# SDEs driven by stable processes


Leonid Mytnik (Technion)
Joint work with J. Berestycki, L. Döring, L. Zambotti


7th International Conference on Lévy Processes
Wrocław

July 18, 2013

# SDE driven by stable noise

$$X_t = X_0 + \int_0^t \sigma(X_s)L(ds) + \int_0^t g(X_s)\,ds,$$

where $\sigma \geq 0$, is Hölder continuous with exponent $\beta \in (0,1)$, $g \geq 0$,

$L$ is spectrally positive $\alpha$-stable, $\alpha \in (1,2)$:

$$Ee^{-\lambda L_t} = e^{-\lambda^\alpha t}, \lambda > 0, t \geq 0.$$

Uniqueness of non-negative solutions? Hitting zero?

# Pathwise uniqueness for SDEs driven by Brownian motion

$$X_t = X_0 + \int_0^t \sigma(X_s) dB_s$$

$B_t$ is a one-dimensional Brownian motion.

Theorem (Yamada, Watanabe (71))

If $\sigma$ is Hölder continuous with exponent $1/2$, then PU holds.

# Pathwise uniqueness for SDEs driven by Brownian motion

$$X_t = X_0 + \int_0^t \sigma(X_s) dB_s$$

$B_t$ is a one-dimensional Brownian motion.

**Theorem (Yamada, Watanabe (71))**

If $\sigma$ is Hölder continuous with exponent $1/2$, then PU holds.

**Remark**

There are counter examples for $\sigma$ which is Hölder continuous with exponent less than $1/2$.

## SDE driven by stable noise. Previous Results

$$X_t = X_0 + \int_0^t \sigma(X_{s-}) L(ds),$$

- $L$ — symmetric $\alpha$-stable noise, $\alpha \in (1,2)$. Pathwise uniqueness **(PU)** holds for $\sigma$ Hölder$(1/\alpha)$ (Komatsu(82), Bass(02)). The result is sharp.

# SDE driven by stable noise. Previous Results

$$X_t = X_0 + \int_0^t \sigma(X_{s-})L(ds),$$

- $L$ — symmetric $\alpha$-stable noise, $\alpha \in (1,2)$.
  Pathwise uniqueness (**PU)** holds for $\sigma$ Hölder($1/\alpha$) (Komatsu(82), Bass(02)). The result is sharp.
- $L$ — spectrally positive $\alpha$-stable, $\alpha \in (1,2)$.
  **PU** holds for $\sigma$ non-decreasing, Hölder($1-1/\alpha$) (Li, M. (11)) Improved by Li, Pu (12), Fournier (13).

# SDE driven by stable noise. Previous Results

$$X_t = X_0 + \int_0^t \sigma(X_{s-}) L(ds),$$

- $L$ — symmetric $\alpha$-stable noise, $\alpha \in (1,2)$.
  Pathwise uniqueness **(PU)** holds for $\sigma$ Hölder$(1/\alpha)$ (Komatsu(82), Bass(02)). The result is sharp.
- $L$ — spectrally positive $\alpha$-stable, $\alpha \in (1,2)$.
  **PU** holds for $\sigma$ non-decreasing, Hölder$(1-1/\alpha)$ (Li, M. (11))
  Improved by Li, Pu (12), Fournier (13).
- $L = a_1 L^1 - a_2 L^2$; $L^i$ — spectrally positive $\alpha$-stable; $a_i \geq 0$ :
  $\exists \gamma = \gamma(\alpha, a_1, a_2) \in [1-1/\alpha, 1/\alpha]$, s.t.
  **PU** holds for $\sigma$ Hölder$(\gamma)$ and $(a_1 - a_2)\sigma$ non-decreasing (Fournier(13))

# SDE driven by stable noise. Add drift

$$X_t = X_0 + \int_0^t \sigma(X_{s-})L(ds) + \int_0^t g(X_{s-})ds.$$

- If $\sigma$ non-decreasing, Hölder$(1-1/\alpha)$, $g$ — Lip, then **PU** holds.

# SDE driven by stable noise. Add drift

$$X_t \quad = \quad X_0 + \int_0^t \sigma(X_{s-})L(ds) + \int_0^t g(X_{s-})ds.$$

- If $\sigma$ non-decreasing, Hölder$(1 - 1/\alpha)$, $g$ — Lip, then **PU** holds.
- We will consider

$$X_t \quad = \quad X_0 + \int_0^t (X_{s-})^\beta L(ds) + \theta \int_0^t (X_{s-})^\eta ds,$$

$L$ — spectrally positive $\alpha$-stable, $\alpha \in (1, 2)$.
$X_0 \geq 0$, $\theta \geq 0$. **PU**?

# SDE driven by stable noise. Add drift

$$X_t \quad = \quad X_0 + \int_0^t \sigma(X_{s-})L(ds) + \int_0^t g(X_{s-})ds.$$

- If $\sigma$ non-decreasing, Hölder$(1-1/\alpha)$, $g$ — Lip, then **PU** holds.
- We will consider

$$X_t \quad = \quad X_0 + \int_0^t (X_{s-})^\beta L(ds) + \theta \int_0^t (X_{s-})^\eta ds,$$

  $L$ — spectrally positive $\alpha$-stable, $\alpha \in (1,2)$.
  $X_0 \geq 0$, $\theta \geq 0$. **PU**?
- Clearly for $\beta \geq 1 - 1/\alpha, \theta = 0$: **PU** holds.

# SDE driven by stable noise. Add drift

$$X_t \;=\; X_0 + \int_0^t \sigma(X_{s-})L(ds) + \int_0^t g(X_{s-})ds.$$

- If $\sigma$ non-decreasing, Hölder$(1-1/\alpha)$, $g$ — Lip, then **PU** holds.
- We will consider

$$X_t \;=\; X_0 + \int_0^t (X_{s-})^\beta L(ds) + \theta \int_0^t (X_{s-})^\eta ds,$$

  $L$ — spectrally positive $\alpha$-stable, $\alpha \in (1,2)$.
  $X_0 \geq 0$, $\theta \geq 0$. **PU**?
- Clearly for $\beta \geq 1 - 1/\alpha, \theta = 0$: **PU** holds.
- $\beta \geq 1 - 1/\alpha; \eta = 0 \text{ or } \eta = 1$: **PU** holds.

# SDE driven by stable noise

$$X_t = X_0 + \int_0^t (X_{s-})^\beta L(ds) + \theta \int_0^t (X_{s-})^\eta ds.$$

From now on:

$$\beta \in [1 - 1/\alpha, 1).$$

For $\eta \in (0,1)$, **PU** holds until

$$T_0 = \inf\{t \geq 0 : X_t = 0\}.$$

- $T_0 < \infty$?

# SDE driven by stable noise

$$X_t = X_0 + \int_0^t (X_{s-})^\beta L(ds) + \theta \int_0^t (X_{s-})^\eta ds.$$

From now on:

$$\beta \in [1 - 1/\alpha, 1).$$

For $\eta \in (0,1)$, **PU** holds until

$$T_0 = \inf\{t \geq 0 : X_t = 0\}.$$

- $T_0 < \infty$?
- If $T_0 < \infty$, does **PU** hold?

# SDE driven by stable noise

- $$X_t \quad = \quad X_0 + \int_0^t (X_{s-})^\beta L(ds) + \theta \int_0^t (X_{s-})^\eta ds.$$

  Let $X_0 > 0$, $\beta \in [1 - 1/\alpha, 1), \eta = 1 - \alpha(1 - \beta)$.

**Theorem 1**

*$T_0 < \infty$, a.s. iff $0 \le \theta < \Gamma(\alpha)$.*
*$T_0 = \infty$, a.s. iff $\theta \ge \Gamma(\alpha)$.*

# SDE driven by stable noise

- $$X_t = X_0 + \int_0^t (X_{s-})^\beta L(ds) + \theta \int_0^t (X_{s-})^\eta ds.$$

  Let $X_0 > 0$, $\beta \in [1 - 1/\alpha, 1), \eta = 1 - \alpha(1-\beta)$.

  **Theorem 1**

  *$T_0 < \infty$, a.s. iff $0 \le \theta < \Gamma(\alpha)$.*
  *$T_0 = \infty$, a.s. iff $\theta \ge \Gamma(\alpha)$.*

- **Theorem 2**

  (i) $\theta \ge \Gamma(\alpha)$. $\exists!$ *strong non-negative solution that never hits zero.*

  (ii) $\theta \le \frac{\Gamma(\alpha\beta)}{\Gamma(\eta)}$. $\exists!$ *strong non-negative solution. Trapped at zero.*

  (iii) $\frac{\Gamma(\alpha\beta)}{\Gamma(\eta)} < \theta < \Gamma(\alpha)$. $\exists!$ *strong solution in $\mathcal{S}$*
  *($\mathcal{S}$: non-negative that spend zero time at zero.)*

- $$X_t \quad = \quad X_0 + \int_0^t (X_{s-})^{\beta} L(ds) + \theta \int_0^t (X_{s-})^{\eta} ds.$$

  Let $X_0 > 0$, $\beta \in [1 - 1/\alpha, 1), \eta = 1 - \alpha(1-\beta)$.

  Theorem 1

  $T_0 < \infty$, *a.s. iff* $0 \le \theta < \Gamma(\alpha)$.

  $T_0 = \infty$, *a.s. iff* $\theta \ge \Gamma(\alpha)$.

- Theorem 2

  (i) $\theta \ge \Gamma(\alpha)$. $\exists!$ *strong non-negative solution that never hits zero.*

  (ii) $\theta \le \frac{\Gamma(\alpha\beta)}{\Gamma(\eta)}$. $\exists!$ *strong non-negative solution. Trapped at zero.*

  (iii) $\frac{\Gamma(\alpha\beta)}{\Gamma(\eta)} < \theta < \Gamma(\alpha)$. $\exists!$ *strong solution in* $\mathcal{S}$

  *(*$\mathcal{S}$*: non-negative that spend zero time at zero.)*

- $\theta > 0$. For $\eta < 1 - \alpha(1-\beta) \Rightarrow T_0 = \infty \Rightarrow \exists!$ strong solution

# SDE driven by stable noise

- $$X_t \quad = \quad X_0 + \int_0^t (X_{s-})^\beta L(ds) + \theta \int_0^t (X_{s-})^\eta ds.$$

  Let $\quad X_0 > 0,\ \beta \in [1 - 1/\alpha, 1), \eta = 1 - \alpha(1-\beta).$

  **Theorem 1**

  $T_0 < \infty$, *a.s. iff* $0 \le \theta < \Gamma(\alpha)$.
  $T_0 = \infty$, *a.s. iff* $\theta \ge \Gamma(\alpha)$.

- **Theorem 2**

  (i) $\theta \ge \Gamma(\alpha)$. $\exists!$ *strong non-negative solution that never hits zero.*
  (ii) $\theta \le \frac{\Gamma(\alpha\beta)}{\Gamma(\eta)}$. $\exists!$ *strong non-negative solution. Trapped at zero.*
  (iii) $\frac{\Gamma(\alpha\beta)}{\Gamma(\eta)} < \theta < \Gamma(\alpha)$. $\exists!$ *strong solution in* $\mathcal{S}$
  *($\mathcal{S}$: non-negative that spend zero time at zero.)*

  - $\theta > 0$. For $\eta < 1 - \alpha(1-\beta) \Rightarrow T_0 = \infty \Rightarrow \exists!$ strong solution
  - For $\eta > 1 - \alpha(1-\beta)$, $\exists!$ strong solution.

# SDE driven by stable noise

$$X_t = X_0 + \int_0^t (X_{s-})^\beta L(ds) + \theta \int_0^t (X_{s-})^\eta ds.$$
$$\beta \geq 1 - 1/\alpha, \eta = 1 - \alpha(1-\beta).$$

**Remarks**

- $\alpha = 2$ ($L$ is Brownian motion) $\Rightarrow \eta = 2\beta - 1$.
  - $T_0 < \infty$, a.s. iff $0 \leq \theta < 1$.
    $T_0 = \infty$, a.s. iff $\theta \geq 1$.
  - (i) $\theta > 1$. $\exists!$ strong non-negative solution that never hits zero.
    (ii) $\theta \leq 2\beta - 1$. $\exists!$ strong non-negative solution. Trapped at zero.
    (iii) $2\beta - 1 < \theta < 1$. $\exists!$ strong solution in $\mathcal{S}$.
    (Cherny, Engelbert (05)).

# SDE driven by stable noise

$$X_t = X_0 + \int_0^t (X_{s-})^\beta L(ds) + \theta \int_0^t (X_{s-})^\eta ds.$$
$$\beta \geq 1 - 1/\alpha, \eta = 1 - \alpha(1-\beta).$$

**Remarks**

- $\alpha = 2$ ($L$ is Brownian motion) $\Rightarrow \eta = 2\beta - 1$.
  - $T_0 < \infty$, a.s. iff $0 \leq \theta < 1$.
    $T_0 = \infty$, a.s. iff $\theta \geq 1$.
  - (i) $\theta > 1$. $\exists!$ strong non-negative solution that never hits zero.
    (ii) $\theta \leq 2\beta - 1$. $\exists!$ strong non-negative solution. Trapped at zero.
    (iii) $2\beta - 1 < \theta < 1$. $\exists!$ strong solution in $\mathcal{S}$.
    (Cherny, Engelbert (05)).
- $\beta = 1/\alpha \Rightarrow \eta = 2 - \alpha$.
  If $\theta = 0$ then $X$ is continuous state branching process (CSBP).
  If $\alpha = 2$, then $X$ is continuous CSBP with immigration $\theta dt$.
  $2X_t$ is a squared Bessel process of dimension$=2\theta$.
  1st dichotomy is well known. **PU** for all $\theta \geq 0$.

# Self-similarity

$$X_t = X_0 + \int_0^t (X_{s-})^\beta L(ds) + \theta \int_0^t (X_{s-})^\eta ds. \qquad (1)$$

$$\beta \geq 1 - 1/\alpha, \eta = 1 - \alpha(1-\beta).$$

For $x > 0$, let $P^x$ be the law of $X_t$ absorbed at 0 with $X_0 = x$.

### Lemma 3

*Let $X_0 = x > 0$. The SDE (1) admits a unique non-negative self-similar solution of index $1/(1-\eta) \geq 1$ absorbed at zero. That is*

$$Law((cX_{c^{-(1-\eta)}t})_{t\geq 0}) = P^{cx}.$$

# Proof of Theorem 1

Lamperti transformation: Let $X_0 = x > 0$. There exists a Lévy process $\xi$ such that

$$(X_{t\wedge T_0})_{t\geq 0} \stackrel{d}{=} \left(x \exp\left(\xi_{\tau(tx^{-(1-\eta)})}\right)\right)_{t\geq 0},$$

where

$$\tau(t) := \inf\{s \geq 0 : A_s > t\} \qquad \text{and} \qquad A_t := \int_0^t \exp\left((1-\eta)\xi_s\right)\, ds.$$

Hence

$$T_0 < \infty \qquad \Longleftrightarrow \qquad \xi \text{ drifts to } -\infty \tag{2}$$

Easy to check, for $\lambda \in [0,1)$,

$$E[\exp(\lambda\xi_1)] = \exp\left(\lambda\left(\theta - \frac{\Gamma(\alpha-\lambda)}{\Gamma(1-\lambda)}\right)\right) \tag{3}$$

$$\Rightarrow \xi \text{ drifts to } -\infty \text{ iff } \theta < \Gamma(\alpha).$$

# Proof of Theorem 2

$$X_t = X_0 + \int_0^t (X_{s-})^\beta L(ds) + \theta \int_0^t (X_{s-})^\eta ds.$$

Representation of $L$:

$$L(ds) = \int_{z=0}^\infty z(\mathcal{N} - \mathcal{N}')(ds, dz),$$

where $\mathcal{N}$ is a PPP on $(0, \infty) \times (0, \infty)$ with intensity measure $\mathcal{N}'(ds, dx) = ds \otimes c_\alpha x^{-1-\alpha}$.

$$X_t = X_0 + \int_0^t \int_{z=0}^\infty (X_{s-})^\beta z(\mathcal{N} - \mathcal{N}')(ds, dz) + \theta \int_0^t (X_{s-})^\eta ds.$$

The proof is based on the simple power transformation $x \mapsto x^{1-\eta}$.

**Lemma 4**
*$V = X^{1-\eta}$ is a solution to*

$$V_t = X_0^{1-\eta} + (1-\eta)\int_0^t \left(\theta - \frac{\Gamma(\alpha\beta)}{\Gamma(\eta)}\right) 1\{V_s \neq 0\}\, ds$$
$$+ \int_0^t \int_0^\infty \left(\left(V_{s-}^{\frac{1}{(1-\eta)}} + V_{s-}^{\frac{\beta}{1-\eta}} z\right)^{1-\eta} - V_{s-}\right)(\mathcal{N} - \mathcal{N}')(ds, dz).$$

**Proof** Itô's formula. ■

If $\theta \leq \frac{\Gamma(\alpha\beta)}{\Gamma(\eta)}$, $V$ is non-neg. supermartingale: trap. at zero $\Rightarrow$ uniqueness.
If $\theta > \frac{\Gamma(\alpha\beta)}{\Gamma(\eta)}$ and $X$ spends zero time at 0, then $V = X^{1-\eta}$ solves

$$V_t = X_0^{1-\eta} + (1-\eta)\left(\theta - \frac{\Gamma(\alpha\beta)}{\Gamma(\eta)}\right) t$$
$$+ \int_0^t \int_0^\infty \left(\left(V_{s-}^{\frac{1}{(1-\eta)}} + V_{s-}^{\frac{\beta}{1-\eta}} z\right)^{1-\eta} - V_{s-}\right)(\mathcal{N} - \mathcal{N}')(ds, dz).$$

# Proof of Theorem 2

**Lemma 5**

*If* $\theta \geq \frac{\Gamma(\alpha\beta)}{\Gamma(\eta)}$ *and* $V_0 \geq 0$, *then* $\exists!$ *non-negative strong solution* $V$ *to*

$$V_t = V_0 + (1-\eta)\left(\theta - \frac{\Gamma(\alpha\beta)}{\Gamma(\eta)}\right)t + \int_0^t \int_0^\infty \left(\left(V_{s-}^{\frac{1}{1-\eta}} + V_{s-}^{\frac{\beta}{1-\eta}}x\right)^{1-\eta} - V_{s-}\right)(\mathcal{N} - \mathcal{N}')(ds, dx).$$

*If* $\theta > \frac{\Gamma(\alpha\beta)}{\Gamma(\eta)}$, *then* $V \in \mathcal{S}$. *Moreover* $V^{1/(1-\eta)}$ *solves SDE for* $X$.

# Proof of Theorem 2

**Lemma 5**

*If* $\theta \geq \frac{\Gamma(\alpha\beta)}{\Gamma(\eta)}$ *and* $V_0 \geq 0$, *then* $\exists!$ *non-negative strong solution* $V$ *to*

$$V_t = V_0 + (1-\eta)\left(\theta - \frac{\Gamma(\alpha\beta)}{\Gamma(\eta)}\right)t + \int_0^t \int_0^\infty \left(\left(V_{s-}^{\frac{1}{1-\eta}} + V_{s-}^{\frac{\beta}{1-\eta}} x\right)^{1-\eta} - V_{s-}\right)(\mathcal{N} - \mathcal{N}')(ds, dx).$$

*If* $\theta > \frac{\Gamma(\alpha\beta)}{\Gamma(\eta)}$, *then* $V \in \mathcal{S}$. *Moreover* $V^{1/(1-\eta)}$ *solves SDE for* $X$.

**Proof** $g(v,x) \equiv \left(v^{\frac{1}{1-\eta}} + v^{\frac{\beta}{1-\eta}} x\right)^{1-\eta} - v$. One can show

$$|g(v,x) - g(u,x)| \leq cx|u-v|^{1-1/\alpha}.$$

Then the proof of **PU** is an adaptation of Yamada-Watanabe argument used in Li, M. (11).

Weak existence is easy to check.
**PU** + weak existence $\Rightarrow$ $\exists!$ strong solution.

$\blacksquare$

By Lemmas 4,5 we finish the proof of Theorem 2.

$$X_t = X_0 + \int_0^t (X_{s-})^\beta L(ds) + \theta \int_0^t (X_{s-})^\eta ds. \qquad (1)$$

Several corollaries of the main results.

**Lemma 6**

*Let $\theta > \frac{\Gamma(\alpha\beta)}{\Gamma(\eta)}$, $X_0 = x \geq 0$. Then the solution $X \in \mathcal{S}$ is self-similar.*

**Remark** Before we knew it just for the solution absorbed at zero. Existence of recurrent non-negative Markovian extension after time $T_0$ for self-similar processes has been studied in the literature (Rivero (05,07), Fitzsimmons (06)). Here we have it for free.

**Lemma 7**

*Let $\Gamma(\alpha) > \theta > \frac{\Gamma(\alpha\beta)}{\Gamma(\eta)}$. Let $(P^x)_{x>0}$ be the laws of solutions to (1). Then there exists $(\bar{P}^x)_{x\geq 0}$ — the unique extension of $(P^x)_{x>0}$ that leaves zero continuously.*

**Proof:** $\bar{P}^0$ describes the unique solution of (1) starting at 0.

$\bar{P}^0$ can be also defined for the case $\theta \geq \Gamma(\alpha)$. In this case we have:

### Lemma 8

*Let $\beta \in [1 - 1/\alpha, 1)$ and $\theta \geq \Gamma(\alpha)$. Then $(\bar{P}^x)_{x \geq 0}$ is weakly continuous in the initial condition.*

# Open Problems

$$X_t = X_0 + \int_0^t (X_{s-})^\beta L(ds) + \theta \int_0^t (X_{s-})^\eta ds.$$

- $\beta < 1 - 1/\alpha$. Conditions for **PU**.

# Open Problems

$$X_t = X_0 + \int_0^t (X_{s-})^\beta L(ds) + \theta \int_0^t (X_{s-})^\eta ds.$$

- $\beta < 1 - 1/\alpha$. Conditions for **PU**.
- Allow more general coefficients:

$$X_t = X_0 + \int_0^t \sigma(X_{s-}) L(ds) + \theta \int_0^t g(X_{s-}) ds.$$

Conditions on $\sigma$, $g$ for **PU**.

# Open Problems

$$X_t = X_0 + \int_0^t (X_{s-})^\beta L(ds) + \theta \int_0^t (X_{s-})^\eta ds.$$

- $\beta < 1 - 1/\alpha$. Conditions for **PU**.
- Allow more general coefficients:

$$X_t = X_0 + \int_0^t \sigma(X_{s-}) L(ds) + \theta \int_0^t g(X_{s-}) ds.$$

  Conditions on $\sigma$, $g$ for **PU**.
- Signed solutions?

Thank You