Hyperbolic Brownian motion with drift exiting strip

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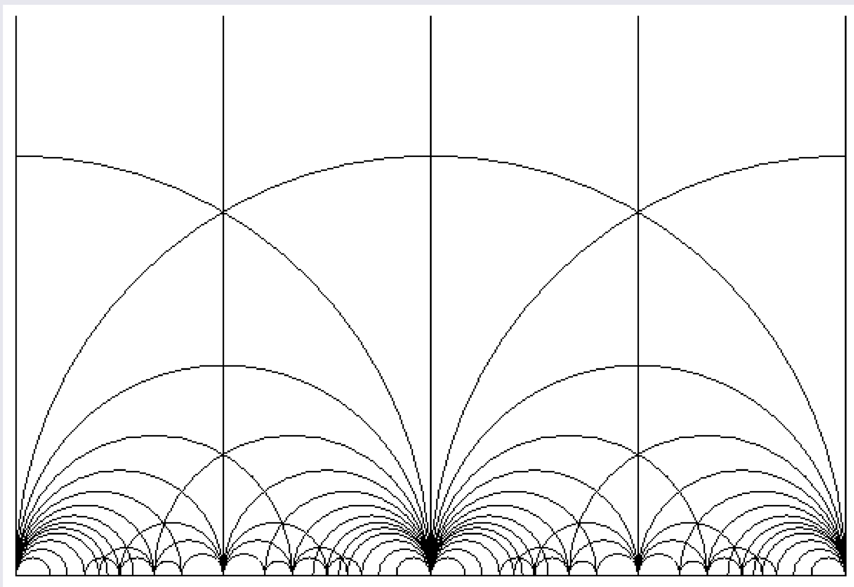
Real hyperbolic space $\mathbb{H}^n$

We use half-space model of hyperbolic space
$\mathbb{H}^n = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n : x_n > 0\}$

- Riemannian metric - $ds^2 = \frac{dx^2}{x_n^2}$,
- Hyperbolic distance - $d_{\mathbb{H}^n}(x, y) = \cosh^{-1}\left(1 + \frac{|x-y|^2}{2x_n y_n}\right)$,
- Laplace-Beltrami operator

$$\Delta_{\mathbb{H}^n} = x_n^2 \sum_{k=1}^{n} \frac{\partial^2}{\partial x_k^2} - (n-2)x_n \frac{\partial}{\partial x_n}. \tag{1}$$

- Geodesics in $\mathbb{H}^2$



Hyperbolic Brownian motion with drift

For every $\mu > 0$, we define the following operator

$$\Delta_\mu := x_n^2 \sum_{k=1}^{n} \frac{\partial^2}{\partial x_k^2} - (2\mu - 1)x_n \frac{\partial}{\partial x_n}.$$

Note that for $\mu = \frac{n-1}{2}$ the operator is the Laplace-Beltrami operator (1). The hyperbolic Brownian motion (HBM) with drift is a diffusion $X^{(\mu)} = \{X^{(\mu)}(t); t \geq 0\}$ on $\mathbb{H}^n$ starting from $X^{(\mu)}(0) = x \in \mathbb{H}^n$ with a generator $\frac{1}{2}\Delta_\mu$. For $\mu = \frac{n-1}{2}$ we obtain the standard HBM on $\mathbb{H}^n$.

Considered sets

We are focused on the sets of the form

$$U = D \times (0, \infty) \subset \mathbb{H}^n, \tag{2}$$

where D is any Lipschitz domain in $\mathbb{R}^{n-1}$. Note that U is unbounded. The boundary ∂U, as a subset of $\mathbb{R}^n$, consists of two parts of totally different nature:

$$\partial_1 U = \{x \in U : x_n \neq 0\} = \partial D \times (0, \infty),$$
$$\partial_2 U = D \times \{0\}.$$

The other part does not belong to $\mathbb{H}^n$ but it can be reached by HBM in infinite time. Example:

$$S_a = \{x \in \mathbb{H}^n : x_1 \in (0, a)\}, \quad a > 0.$$

We call it hyperbolic strip, because reflections with respect to hyperplanes $\{x \in \mathbb{H}^n : x_1 = 0\}$ and $\{x \in \mathbb{H}^n : x_1 = a\}$ are isometries of the $\mathbb{H}^n$.

Dirichlet problem

Because of complicated form of the boundary of the sets of the form (2), it is hard to investigate behavior of λ - harmonic function. Some observations lead us to consider the following Dirichlet problem. For given $f \in \mathcal{C}_b(\partial U)$ and $\lambda > 0$ we are looking for $u \in \mathcal{C}^2(U)$ satisfying

$$\left(\frac{1}{2}\Delta_\mu u\right)(x) = \lambda u(x), \quad x \in U, \tag{3}$$

such that $x_n^{\sqrt{2\lambda+\mu^2}-\mu} u(x)$ is bounded and

$$\lim_{\substack{x \to z \\ x \in U}} x_n^{\sqrt{2\lambda+\mu^2}-\mu} u(x) = f(z), \quad z \in \partial U. \tag{4}$$

λ - Poisson kernel

Let us define first exit time τ_U^μ of the proces $X^{(\mu)}$ from the set U as follows

$$\tau_U^\mu = \inf\left\{t > 0 : X_t^{(\mu)} \notin U\right\}.$$

For any $\lambda \geq 0$, the λ-Poisson kernel of the set U for HBM with drift $\mu > 0$ is a function $P_U^{\mu,\lambda} : U \times \partial D \to [0, \infty)$ such that for given $f \in \mathcal{C}_b(\partial U)$

$$u(x) = \int_{\partial D} f(y) P_D^{\mu,\lambda}(x, y) dy$$

satisfies conditions (3) and (4).

Solution

We have

$$P_U^{\mu,\lambda}(x, y) = x_n^{\mu-\eta} \mathbb{P}^x\left(X^\eta\left(\tau_D^\eta\right) \in dy\right),$$

where $\eta = \sqrt{2\lambda + \mu^2}$. It means that studying λ - Poisson kernel is just studying Poisson kernel for the process with suitable drift.

Poisson kernel of the stripe - asymptotics

We write $f \stackrel{c}{\approx} g, x \in A$ whenever $\frac{1}{c}f(x) < g(x) < cf(x)$ for every $x \in A$. We have

$$P_{S_a}^\mu(x, y) \stackrel{c}{\approx} \frac{e^{-\frac{\pi}{a}|x-y|}}{a^{\mu+\nu+2}}\left(a^{\mu+\nu+2} + |x-y|^{\mu+\nu+2}\right) \times$$
$$\times \begin{cases} \dfrac{\delta_a(x_1)}{|x-y|^n} \dfrac{x_n^{2\nu-\mu} y_n^{-1-\mu}}{\left(\cosh\rho + \frac{1}{a}|x-y|\right)^{\mu+1/2}}, & y \in \partial_1 S_a, \\ x_n^{2\mu} \dfrac{|x-y|^2 \wedge [\delta_a(x_1)\delta_a(y_1)]}{|x-y|^{2(\mu+\nu+1)}}, & y \in \partial_2 S_a. \end{cases}$$

where $\nu = \frac{n-1}{2}$, $\delta_a(x_1) = x_a \wedge (a - x_1)$, $\rho = d_{\mathbb{H}^n}(x, y)$.

Green function of the stripe - asymptotics

We define Green function $G_U^\mu(x, y)$ of the set U for proces $X^{(\mu)}$ as follows

$$G_U^\mu(x, y) = \int_0^{\tau_U^\mu} \mathbb{P}^x\left(X^{(\mu)}(t) \in dy\right) dt.$$

We have

$$G_{s_a}^\mu(x, y) \stackrel{c}{\approx} \frac{e^{-\frac{\pi}{a}|x-y|}}{a^{\mu+\nu+2}} x_n^{2\nu-\mu} y_n^{-1-\mu} \frac{a^{\mu+\nu+2} + |x-y|^{\mu+\nu+2}}{|x-y|^n}$$
$$\times \frac{|x-y|^2 \wedge (\delta_a(x_1)\delta_a(y_1))}{\left(\cosh\rho + \frac{1}{a}|x-y|\right)^{\mu+1/2}}, \quad x, y \in S_a.$$

When a tends to ∞ we obtain estimates of Green function and Poisson kernel of hyperbolic halfspace $S_\infty = \{x \in \mathbb{H}^n : x_1 > 0\}$.

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