First Hitting times of Bessel processes and zeros of modified Bessel functions

joint work with Y. Hamana (and T. Shirai)

2013/7/15 (Wroclaw)

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All results are found in the arXiv.

1 Plan of talk

$\{R_t^{(\nu)}\}$: a $2(\nu+1)$-dim Bessel process with index ν ($\nu > -1$ for simplicity) generated by

$$\mathscr{G}^{(\nu)} = \frac{1}{2}\frac{d^2}{dx^2} + \frac{2\nu+1}{2x}\frac{d}{dx} = \frac{1}{2x^{2\nu+1}}\frac{d}{dx}\Big(\frac{1}{x^{-2\nu-1}}\frac{d}{dx}\Big) \quad (x>0)$$

First hitting time : $\tau_{a,b}^{(\nu)} := \inf\{t>0; R_t^{(\nu)} = b\} \quad (R_0^{(\nu)} = a)$

1. explicit expressions for the distribution functions (and the densities) for $\tau_{a,b}^{(\nu)}$

   by means of the zeros of I_ν and K_ν
2. asymptotic behaviors of the tail probabilities (and the densities)
3. explicit expressions for the Lévy measures
4. applications to the asymptotics of the expectations of the volumes of Wiener sausages
5. study on the zeros of K_ν

2 Laplace transforms

$x \mapsto \mathbf{E}[\exp(-\lambda \tau_{x,b}^{(\nu)})]$: increasing solution for $\mathscr{G}^{(\nu)} u = \lambda u$ for $0 \leqq x < b$

decreasing solution for $\mathscr{G}^{(\nu)} u = \lambda u$ for $x > b$

In fact,

$$a < b \quad \Longrightarrow \quad \mathbf{E}[\exp(-\lambda \tau_{a,b}^{(\nu)})] = \frac{a^{-\nu} I_\nu(a\sqrt{2\lambda})}{b^{-\nu} I_\nu(b\sqrt{2\lambda})} \qquad (0 \text{ is not a natural boundary})$$

$$b < a \quad \Longrightarrow \quad \mathbf{E}[\exp(-\lambda \tau_{a,b}^{(\nu)})] = \frac{a^{-\nu} K_\nu(a\sqrt{2\lambda})}{b^{-\nu} K_\nu(b\sqrt{2\lambda})} \qquad (\infty \text{ is a natural boundary})$$

Here I_ν and K_ν is the modified Bessel function :

$$I_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{n=0}^\infty \frac{(z/2)^{2n}}{n!\Gamma(\nu+n+1)},$$

$$K_\nu(z) = \frac{\pi}{2} \frac{I_{-\nu}(z) - I_\nu(z)}{\sin \nu\pi} \; (\nu \notin \mathbf{Z}), \qquad K_n(z) = \frac{(-1)^n}{2} \left[\frac{\partial I_{-\nu}}{\partial \nu} - \frac{\partial I_\nu(z)}{\partial \nu}\right]_{\nu=n}$$

Bessel's differential equation : $\displaystyle \frac{d^2u}{dz^2} + \frac{1}{z}\frac{du}{dz} - \left(1 + \frac{\nu^2}{z^2}\right)u = 0$

3 When $a < b$ (by Kent 1980)

From formula

$$\mathbf{E}[\exp(-\lambda\tau_{a,b}^{(\nu)})] = \frac{a^{-\nu}I_\nu(a\sqrt{2\lambda})}{b^{-\nu}I_\nu(b\sqrt{2\lambda})} = \text{const.} \prod_{k=1}^{\infty} \frac{1-2a^2\lambda/\lambda_{\nu,k}}{1-2b^2\lambda/\lambda_{\nu,k}},$$

we get

$$\mathbf{P}(\tau_{a,b}^{(\nu)} > t) = 2\Big(\frac{b}{a}\Big)^{\nu} \sum_{k=1}^{\infty} \frac{J_\nu(aj_{\nu,k}/b)}{j_{\nu,k}J_{\nu+1}(j_{\nu,k})} e^{-(j_{\nu,k}^2/2b^2)t},$$

where $j_{\nu,k}$ $(k=1,2,...)$ are the zeros of J_ν $\qquad (I_\nu(z) = e^{-\nu i\pi/2}J_\nu(e^{\pi i/2}z))$.

Hence,

$$\mathbf{P}(\tau_{a,b}^{(\nu)} > t) = 2\Big(\frac{b}{a}\Big)^{\nu} \frac{J_\nu(aj_{\nu,1}/b)}{j_{\nu,k}J_{\nu+1}(j_{\nu,1})} e^{-(j_{\nu,1}^2/2b^2)t}(1+o(1)) \qquad \text{as } t\to\infty$$

4 When $b < a$

we need to consider the natural boundary ∞:

$$\mathbf{E}[\exp(-\lambda\tau_{a,b}^{(\nu)})] = \frac{a^{-\nu}K_\nu(a\sqrt{2\lambda})}{b^{-\nu}K_\nu(b\sqrt{2\lambda})}$$

We invert this Laplace transform.

Known facts on the zeros of K_ν ($K_{-\nu} = K_\nu$):

For $0 \leqq \nu < 3/2$, $K_\nu(z)$ has no zero, for $\nu = 3/2$, $K_{3/2}(z)$ has one zero -1

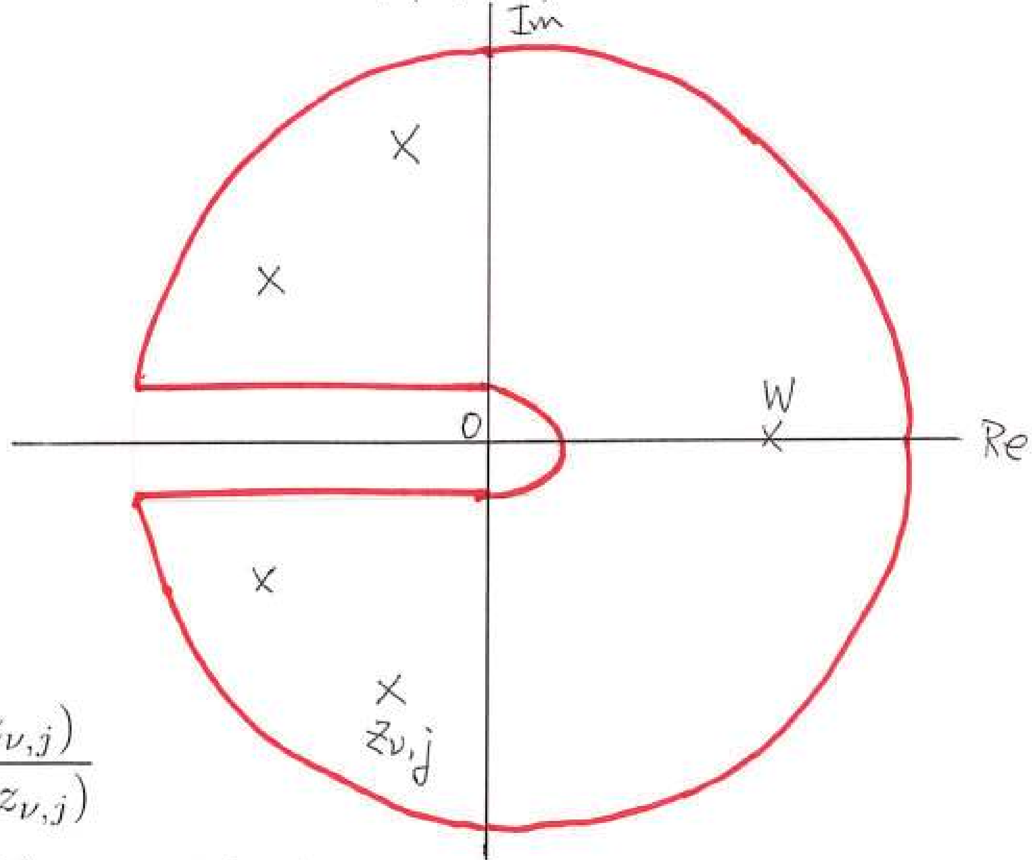
for $3/2 < \nu < 7/2$, $K_\nu(z)$ has 2 zeros, for $\nu = 7/2$, $K_{7/2}(z)$ has 3 zeros (one is negative)

for $7/2 < \nu < 11/2$, $K_\nu(z)$ has 4 zeros, $K_{11/2}(z)$ has 5 zeros (one is negative)

We denote the zeros of $K_\nu(z)$ by $z_{\nu,1}, z_{\nu,2}, ..., z_{\nu,N(\nu)}$

$\mathrm{Re}(z_{\nu,j}) < 0$. $\overline{z_{\nu,j}}$ is also a zero of K_ν.

Laplace inversion for $\quad 0 < \lambda \mapsto \mathbf{E}[\exp(-\lambda \tau_{a,b}^{(\nu)})] = \dfrac{a^{-\nu} K_\nu(a\sqrt{2\lambda})}{b^{-\nu} K_\nu(b\sqrt{2\lambda})}$



We set $c = \dfrac{a}{b} > 1$ and

consider a contour integral of

$$g_{\nu,c}^{w}(z) = \frac{we^{(c-1)z}}{z(z-w)} \frac{K_\nu(cz)}{K_\nu(z)}.$$

$$\mathrm{Res}(g_{\nu,c}^{w}; z = w) = e^{(c-1)z} \frac{K_\nu(cw)}{K_\nu(w)},$$

which we want to study.

$$\mathrm{Res}(g_{\nu,c}^{w}; z = z_{\nu,j}) = \frac{we^{(c-1)z_{\nu,j}}}{z_{\nu,j}(w - z_{\nu,j})} \frac{K_\nu(cz_{\nu,j})}{K_{\nu+1}(z_{\nu,j})}$$

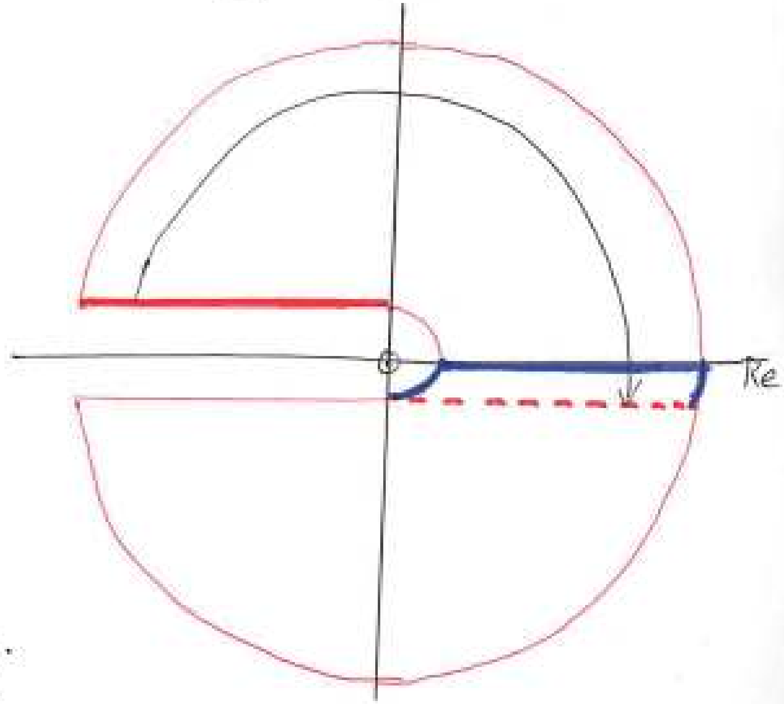
$$\int_{-R}^{0} g_{\nu,c}^{w}(x+i\varepsilon)dx = \int_0^R \frac{we^{-(c-1)(x-i\varepsilon)}}{(x-i\varepsilon)(x-i\varepsilon+w)} \frac{K_\nu(e^{i\pi}c(x-i\varepsilon))}{K_\nu(e^{i\pi}(x-i\varepsilon))} dx,$$

Here $K_\nu(e^{i\pi}z) = e^{-i\pi\nu}K_\nu(z) - i\pi I_\nu(z)$

For the Laplace inversion for $\quad 0 < \lambda \mapsto \mathbf{E}[\exp(-\lambda\tau_{a,b}^{(\nu)}] = \dfrac{a^{-\nu}K_\nu(a\sqrt{2\lambda})}{b^{-\nu}K_\nu(b\sqrt{2\lambda})}$,

we use (if $\nu > 0, \nu \notin \mathbf{Z}$)

$$\begin{aligned}\frac{K_\nu(cw)}{K_\nu(w)} &= \frac{e^{-(c-1)w}}{c^\nu} \\ &\quad -e^{-(c-1)w}\sum_{j=1}^{N(\nu)}\frac{we^{(c-1)z_{\nu,j}}}{z_{\nu,j}(w-z_{\nu,j})}\frac{K_\nu(cz_{\nu,j})}{K_{\nu+1}(z_{\nu,j})} \\ &\quad -e^{-(c-1)w}\int_0^\infty \frac{we^{-(c-1)x}L_{\nu,c}(x)}{x(x+w)}dx,\end{aligned}$$



where

$$L_{\nu,c}(x) = \frac{\cos(\pi\nu)\{I_\nu(cx)K_\nu(x) - I_\nu(x)K_\nu(cx)\}}{K_\nu(x)^2 + \pi^2 I_\nu(x)^2 + 2\pi\sin(\pi\nu)K_\nu(x)I_\nu(x)}.$$

(Recall $K_\nu(e^{i\pi}z) = e^{-i\pi\nu}K_\nu(z) - i\pi I_\nu(z), \quad K_\nu(e^{-i\pi}z) = e^{i\pi\nu}K_\nu(z) + i\pi I_\nu(z)$

5 Explicit form for the distribution (Main Reult)

If $\nu > 3/2$ is not an integer (for example) and $b < a$,

$$P(\tau_{a,b}^{(\nu)} \leqq t) = \left(\frac{b}{a}\right)^{2\nu} \int_0^t \frac{a-b}{\sqrt{2\pi s^3}} e^{-\frac{(a-b)^2}{2s}} ds$$
$$-\left(\frac{b}{a}\right)^{\nu} \sum_{j=1}^{N(\nu)} \frac{K_\nu(a z_{\nu,j}/b)}{z_{\nu,j} K_{\nu+1}(z_{\nu,j})} \int_0^t \frac{a-b}{\sqrt{2\pi s^3}} e^{-\frac{(a-b)^2}{2s} + \frac{(a-b) z_{\nu,j}\sqrt{t}}{b\sqrt{s}}} ds$$
$$-\left(\frac{b}{a}\right)^{\nu} \int_0^t \frac{a-b}{\sqrt{2\pi s^3}} e^{-\frac{(a-b)^2}{2s}} \left[\int_0^\infty \frac{e^{-\frac{x(a-b)\sqrt{t}}{b\sqrt{s}}} L_{\nu,a/b}(x)}{x} dx\right] ds,$$

where $L_{\nu,c}(x) = \dfrac{\cos(\pi\nu)\{I_\nu(cx)K_\nu(x) - I_\nu(x)K_\nu(cx)\}}{K_\nu(x)^2 + \pi^2 I_\nu(x)^2 + 2\pi\sin(\pi\nu)K_\nu(x)I_\nu(x)}$.

Asymptotics of tail probability after Byczkowski–Ryznar

$$P(\tau_{a,b}^{(\nu)} > t) = 1 - \left(\frac{b}{a}\right)^{2\nu} \qquad \text{(this is } P(\tau_{a,b}^{(\nu)} = \infty))$$
$$-\left(\frac{b^3}{2a}\right)^{\nu} \left\{\left(\frac{a}{b}\right)^{\nu} - \left(\frac{b}{a}\right)^{\nu}\right\} \frac{1}{\Gamma(\nu+1)} \cdot t^{-\nu} + o(t^{-\nu}).$$

$P(\tau_{a,b}^{(0)} > t) = \dfrac{2\log(a/b)}{\log t} \cdot (1 + o(1))$, when $\nu = 0$ (2-dim BM case)

6 Lévy measure for $\tau_{a,b}^{(\nu)}$ (infinitely divisible)

If $\nu > -1$ and $a < b$ (resp. $b < a$),

$$\mathbf{E}[\exp(-\lambda\tau_{a,b}^{(\nu)})] = \frac{a^{-\nu}I_\nu(a\sqrt{2\lambda})}{b^{-\nu}I_\nu(b\sqrt{2\lambda})} \quad \text{or} \quad \frac{a^{-\nu}K_\nu(a\sqrt{2\lambda})}{b^{-\nu}K_\nu(b\sqrt{2\lambda})}$$

$$= \exp\left(\int_0^\infty (e^{-\lambda s} - 1)m_{a,b}^{(\nu)}(ds)\right).$$

To find the Lévy measure, we need compute

$$\frac{d}{dz}\log I(z) = \frac{I_\nu'(z)}{I_\nu(z)} \quad \text{This is easy ! since } I_\nu(z) = \left(\frac{z}{2}\right)^\nu \frac{1}{\Gamma(\nu+1)} \prod_{n=1}^\infty \left(1 + \frac{z^2}{j_{\nu,j}^2}\right)$$

$$\frac{d}{dz}\log K_\nu(z) = \frac{K_\nu'(z)}{K_\nu(z)} = \frac{\nu}{z} - \frac{K_{\nu+1}(z)}{K_\nu(z)}.$$

We can apply a similar residue calculus for $\dfrac{K_{\nu+1}(z)}{K_\nu(z)}$.

If $b < a$, $\displaystyle\int_0^\infty s e^{-\lambda s} m_{a,b}^{(\nu)}(ds) = -\frac{d}{d\lambda}\Big(\log \frac{K_\nu(a\sqrt{2\lambda})}{K_\nu(b\sqrt{2\lambda})}\Big)$.

From formula ($z_{\nu,j}$'s are the zeros of K_ν)

$$\frac{K_{\nu+1}(z)}{K_\nu(z)} = 1 + \frac{2\nu}{z} + \sum_{j=1}^{N(\nu)} \frac{1}{z_{\nu,j} - z} + \cos(\pi\nu) \int_0^\infty \frac{dx}{x(x+z)G_\nu(x)},$$

the Lévy measure is given by

$$\frac{m_{a,b}^{(\nu)}(ds)}{ds} = \frac{a-b}{\sqrt{2\pi s^3}} - \frac{1}{2\sqrt{\pi s^3}} \sum_{j=1}^{N(\nu)} \int_0^\infty e^{-\frac{\xi^2}{4s}} \left(e^{\frac{z_{\nu,j}\xi}{\sqrt{2}a}} - e^{\frac{z_{\nu,j}\xi}{\sqrt{2}b}} \right) d\xi$$
$$+ \frac{\cos(\pi\nu)}{2\sqrt{\pi s^3}} \int_0^\infty \int_0^\infty \frac{1}{\eta G_\nu(\eta)} e^{-\frac{\xi^2}{4s}} \left(e^{-\frac{\xi\eta}{\sqrt{2}a}} - e^{-\frac{\xi\eta}{\sqrt{2}b}} \right) d\xi d\eta.$$

Here

$$G_\nu(x) = K_\nu(x)^2 + \pi^2 I_\nu(x)^2 + 2\pi \sin(\pi\nu) K_\nu(x) I_\nu(x).$$

The simple formula above for $\dfrac{K_{\nu+1}(z)}{K_\nu(z)}$ has nice applications.

7 The exectation of the volume of Wiener sausage

For a d-dim BM $\{B(t)\}_{t \geqq 0}$ with $B(0) = 0$ and a ball U_r with radius $r > 0$,

we denote the Wiener sausage by $W(t) := \{y \in \mathbf{R}^d; \exists s \in [0,t] \text{ s.t. } |y - B(s)| < r\}$.

It is known that $\mathbf{E}[\text{vol}(W(t))] = \text{vol}(U_r) + L^{(d)}(t), \qquad L^{(d)}(t) = \displaystyle\int_{\mathbf{R}^d \setminus U_r} \mathbf{P}_x(\tau \leqq t)dx,$

where $\tau = \inf\{t \geqq 0; B(t) \in U_r\} \Big(= \tau^{((d/2)-2)}_{|x|,r}$: hitting time of a Bessel process$\Big)$.

Known Facts: $\quad L^{(1)}(t) = 2\sqrt{\dfrac{2t}{\pi}}, \qquad L^{(3)}(t) = 2\pi r t + 4r^2\sqrt{2\pi t}$ (classical!)

$$\int_0^\infty e^{-\lambda t} L^{(d)}(t)dt = \frac{\text{vol}(\partial U_r)}{\sqrt{2\lambda^3}} \frac{K_{d/2}(r\sqrt{2\lambda})}{K_{d/2-1}(r\sqrt{2\lambda})},$$

$$K_{n+1/2}(z) = \sqrt{\frac{\pi}{2z}} e^{-z} \sum_{k=0}^{n} \frac{(n+k)!}{k!(n-k)!} \frac{1}{(2z)^k}$$

If $d \geqq 5$ is odd, using the zeros $z_j^{(d)}$ of $K_{d/2-1}$,

$$L^{(d)}(t) = S_{d-1} r^{d-2} \left[\frac{(d-2)t}{2} + \frac{r^2}{d-4} - \frac{\sqrt{2} r^3}{\sqrt{\pi t}} \sum_{j=1}^{N_d} \frac{1}{(z_j^{(d)})^2} \int_0^\infty e^{-\frac{r^2 x^2}{2t} + z_j^{(d)} x} dx \right].$$(Hamana)

(1) If $d=2$, $L^{(2)}(t) = 2\pi r\left[\sqrt{\frac{2t}{\pi}} + \frac{\sqrt{2}r^2}{\sqrt{\pi t}}\int_0^\infty\int_0^\infty \frac{xy-1+e^{-xy}}{y^3 G^{(2)}(y)} e^{-\frac{r^2x^2}{2t}}\,dxdy\right]$.

(2) If $d=4$, $L^{(4)}(t) = 2\pi^2 r^2\left[t + \frac{\sqrt{2}r^3}{\sqrt{\pi t}}\int_0^\infty\int_0^\infty \frac{1-e^{-xy}}{y^3 G^{(4)}(y)} e^{-\frac{r^2x^2}{2t}}\,dxdy\right]$.

(3) If $d \geqq 6$ and d is even,

$$L^{(d)}(t) = S_{d-1}r^{d-2}\left[\frac{(d-2)t}{2} + \frac{r^2}{d-4} - \frac{\sqrt{2}r^3}{\sqrt{\pi t}}\sum_{j=1}^{N_d}\frac{1}{(z_j^{(d)})^2}\int_0^\infty e^{-\frac{r^2x^2}{2t}+z_j^{(d)}x}\,dx\right.$$
$$\left. + \frac{(-1)^{d/2-1}\sqrt{2}r^3}{\sqrt{\pi t}}\int_0^\infty\int_0^\infty \frac{e^{-xy}}{y^3 G^{(d)}(y)} e^{-\frac{r^2x^2}{2t}}\,dxdy\right].$$

Large time asymptotics For some explicitly written constants α_i's

$$L^{(d)}(t) = S_{d-1}r^{d-2}\left[\frac{(d-2)t}{2} + \frac{r^2}{d-4} - \frac{\alpha_d\, r^{d-2}}{2^{d/2-2}(d-4)\Gamma(d/2-1)}\frac{1}{t^{d/2-2}}\right.$$
$$\left. + \frac{1}{t^{d/2-1}}\sum_{n=0}^{d-6}\frac{\alpha_n^{(d)}}{t^{n/2}} + \frac{\Gamma((d-3)/2)r^{2d-4}}{\sqrt{\pi}(d-2)\Gamma(d/2-1)^3}\frac{\log t}{t^{d-3}} + O\left(\frac{1}{t^{d-3}}\right)\right],$$

which is a generalization of a result by Le Gall (But note Le Gall considered a Wiener

$$L^{(d)}(t) = c_1^{(d)}t + c_2^{(d)} + c_3^{(d)}t^{2-d/2} + O(t^{1-d/2}) \quad (d\text{: odd } (\geqq 5)).$$

sausage with

$$L^{(4)}(t) = c_1' t + c_2'\log t + c_3' + c_4'\frac{\log t}{t} + o\left(\frac{\log t}{t}\right) \quad (d=4)$$

a general compact set)

8 The zeros of K_ν are the roots of some algebraic equations

Consider the asymptotic expansion as $z \to \infty$ for

$$\frac{K_{\nu+1}(z)}{K_\nu(z)} = 1 + \frac{2\nu}{z} + \sum_{j=1}^{N(\nu)} \frac{1}{z_{\nu,j} - z} + \cos(\pi\nu) \int_0^\infty \frac{dx}{x(x+z)G_\nu(x)}.$$

Then, $\displaystyle\sum_{j=1}^{N(\nu)} \frac{1}{z_{\nu,j} - z} = \sum_{j=1}^{N(\nu)} \frac{-1}{z(1 - z_{\nu,j}/z)} = -\sum_{n=0}^{\infty}\Bigg(\sum_{j=1}^{N(\nu)} (z_{\nu,j})^n\Bigg)\frac{1}{z^{n+1}},$

and it is well known that for $K_\nu(z), K_{\nu+1}(z)$ as $z \to \infty$,

$$K_\nu(z) = \sqrt{\frac{\pi}{2z}}e^{-z} \sum_{n=0}^{M+1} \frac{\Gamma(\nu+n+1/2)}{n!\Gamma(\nu-n+1/2)} \frac{1}{(2z)^n} + O\Big(\frac{1}{z^{M+2}}\Big).$$

Hence we obtain $\boxed{\displaystyle\sum_{j=1}^{N(\nu)} (z_{\nu,j})^n = -a_{n+1} + (-1)^n \cos(\pi\nu) \int_0^\infty \frac{x^{n-1}}{G_\nu(x)} dx}$ $(n = 1, 2, \dots),$

where a_n's are determined recursively (e.g.,$a_0 = 1,\ a_1 = \nu + \frac{1}{2},\ a_2 = \frac{1}{2}(\nu^2 - \frac{1}{4}),$

$a_3 = -\frac{1}{2}(\nu^2 - \frac{1}{4}),\ a_4 = -\frac{1}{8}(\nu^2 - \frac{1}{4})(\nu^2 - \frac{25}{4}),\ a_5 = \frac{1}{2}(\nu^2 - \frac{1}{4})(\nu^2 - \frac{13}{4}),$

$G_\nu(x) = K_\nu(x)^2 + \pi^2 I_\nu(x)^2 + 2\pi\sin(\pi\nu)K_\nu(x)I_\nu(x).$

Mathematica easily solves this algebraic equation numerically.

Asymptotics as $z \downarrow 0$ of

$$\frac{K_{\nu+1}(z)}{K_\nu(z)} = 1 + \frac{2\nu}{z} + \sum_{j=1}^{N(\nu)} \frac{1}{z_{\nu,j} - z} + \cos(\pi\nu) \int_0^\infty \frac{dx}{x(x+z)G_\nu(x)},$$

gives us the algebraic equations for the reciprocals $z_{\nu,j}^{-1}$. (Good check !)

Example (2 Zeros of K_2) When $\nu = 2$, we have

$$\begin{cases} z_{2,1} + z_{2,2} = -\dfrac{15}{8} - \displaystyle\int_0^\infty \frac{dy}{G_2(y)}, \\ z_{2,1}^2 + z_{2,2}^2 = \dfrac{15}{8} + \displaystyle\int_0^\infty \frac{y\,dy}{G_2(y)}, \end{cases} \qquad \begin{cases} \dfrac{1}{z_{2,1}} + \dfrac{1}{z_{2,2}} = -1 - \displaystyle\int_0^\infty \frac{dy}{y^2 G_2(y)}, \\ \dfrac{1}{z_{2,1}^2} + \dfrac{1}{z_{2,2}^2} = \dfrac{1}{2} + \displaystyle\int_0^\infty \frac{dy}{y^3 G_2(y)}, \end{cases}$$

where $G_2(x) = K_2(x)^2 + \pi^2 I_2(x)^2$.

By using Mathematica we obtain from each system of equations that the zeros are

approximately $\boxed{-1.28137 \pm 0.429485\, i}$, and check the comment in Watson's book

"The two zeros of $K_2(z)$ are not very far from the points $\boxed{-1.29 \pm 0.44\, i}$."

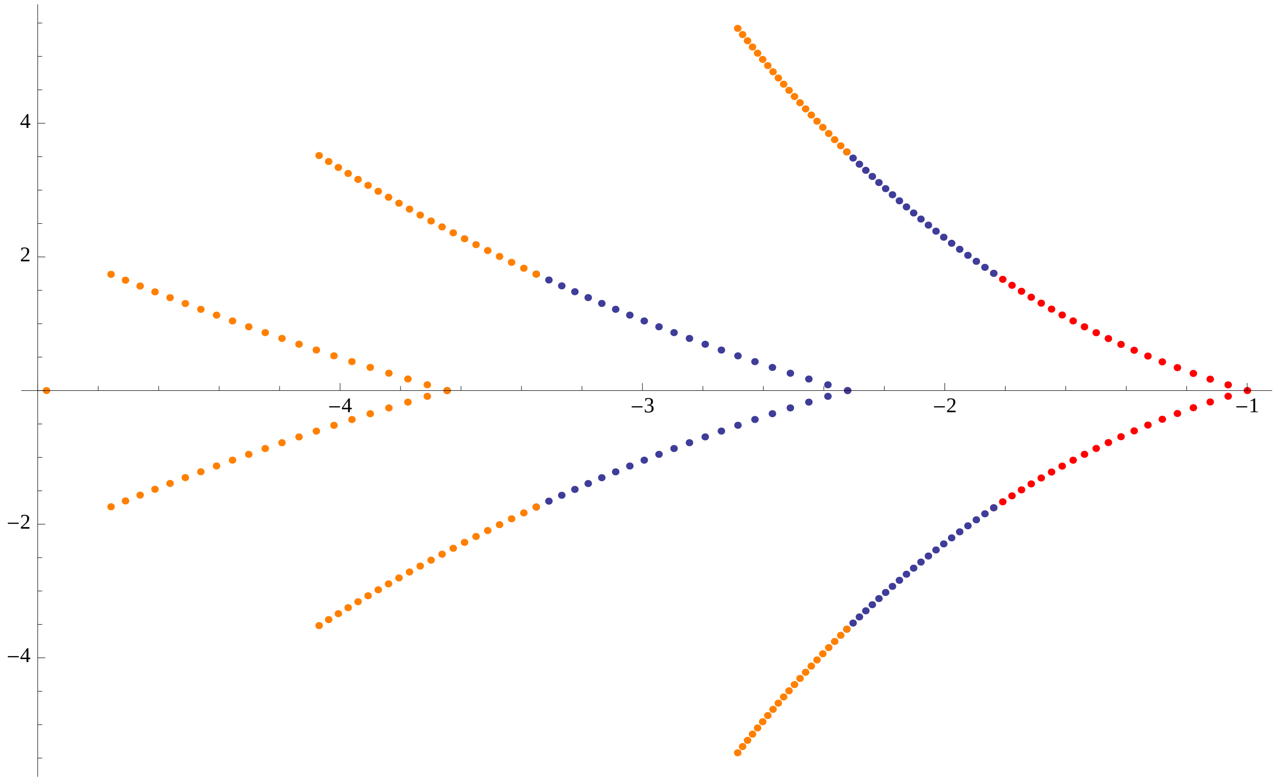


図1　K_ν has 2 zeros when $\frac{3}{2}<\nu<\frac{7}{2}$, 4 zeros when $\frac{7}{2}<\nu<\frac{11}{2}$, 6 zeros when $\frac{11}{2}<\nu<\frac{15}{2}$...

References

M.K.Kerimov and S.L.Skorokhodov, U.S.S.R. Comput. Math. and Math. Phys., 1984.

M.V.Zavolzhenskii and A.KH.Terskov, U.S.S.R. Comput. Math. and Math. Phys., 1977.