

**First Hitting times of Bessel processes and  
zeros of modified Bessel functions**

**joint work with Y. Hamana (and T. Shirai)**

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All results are found in the arXiv.

# 1 Plan of talk

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$\{R_t^{(\nu)}\}$  : a  $2(\nu + 1)$ -dim Bessel process with index  $\nu$  ( $\nu > -1$  for simplicity) generated by

$$\mathcal{G}^{(\nu)} = \frac{1}{2} \frac{d^2}{dx^2} + \frac{2\nu + 1}{2x} \frac{d}{dx} = \frac{1}{2x^{2\nu+1}} \frac{d}{dx} \left( \frac{1}{x^{-2\nu-1}} \frac{d}{dx} \right) \quad (x > 0)$$

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First hitting time :  $\tau_{a,b}^{(\nu)} := \inf\{t > 0; R_t^{(\nu)} = b\} \quad (R_0^{(\nu)} = a)$

1. explicit expressions for the distribution functions (and the densities) for  $\tau_{a,b}^{(\nu)}$   
by means of the zeros of  $I_\nu$  and  $K_\nu$
2. asymptotic behaviors of the tail probabilities (and the densities)
3. explicit expressions for the Lévy measures
4. applications to the asymptotics of the expectations of the volumes of Wiener sausages
5. study on the zeros of  $K_\nu$

## 2 Laplace transforms

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$x \mapsto \mathbf{E}[\exp(-\lambda\tau_{x,b}^{(\nu)})]$  : increasing solution for  $\mathcal{G}^{(\nu)}u = \lambda u$  for  $0 \leq x < b$

decreasing solution for  $\mathcal{G}^{(\nu)}u = \lambda u$  for  $x > b$

In fact,

$$a < b \implies \mathbf{E}[\exp(-\lambda\tau_{a,b}^{(\nu)})] = \frac{a^{-\nu} I_{\nu}(a\sqrt{2\lambda})}{b^{-\nu} I_{\nu}(b\sqrt{2\lambda})} \quad (0 \text{ is not a natural boundary})$$

$$b < a \implies \mathbf{E}[\exp(-\lambda\tau_{a,b}^{(\nu)})] = \frac{a^{-\nu} K_{\nu}(a\sqrt{2\lambda})}{b^{-\nu} K_{\nu}(b\sqrt{2\lambda})} \quad (\infty \text{ is a natural boundary})$$

Here  $I_{\nu}$  and  $K_{\nu}$  is the modified Bessel function :

$$I_{\nu}(z) = \left(\frac{z}{2}\right)^{\nu} \sum_{n=0}^{\infty} \frac{(z/2)^{2n}}{n! \Gamma(\nu + n + 1)},$$

$$K_{\nu}(z) = \frac{\pi}{2} \frac{I_{-\nu}(z) - I_{\nu}(z)}{\sin \nu \pi} \quad (\nu \notin \mathbf{Z}), \quad K_n(z) = \frac{(-1)^n}{2} \left[ \frac{\partial I_{-n}}{\partial \nu} - \frac{\partial I_n(z)}{\partial \nu} \right]_{\nu=n}$$

Bessel's differential equation :  $\frac{d^2 u}{dz^2} + \frac{1}{z} \frac{du}{dz} - \left(1 + \frac{\nu^2}{z^2}\right) u = 0$

### 3 When $a < b$ (by Kent 1980)

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From formula

$$\mathbf{E}[\exp(-\lambda\tau_{a,b}^{(\nu)})] = \frac{a^{-\nu} I_{\nu}(a\sqrt{2\lambda})}{b^{-\nu} I_{\nu}(b\sqrt{2\lambda})} = \text{const.} \prod_{k=1}^{\infty} \frac{1 - 2a^2\lambda/\lambda_{\nu,k}}{1 - 2b^2\lambda/\lambda_{\nu,k}},$$

we get

$$\mathbf{P}(\tau_{a,b}^{(\nu)} > t) = 2\left(\frac{b}{a}\right)^{\nu} \sum_{k=1}^{\infty} \frac{J_{\nu}(aj_{\nu,k}/b)}{j_{\nu,k}J_{\nu+1}(j_{\nu,k})} e^{-(j_{\nu,k}^2/2b^2)t},$$

where  $j_{\nu,k}$  ( $k = 1, 2, \dots$ ) are the zeros of  $J_{\nu}$  ( $I_{\nu}(z) = e^{-\nu i\pi/2} J_{\nu}(e^{\pi i/2} z)$ ).

Hence,

$$\mathbf{P}(\tau_{a,b}^{(\nu)} > t) = 2\left(\frac{b}{a}\right)^{\nu} \frac{J_{\nu}(aj_{\nu,1}/b)}{j_{\nu,1}J_{\nu+1}(j_{\nu,1})} e^{-(j_{\nu,1}^2/2b^2)t} (1 + o(1)) \quad \text{as } t \rightarrow \infty$$

## 4 When $b < a$

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we need to consider the natural boundary  $\infty$ :

$$\mathbf{E}[\exp(-\lambda\tau_{a,b}^{(\nu)})] = \frac{a^{-\nu} K_{\nu}(a\sqrt{2\lambda})}{b^{-\nu} K_{\nu}(b\sqrt{2\lambda})}$$

**We invert this Laplace transform.**

Known facts on the zeros of  $K_{\nu}$  ( $K_{-\nu} = K_{\nu}$ ):

For  $0 \leq \nu < 3/2$ ,  $K_{\nu}(z)$  has no zero, for  $\nu = 3/2$ ,  $K_{3/2}(z)$  has one zero  $-1$

for  $3/2 < \nu < 7/2$ ,  $K_{\nu}(z)$  has 2 zeros, for  $\nu = 7/2$ ,  $K_{7/2}(z)$  has 3 zeros (one is negative)

for  $7/2 < \nu < 11/2$ ,  $K_{\nu}(z)$  has 4 zeros,  $K_{11/2}(z)$  has 5 zeros (one is negative)

We denote the zeros of  $K_{\nu}(z)$  by  $z_{\nu,1}, z_{\nu,2}, \dots, z_{\nu,N(\nu)}$

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$\operatorname{Re}(z_{\nu,j}) < 0$ .  $\overline{z_{\nu,j}}$  is also a zero of  $K_{\nu}$ .

Laplace inversion for  $0 < \lambda \mapsto \mathbf{E}[\exp(-\lambda \tau_{a,b}^{(\nu)})] = \frac{a^{-\nu} K_{\nu}(a\sqrt{2\lambda})}{b^{-\nu} K_{\nu}(b\sqrt{2\lambda})}$

We set  $c = \frac{a}{b} > 1$  and

consider a contour integral of

$$g_{\nu,c}^w(z) = \frac{we^{(c-1)z} K_{\nu}(cz)}{z(z-w) K_{\nu}(z)}$$

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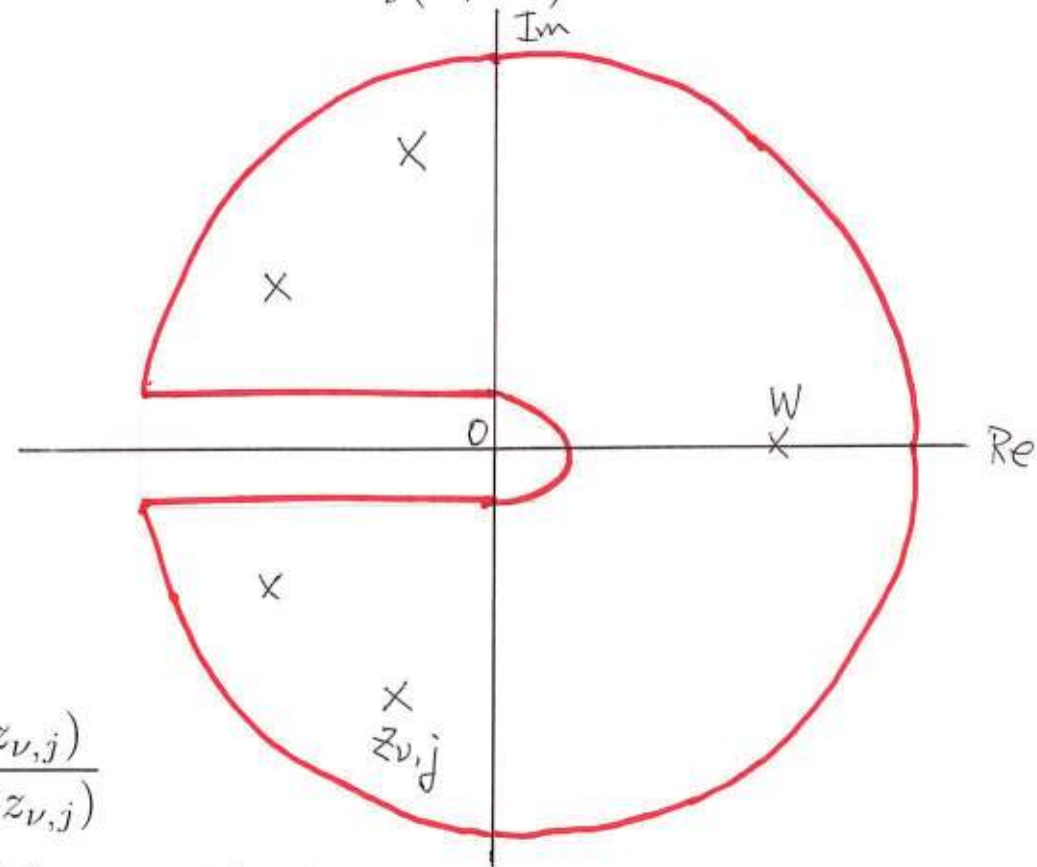

$$\text{Res}(g_{\nu,c}^w; z = w) = e^{(c-1)z} \frac{K_{\nu}(cw)}{K_{\nu}(w)},$$

which we want to study.

$$\text{Res}(g_{\nu,c}^w; z = z_{\nu,j}) = \frac{we^{(c-1)z_{\nu,j}} K_{\nu}(cz_{\nu,j})}{z_{\nu,j}(w - z_{\nu,j}) K_{\nu+1}(z_{\nu,j})}$$

$$\int_{-R}^0 g_{\nu,c}^w(x + i\varepsilon) dx = \int_0^R \frac{we^{-(c-1)(x-i\varepsilon)} K_{\nu}(e^{i\pi}c(x-i\varepsilon))}{(x-i\varepsilon)(x-i\varepsilon+w) K_{\nu}(e^{i\pi}(x-i\varepsilon))} dx,$$

Here  $K_{\nu}(e^{i\pi}z) = e^{-i\pi\nu} K_{\nu}(z) - i\pi I_{\nu}(z)$



For the Laplace inversion for  $0 < \lambda \mapsto \mathbf{E}[\exp(-\lambda \tau_{a,b}^{(\nu)})] = \frac{a^{-\nu} K_\nu(a\sqrt{2\lambda})}{b^{-\nu} K_\nu(b\sqrt{2\lambda})}$ ,

we use (if  $\nu > 0, \nu \notin \mathbf{Z}$ )

$$\frac{K_\nu(cw)}{K_\nu(w)} = \frac{e^{-(c-1)w}}{c^\nu}$$

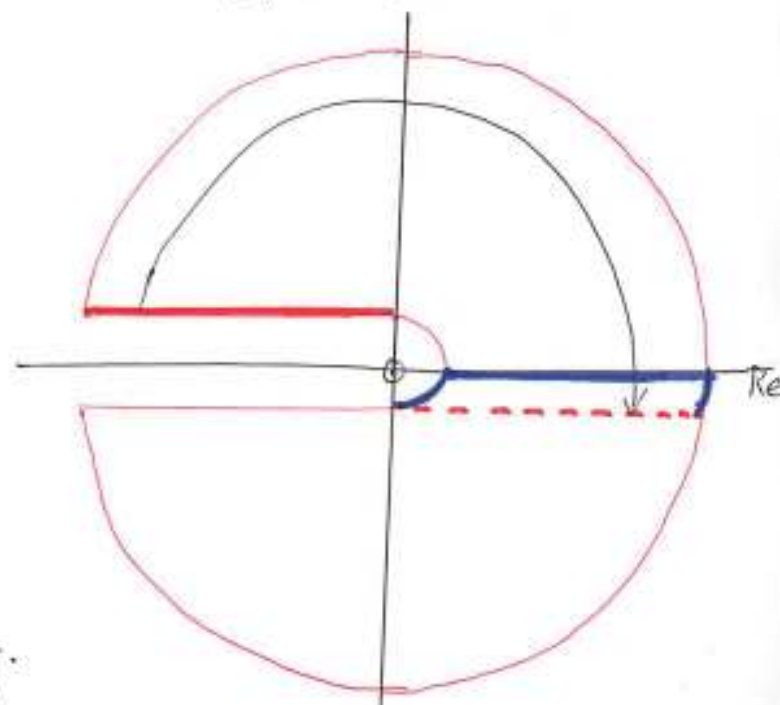
$$-e^{-(c-1)w} \sum_{j=1}^{N(\nu)} \frac{we^{(c-1)z_{\nu,j}}}{z_{\nu,j}(w - z_{\nu,j})} \frac{K_\nu(cz_{\nu,j})}{K_{\nu+1}(z_{\nu,j})}$$

$$-e^{-(c-1)w} \int_0^\infty \frac{we^{-(c-1)x} L_{\nu,c}(x)}{x(x+w)} dx,$$

where

$$L_{\nu,c}(x) = \frac{\cos(\pi\nu) \{I_\nu(cx)K_\nu(x) - I_\nu(x)K_\nu(cx)\}}{K_\nu(x)^2 + \pi^2 I_\nu(x)^2 + 2\pi \sin(\pi\nu) K_\nu(x) I_\nu(x)}.$$

(Recall  $K_\nu(e^{i\pi} z) = e^{-i\pi\nu} K_\nu(z) - i\pi I_\nu(z)$ ,  $K_\nu(e^{-i\pi} z) = e^{i\pi\nu} K_\nu(z) + i\pi I_\nu(z)$ )



## 5 Explicit form for the distribution (Main Reult)

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If  $\nu > 3/2$  is not an integer (for example) and  $b < a$ ,

$$\begin{aligned}
 P(\tau_{a,b}^{(\nu)} \leq t) &= \left(\frac{b}{a}\right)^{2\nu} \int_0^t \frac{a-b}{\sqrt{2\pi s^3}} e^{-\frac{(a-b)^2}{2s}} ds \\
 &\quad - \left(\frac{b}{a}\right)^\nu \sum_{j=1}^{N(\nu)} \frac{K_\nu(az_{\nu,j}/b)}{z_{\nu,j} K_{\nu+1}(z_{\nu,j})} \int_0^t \frac{a-b}{\sqrt{2\pi s^3}} e^{-\frac{(a-b)^2}{2s} + \frac{(a-b)z_{\nu,j}\sqrt{t}}{b\sqrt{s}}} ds \\
 &\quad - \left(\frac{b}{a}\right)^\nu \int_0^t \frac{a-b}{\sqrt{2\pi s^3}} e^{-\frac{(a-b)^2}{2s}} \left[ \int_0^\infty \frac{e^{-\frac{x(a-b)\sqrt{t}}{b\sqrt{s}}} L_{\nu,a/b}(x)}{x} dx \right] ds,
 \end{aligned}$$

where  $L_{\nu,c}(x) = \frac{\cos(\pi\nu)\{I_\nu(cx)K_\nu(x) - I_\nu(x)K_\nu(cx)\}}{K_\nu(x)^2 + \pi^2 I_\nu(x)^2 + 2\pi \sin(\pi\nu)K_\nu(x)I_\nu(x)}$ .

Asymptotics of tail probability after Byczkowski–Ryznar

$$\begin{aligned}
 P(\tau_{a,b}^{(\nu)} > t) &= 1 - \left(\frac{b}{a}\right)^{2\nu} \quad (\text{this is } P(\tau_{a,b}^{(\nu)} = \infty)) \\
 &\quad - \left(\frac{b^3}{2a}\right)^\nu \left\{ \left(\frac{a}{b}\right)^\nu - \left(\frac{b}{a}\right)^\nu \right\} \frac{1}{\Gamma(\nu+1)} \cdot t^{-\nu} + o(t^{-\nu}).
 \end{aligned}$$

$$P(\tau_{a,b}^{(0)} > t) = \frac{2 \log(a/b)}{\log t} \cdot (1 + o(1)), \text{ when } \nu = 0 \text{ (2-dim BM case)}$$



## 6 Lévy measure for $\tau_{a,b}^{(\nu)}$ (infinitely divisible)

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If  $\nu > -1$  and  $a < b$  (resp.  $b < a$ ),

$$\mathbf{E}[\exp(-\lambda\tau_{a,b}^{(\nu)})] = \frac{a^{-\nu} I_{\nu}(a\sqrt{2\lambda})}{b^{-\nu} I_{\nu}(b\sqrt{2\lambda})} \quad \text{or} \quad \frac{a^{-\nu} K_{\nu}(a\sqrt{2\lambda})}{b^{-\nu} K_{\nu}(b\sqrt{2\lambda})}$$
$$= \exp\left(\int_0^{\infty} (e^{-\lambda s} - 1) m_{a,b}^{(\nu)}(ds)\right).$$

To find the Lévy measure, we need compute

$$\frac{d}{dz} \log I(z) = \frac{I'_{\nu}(z)}{I_{\nu}(z)} \quad \text{This is easy ! since } I_{\nu}(z) = \left(\frac{z}{2}\right)^{\nu} \frac{1}{\Gamma(\nu+1)} \prod_{n=1}^{\infty} \left(1 + \frac{z^2}{j_{\nu,j}^2}\right)$$

$$\frac{d}{dz} \log K_{\nu}(z) = \frac{K'_{\nu}(z)}{K_{\nu}(z)} = \frac{\nu}{z} - \frac{K_{\nu+1}(z)}{K_{\nu}(z)}.$$

**We can apply a similar residue calculus for**  $\frac{K_{\nu+1}(z)}{K_{\nu}(z)}$ .

If  $b < a$ , 
$$\int_0^\infty se^{-\lambda s} m_{a,b}^{(\nu)}(ds) = -\frac{d}{d\lambda} \left( \log \frac{K_\nu(a\sqrt{2\lambda})}{K_\nu(b\sqrt{2\lambda})} \right).$$

From formula ( $z_{\nu,j}$ 's are the zeros of  $K_\nu$ )

$$\frac{K_{\nu+1}(z)}{K_\nu(z)} = 1 + \frac{2\nu}{z} + \sum_{j=1}^{N(\nu)} \frac{1}{z_{\nu,j} - z} + \cos(\pi\nu) \int_0^\infty \frac{dx}{x(x+z)G_\nu(x)},$$

the Lévy measure is given by

$$\begin{aligned} \frac{m_{a,b}^{(\nu)}(ds)}{ds} &= \frac{a-b}{\sqrt{2\pi s^3}} - \frac{1}{2\sqrt{\pi s^3}} \sum_{j=1}^{N(\nu)} \int_0^\infty e^{-\frac{\xi^2}{4s}} \left( e^{\frac{z_{\nu,j}\xi}{\sqrt{2a}}} - e^{\frac{z_{\nu,j}\xi}{\sqrt{2b}}} \right) d\xi \\ &\quad + \frac{\cos(\pi\nu)}{2\sqrt{\pi s^3}} \int_0^\infty \int_0^\infty \frac{1}{\eta G_\nu(\eta)} e^{-\frac{\xi^2}{4s}} \left( e^{-\frac{\xi\eta}{\sqrt{2a}}} - e^{-\frac{\xi\eta}{\sqrt{2b}}} \right) d\xi d\eta. \end{aligned}$$

Here

$$G_\nu(x) = K_\nu(x)^2 + \pi^2 I_\nu(x)^2 + 2\pi \sin(\pi\nu) K_\nu(x) I_\nu(x).$$

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The simple formula above for  $\frac{K_{\nu+1}(z)}{K_\nu(z)}$  has nice applications.

## 7 The expectation of the volume of Wiener sausage

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For a  $d$ -dim BM  $\{B(t)\}_{t \geq 0}$  with  $B(0) = 0$  and a ball  $U_r$  with radius  $r > 0$ ,

we denote the Wiener sausage by  $W(t) := \{y \in \mathbf{R}^d; \exists s \in [0, t] \text{ s.t. } |y - B(s)| < r\}$ .

It is known that  $\mathbf{E}[\text{vol}(W(t))] = \text{vol}(U_r) + L^{(d)}(t)$ ,  $L^{(d)}(t) = \int_{\mathbf{R}^d \setminus U_r} \mathbf{P}_x(\tau \leq t) dx$ ,

where  $\tau = \inf\{t \geq 0; B(t) \in U_r\}$  ( $= \tau_{|x|, r}^{((d/2)-2)}$ : hitting time of a Bessel process).

Known Facts:  $L^{(1)}(t) = 2\sqrt{\frac{2t}{\pi}}$ ,  $L^{(3)}(t) = 2\pi r t + 4r^2\sqrt{2\pi t}$  (classical!)

$$\int_0^\infty e^{-\lambda t} L^{(d)}(t) dt = \frac{\text{vol}(\partial U_r)}{\sqrt{2\lambda^3}} \frac{K_{d/2}(r\sqrt{2\lambda})}{K_{d/2-1}(r\sqrt{2\lambda})},$$

$$K_{n+1/2}(z) = \sqrt{\frac{\pi}{2z}} e^{-z} \sum_{k=0}^n \frac{(n+k)!}{k!(n-k)!} \frac{1}{(2z)^k}$$

If  $d \geq 5$  is odd, using the zeros  $z_j^{(d)}$  of  $K_{d/2-1}$ ,

$$L^{(d)}(t) = S_{d-1} r^{d-2} \left[ \frac{(d-2)t}{2} + \frac{r^2}{d-4} - \frac{\sqrt{2}r^3}{\sqrt{\pi t}} \sum_{j=1}^{N_d} \frac{1}{(z_j^{(d)})^2} \int_0^\infty e^{-\frac{r^2 x^2}{2t} + z_j^{(d)} x} dx \right]. \text{(Hamana)}$$

$$(1) \text{ If } d = 2, L^{(2)}(t) = 2\pi r \left[ \sqrt{\frac{2t}{\pi}} + \frac{\sqrt{2}r^2}{\sqrt{\pi t}} \int_0^\infty \int_0^\infty \frac{xy - 1 + e^{-xy}}{y^3 G^{(2)}(y)} e^{-\frac{r^2 x^2}{2t}} dx dy \right].$$

$$(2) \text{ If } d = 4, L^{(4)}(t) = 2\pi^2 r^2 \left[ t + \frac{\sqrt{2}r^3}{\sqrt{\pi t}} \int_0^\infty \int_0^\infty \frac{1 - e^{-xy}}{y^3 G^{(4)}(y)} e^{-\frac{r^2 x^2}{2t}} dx dy \right].$$

(3) If  $d \geq 6$  and  $d$  is even,

$$L^{(d)}(t) = S_{d-1} r^{d-2} \left[ \frac{(d-2)t}{2} + \frac{r^2}{d-4} - \frac{\sqrt{2}r^3}{\sqrt{\pi t}} \sum_{j=1}^{N_d} \frac{1}{(z_j^{(d)})^2} \int_0^\infty e^{-\frac{r^2 x^2}{2t} + z_j^{(d)} x} dx \right. \\ \left. + \frac{(-1)^{d/2-1} \sqrt{2}r^3}{\sqrt{\pi t}} \int_0^\infty \int_0^\infty \frac{e^{-xy}}{y^3 G^{(d)}(y)} e^{-\frac{r^2 x^2}{2t}} dx dy \right].$$

Large time asymptotics

For some explicitly written constants  $\alpha_i$ 's

$$L^{(d)}(t) = S_{d-1} r^{d-2} \left[ \frac{(d-2)t}{2} + \frac{r^2}{d-4} - \frac{\alpha_d r^{d-2}}{2^{d/2-2} (d-4) \Gamma(d/2-1)} \frac{1}{t^{d/2-2}} \right. \\ \left. + \frac{1}{t^{d/2-1}} \sum_{n=0}^{d-6} \frac{\alpha_n^{(d)}}{t^{n/2}} + \frac{\Gamma((d-3)/2) r^{2d-4}}{\sqrt{\pi} (d-2) \Gamma(d/2-1)^3} \frac{\log t}{t^{d-3}} + O\left(\frac{1}{t^{d-3}}\right) \right],$$

which is a generalization of a result by Le Gall (But note Le Gall considered a Wiener

$$L^{(d)}(t) = c_1^{(d)} t + c_2^{(d)} + c_3^{(d)} t^{2-d/2} + O(t^{1-d/2}) \quad (d: \text{ odd } (\geq 5)). \quad \text{sausage with}$$

$$L^{(4)}(t) = c'_1 t + c'_2 \log t + c'_3 + c'_4 \frac{\log t}{t} + o\left(\frac{\log t}{t}\right) \quad (d=4) \quad \text{a general compact set}$$

## 8 The zeros of $K_\nu$ are the roots of some algebraic equations

Consider the asymptotic expansion as  $z \rightarrow \infty$  for

$$\frac{K_{\nu+1}(z)}{K_\nu(z)} = 1 + \frac{2\nu}{z} + \sum_{j=1}^{N(\nu)} \frac{1}{z_{\nu,j} - z} + \cos(\pi\nu) \int_0^\infty \frac{dx}{x(x+z)G_\nu(x)}.$$

$$\text{Then, } \sum_{j=1}^{N(\nu)} \frac{1}{z_{\nu,j} - z} = \sum_{j=1}^{N(\nu)} \frac{-1}{z(1 - z_{\nu,j}/z)} = - \sum_{n=0}^{\infty} \left( \sum_{j=1}^{N(\nu)} (z_{\nu,j})^n \right) \frac{1}{z^{n+1}},$$

and it is well known that for  $K_\nu(z), K_{\nu+1}(z)$  as  $z \rightarrow \infty$ ,

$$K_\nu(z) = \sqrt{\frac{\pi}{2z}} e^{-z} \sum_{n=0}^{M+1} \frac{\Gamma(\nu + n + 1/2)}{n! \Gamma(\nu - n + 1/2)} \frac{1}{(2z)^n} + O\left(\frac{1}{z^{M+2}}\right).$$

$$\text{Hence we obtain } \boxed{\sum_{j=1}^{N(\nu)} (z_{\nu,j})^n = -a_{n+1} + (-1)^n \cos(\pi\nu) \int_0^\infty \frac{x^{n-1}}{G_\nu(x)} dx} \quad (n = 1, 2, \dots),$$

where  $a_n$ 's are determined recursively (e.g.,  $a_0 = 1$ ,  $a_1 = \nu + \frac{1}{2}$ ,  $a_2 = \frac{1}{2}(\nu^2 - \frac{1}{4})$ ,

$$a_3 = -\frac{1}{2}(\nu^2 - \frac{1}{4}), \quad a_4 = -\frac{1}{8}(\nu^2 - \frac{1}{4})(\nu^2 - \frac{25}{4}), \quad a_5 = \frac{1}{2}(\nu^2 - \frac{1}{4})(\nu^2 - \frac{13}{4}),$$

$$G_\nu(x) = K_\nu(x)^2 + \pi^2 I_\nu(x)^2 + 2\pi \sin(\pi\nu) K_\nu(x) I_\nu(x).$$

*Mathematica* easily solves this algebraic equation numerically.

Asymptotics as  $z \downarrow 0$  of

$$\frac{K_{\nu+1}(z)}{K_{\nu}(z)} = 1 + \frac{2\nu}{z} + \sum_{j=1}^{N(\nu)} \frac{1}{z_{\nu,j} - z} + \cos(\pi\nu) \int_0^{\infty} \frac{dx}{x(x+z)G_{\nu}(x)},$$

gives us the algebraic equations for the reciprocals  $z_{\nu,j}^{-1}$ . (Good check !)

**Example (2 Zeros of  $K_2$ )** When  $\nu = 2$ , we have

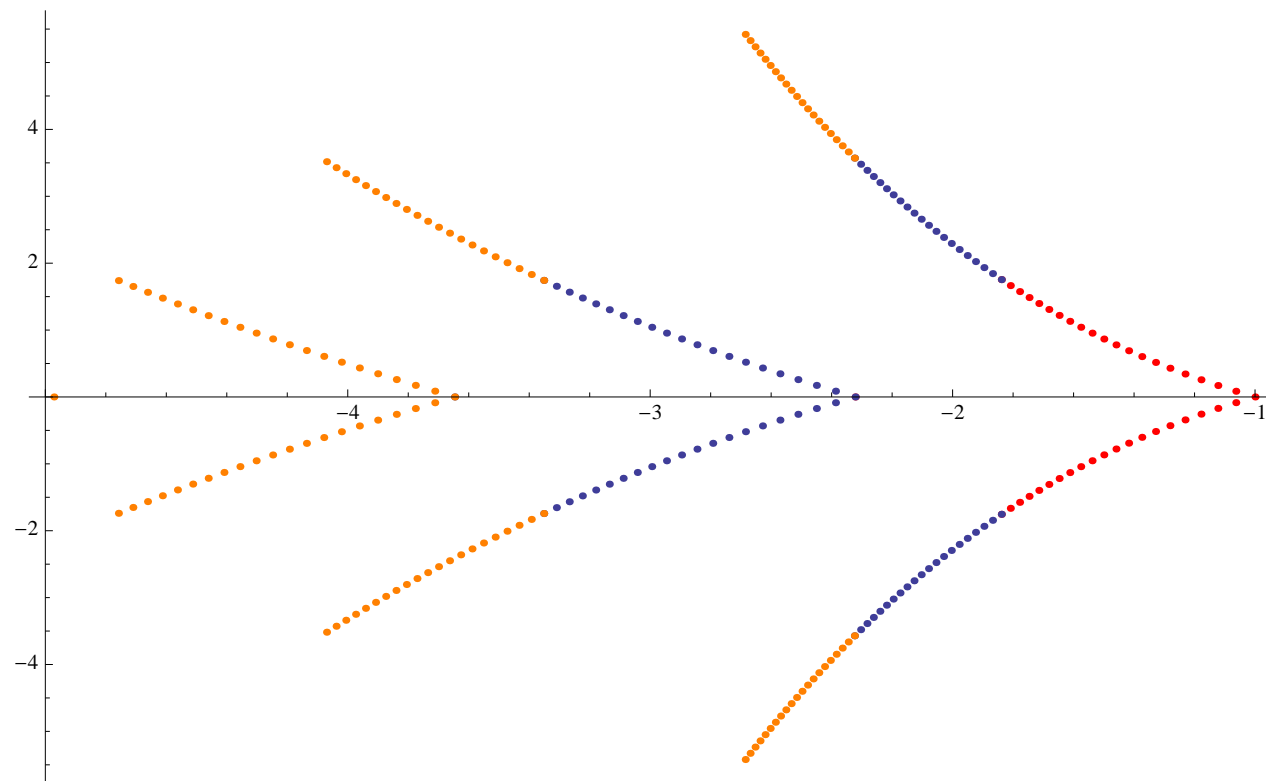
$$\begin{cases} z_{2,1} + z_{2,2} = -\frac{15}{8} - \int_0^{\infty} \frac{dy}{G_2(y)}, & \begin{cases} \frac{1}{z_{2,1}} + \frac{1}{z_{2,2}} = -1 - \int_0^{\infty} \frac{dy}{y^2 G_2(y)}, \\ \frac{1}{z_{2,1}^2} + \frac{1}{z_{2,2}^2} = \frac{1}{2} + \int_0^{\infty} \frac{dy}{y^3 G_2(y)}, \end{cases} \\ z_{2,1}^2 + z_{2,2}^2 = \frac{15}{8} + \int_0^{\infty} \frac{y dy}{G_2(y)}, \end{cases}$$

where  $G_2(x) = K_2(x)^2 + \pi^2 I_2(x)^2$ .

By using Mathematica we obtain from each system of equations that the zeros are

approximately  $\boxed{-1.28137 \pm 0.429485 i}$ , and check the comment in Watson's book

"The two zeros of  $K_2(z)$  are not very far from the points  $\boxed{-1.29 \pm 0.44 i}$ ."



☒1  $K_\nu$  has 2 zeros when  $\frac{3}{2} < \nu < \frac{7}{2}$ , 4 zeros when  $\frac{7}{2} < \nu < \frac{11}{2}$ , 6 zeros when  $\frac{11}{2} < \nu < \frac{15}{2} \dots$

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