $n \in \mathbf{N}$

$$P(N_T > n) \leq (1 + \delta) \exp(-n \ln n) \quad (\text{Stirling's formula})$$

choose $n = n_\varepsilon$ suitably

22. Heuristics of exits: climbing versus jumping

essential term to estimate for $1 \leq k \leq n_\varepsilon$

$$P\left(\sum_{i=1}^k |\varepsilon W_i| > 1 - \delta\right)$$

law of i.i.d. random variables $(|W_i|)_{i \in \mathbb{N}}$: $\beta_\varepsilon^{-1} 2\nu|_{[g_\varepsilon, \infty[}$

hence

$$\begin{aligned} P\left(\sum_{i=1}^k |\varepsilon W_i| > 1 - \delta\right) &\leq \beta_\varepsilon^{-k} \exp\left(-\inf\left\{\sum_{i=1}^k x_i^\alpha : \sum_{i=1}^k x_i \geq \frac{1 - \delta}{\varepsilon}, x_i \in [g_\varepsilon, \infty[\right\}\right) \\ &= \beta_\varepsilon^{-k} \exp\left(-\inf\left\{\sum_{i=1}^k x_i^\alpha : \sum_{i=1}^k x_i = \frac{1 - \delta}{\varepsilon}, x_i \in [g_\varepsilon, \infty[\right\}\right) \end{aligned}$$

minimization problem in the exponent of this estimate causes phase transition

22. Heuristics of exits: climbing versus jumping

By suitable choice of g_ε : lower boundary for x_i in inf can be taken 0.

sub-exponential tails

$$\inf\left\{\sum_{i=1}^k x_i^\alpha : \sum_{i=1}^k x_i = 1, x_i \geq 0\right\} = 1$$

The **minimum** is taken on the **boundary of the simplex**, and $x_i = \frac{1}{n}, 1 \leq i \leq n$, corresponds to **maximum** of the function

$$(x_1, \dots, x_n) \mapsto \sum_{i=1}^n x_i^\alpha$$

22. Heuristics of exits: climbing versus jumping

Super-exponential tails

$$\inf \left\{ \sum_{i=1}^k x_i^\alpha : \sum_{i=1}^k x_i = 1, x_i \geq 0 \right\} = n \left(\frac{1}{n} \right)^\alpha$$

The **minimum** is taken for $x_i = \frac{1}{n}, 1 \leq i \leq n$, the **unique local minimum** of the function

$$(x_1, \dots, x_n) \mapsto \sum_{i=1}^n x_i^\alpha$$

Bifurcation in the asymptotic behavior:

phase transition due to switch from **concavity** to **convexity** at $\alpha = 1$ of

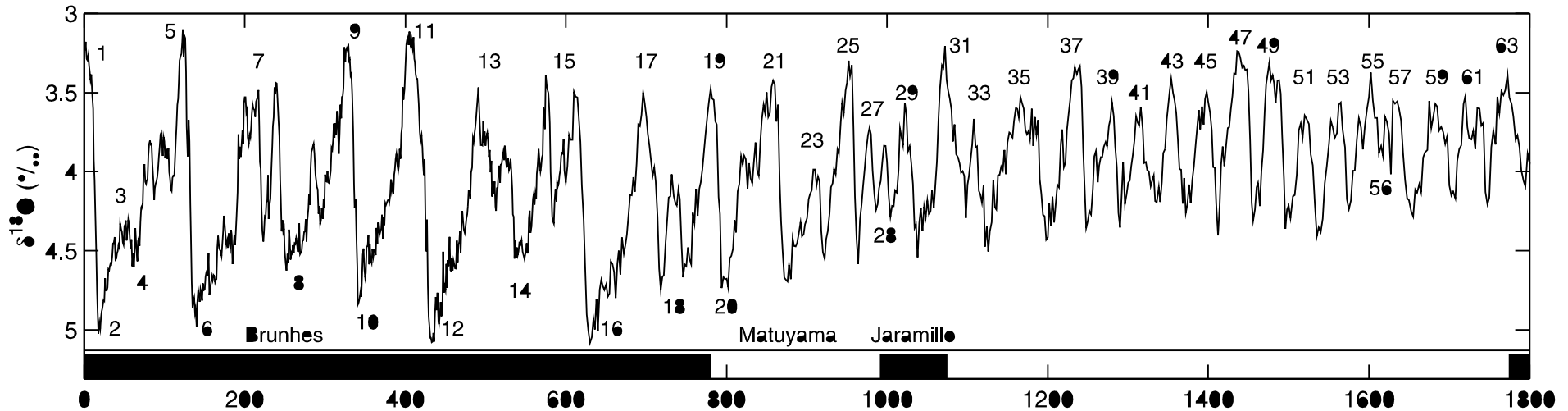
$$x \mapsto x^\alpha, \quad x \geq 0,$$

big jumps of the Lévy process **govern asymptotic behavior**

$\alpha < 1$: **biggest jump** responsible for exit

$\alpha > 1$: cumulative action of **several large jumps** responsible for exit

23. Paleoclimatic time series



Lisiecki, Raymo, *Paleoceanography* 2005 concentration variation of ^{18}O to ^{16}O taken from marine sediments at 57 globally distributed sites (e.g. Brunhes, Matuyama, Jaramillo):

global average temperature time series

basic feature:

- from 0 to -1 Myr **periodicity** $\sim 100\,000\text{ y}$
- from -1 Myr to -1.8 Myr **periodicity** $\sim 44\,000\text{ y}$
- Milankovich cycles: **axial tilt** (41 000 y) and **eccentricity** (100 000 y)