Analysis on the Wiener-Poisson space and its application to the asymptotic expansion

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1 Analysis on the Wiener-Poisson space

1.1 SDE on the Wiener-Poisson space

Let $z(t)$ be a Lévy process, $\mathbf{R}^m$-valued, with Lévy measure $\mu(dz)$ such that the characteristic function ψ_t is given by

$$\psi_t(\xi) = E[e^{i(\xi,z(t))}] = \exp(t\int(e^{i(\xi,z)} - 1 - i(\xi,z)\frac{1}{1+|z|^2})\mu(dz)).$$

We may write

$$z(t) = (z_1(t),...,z_m(t)) = \int_0^t \int_{\mathbf{R}^m\setminus\{0\}} z\ \{N(dsdz) - \frac{1}{1+|z|^2}.\mu(dz)ds\},$$

where $N(dsdz)$ is a Poisson random measure on $\mathrm{T} \times (\mathbf{R}^m \setminus \{0\})$ with mean $ds \times \mu(dz)$. We define a jump-diffusion process X_t by an SDE

$$X_t = x + \int_0^t b(X_{s-})ds + \int_0^t \sigma(X_{s-})dW(s) + \int_0^t \int_{\mathbf{R}^m\setminus\{0\}} g(X_{s-},z)\tilde{N}(dsdz). \qquad (*)$$

Here, $b(x) = (b^i(x))$ is a continuous functions on $\mathbf{R}^d$, Lipschits continuous and $\mathbf{R}^d$ valued, $\sigma(x) = (\sigma^{ij}(x))$ is a continuous $d \times m$ matrix on $\mathbf{R}^d$, Lipschits continuous, and $g(x,z)$ is a continuous functions on $\mathbf{R}^d \times \mathbf{R}^m$ and $\mathbf{R}^d$ valued, We assume

$$|b(x)| \le K(1+|x|),\ |\sigma(x)| \le K(1+|x|),\ |g(x,z)| \le K(z)(1+|x|),$$

and

$$|b(x)-b(y)| \le L|x-y|,\ |\sigma(x)-\sigma(y)| \le L|x-y|,$$
$$|g(x,z)-g(y,z)| \le L(z)|x-y|.$$

Here K, L are positive constants, and $K(z), L(z)$ are positive functions satisfying

$$\int_{\mathbf{R}^m\setminus\{0\}} \{K^p(z) + L^p(z)\}\mu(dz) < +\infty,$$

where $p \ge 2$.

Under the above assumptions the SDE has a solution which is càdlàg, $\mathcal{F}_t$-adapted, and $x \mapsto X_t(x)$ is continuous for $a.a.\ \omega$. In case the mapping $x \mapsto X_t(x)$ has a property of the flow of homeomorphisms (diffeomorphisms) property a.s., we write $X_{s,t} = X_t \circ X_s^{-1}$.

[1] Some parts of this talk are based on joint works with Dr. M. Hayashi and with Prof. H. Kunita.

We put

$$\phi_z : x \mapsto x + g(x,z).$$

It is called a *connactor*. A sufficient condition that $x \mapsto X_t(x)$ is a flow of diffeomorphisms a.s. is that the mapping ϕ_z is a diffeomorphis for any $z \in \text{supp}\ \mu$ (Fujiwara-Kunita [2] Theorems 2.3, 2.4). In what follows we assume ϕ_z is a diffeomorphism, and write $\xi_{s,t} = X_{s,t}(x)$. We also assume μ satisfies the order condition.

We make a perturbation of the trajectory $F = \xi_s, 0 < s < t$ by ε_u^+, which will be introduced in Sect. 1.2.

Let $\mathbf{u} = (u_1, ..., u_k)$. Then

$$\xi_{s,t}(x) \circ \varepsilon_{\mathbf{u}}^+ = \xi_{t_k,t} \circ \phi_{z_k} \circ \xi_{t_{k-1},t_k} \circ \phi_{z_{k-1}} \cdots \circ \phi_{z_1} \circ \xi_{s,t_1} \qquad (**)$$

Example of SDE (Linear SDE) X_t is given by

$$dX_t = V_0(X_{t-})dt + \sum_{j=1}^{m} V_j(X_{t-})dz_j(t). \qquad (1.1)$$

Here $V_0, V_1, ..., V_m$ are smooth vector fields on $\mathbf{R}^d$.

In what follows we assume $b(x)$ and $\sigma(x)$ are $g(x,z)$ functions having bounded derivatives of all orders, for simplicity.

1.2 Sobolev spaces on the Wiener-Poisson space

Let T be a positive number and let $\mathrm{T} = [0,T]$. Let Ω_1 be the set of all continuous maps $\omega_1 : \mathrm{T} \to \mathbf{R}^m$ such that $\omega_1(0) = 0$ and let $\mathcal{F}_1$ be the smallest σ-field of Ω_1 with respect to which $\{w_1(t), t \in \mathrm{T}\}$ are measurable. Let P_1 be a probability measure on $(\Omega_1, \mathcal{F}_1)$ such that $W(t) := \omega_1(t)$ is a standard m-dimensional Brownian motion.

Let Ω_2 be the set of all non-negative integer valued measures on $U = \mathrm{T} \times \mathbf{R}^m$ such that $\omega_2(\mathrm{T} \times \{0\}) = 0$ and let $\mathcal{F}_2$ be the smallest σ-field of Ω_2 with respect to which $\{w_2(E);\ E$ are Borel sets in $U\}$ are measurable. Let P_2 be a probability measure on $(\Omega_2, \mathcal{F}_2)$ such that $N(dtdz) := \omega_2(dtdz)$ is a Poisson random measure with intensity measure $\hat{N}(dtdz) := dt\mu(dz)$, where μ is a Lévy measure. In the following, we set $u = (t,z)$ and $\hat{N}(du) = \hat{N}(dtdz)$, $\tilde{N}(du) = N(du) - \hat{N}(du)$.

We shall introduce assumptions concerning the Lévy measure. Set

$$\varphi(\rho) = \int_{|z| \le \rho} |z|^2 \mu(dz). \qquad (1.2)$$

We say that the measure μ satisfies an *order condition* if there exists $0 < \alpha < 2$ such that

$$\liminf_{\rho \to 0} \frac{\varphi(\rho)}{\rho^\alpha} > 0. \qquad (1.3)$$

On $(\Omega_1, \mathcal{F}_1, P_1)$, the Malliavin-Shigekawa derivative D_t is defined.

Next, we shall introduce difference operators $\tilde{D}_u, u \in U$, acting on the Poisson space. For each $u = (t, z) = (t, z_1, ..., z_m) \in U$, we define a map $\varepsilon_u^- : \Omega_2 \to \Omega_2$ by $\varepsilon_u^- \omega_2(A) = \omega_2(A \cap \{u\}^c)$, and $\varepsilon_u^+ : \Omega_2 \to \Omega_2$ by $\varepsilon_u^+ \omega_2(A) = \omega_2(A \cap \{u\}^c) + 1_A(u)$. (These are extended to Ω by setting $\varepsilon_u^{\pm}(\omega_1, \omega_2) = (\omega_1, \varepsilon_u^{\pm}\omega_2)$) It holds $\varepsilon_u^- \omega = \omega$ a.s. P for any u since $\omega_2(\{u\}) = 0$ holds for almost all ω_2 for any u. The difference operators $\tilde{D}_u$ for a $\mathcal{F}_2$-measurable random variable X is defined after Picard (Carlen-Pardoux, Nualart-Vives) by

$$\tilde{D}_u X = X \circ \varepsilon_u^+ - X. \tag{1.4}$$

Let $\mathbf{u} = (u_1, ..., u_k) = ((t_1, z^1), ..., (t_k, z^k)) = (\mathbf{t}, \mathbf{z})$. We set $|\mathbf{u}| = |\mathbf{z}| = \max_{1 \le i \le k} |z^i|$ and $\gamma(\mathbf{u}) = \gamma(\mathbf{z}) = |z^1| \cdots |z^k|$. We define $\varepsilon_{\mathbf{u}}^+ = \varepsilon_{u_1}^+ \circ \cdots \circ \varepsilon_{u_k}^+$ and $\tilde{D}_{\mathbf{u}} = \tilde{D}_{\mathbf{u}}^k = \tilde{D}_{u_1} \cdots \tilde{D}_{u_k}$.

For $\mathbf{t} = (t_1, ..., t_k), \mathbf{u} = (u_1, ..., u_k)$, let

$$D_{(\mathbf{t}, \mathbf{u})} = D_{(t_k, u_k)} \cdots D_{(t_2, u_2)} D_{(t_1, u_1)}.$$

Let $\mathcal{P} = \mathcal{P}_1 \otimes \mathcal{P}_2$. Spaces $\mathcal{P}_1, \mathcal{P}_2$ are identifies with $\mathcal{P}_1 \otimes 1, 1 \otimes \mathcal{P}_2$ respectively. We put for $p \ge 2$

$$\mathbf{D}_{k,l,p} = \bar{\mathcal{P}}^{|\cdot|_{k,l,p}} = \{F; |F|_{k,l,p} < +\infty\},$$

where

$$|F|_{k,l,p} := \left(|F|_{0,l,p}^p + \sum_{k'=1}^{k} \sum_{l'=0}^{l} \mathbf{E}\left[\int_{\mathbf{A}(\rho)^{\mathbf{k'}}} \left(\int_{\mathbf{T}^{\mathbf{l'}}} \left| \frac{\mathbf{D}_{\mathbf{t}}^{\mathbf{l'}} \tilde{\mathbf{D}}_{\mathbf{u}}^{\mathbf{k'}} \mathbf{F}}{\gamma(\mathbf{u})} \right|^{\mathbf{2}} \mathbf{dt} \right)^{\mathbf{p/2}} \hat{\mathbf{M}}(\mathbf{du}) \right] \right)^{\mathbf{1/p}}. \tag{1.5}$$

Here $A(\rho) = \{(t, z); |z| \le \rho\}$, and $\hat{M}(\mathbf{du}) = \hat{M}(du_1) \cdots \hat{M}(du_k)$, $\hat{M}(du) = |z|^2 \hat{N}(du)$. The operator $D_{(t,u)}$ is extended continuously to $\mathbf{D}_{k,l,p}$. Let

$$\mathbf{D}_\infty = \cap_{k,l=0}^{\infty} \cap_{p \ge 2} \mathbf{D}_{k,l,p}.$$

We denote by $|\Phi|'_{k,l,p}$ the dual norm of $|.|_{k,l,p}$ for $\Phi \in \mathbf{D}'_\infty$. That is, $|\Phi|'_{k,l,p} = \sup_{|G|_{\mathbf{D}_\infty} = 1} |\langle \Phi, G \rangle|$.

We denote by $\mathcal{S}$ the space of rapidly decreasing distributions, and by $\mathcal{S}'$ its dual space (the space of tempered distributions).

A basic idea to state the existence of the smooth density is as follows. We shall define the composition $T \circ F$ of F with $T \in \mathcal{S}'$, by putting

$$\langle T \circ F, G \rangle = {}_{\mathcal{S}'}\langle \mathcal{F}T, E[G e_\xi(F)] \rangle_{\mathcal{S}}, G \in \mathbf{D}_\infty. \tag{1.6}$$

Here $e_\xi(x) = e^{i(x,\xi)}$, and $\mathcal{F}T$ denotes the Fourier transform of T in $\mathcal{S}'$.

Among smooth functionals, we are particularly interested in *nondegerate functionals* in the sense of Malliavin-Picard (functionals satisfying the Condition **(ND)** stated below). An important property of a nondegenerate functional F is that for any $n \in \mathbf{N}$ there exist $k, l \in \mathbf{N}, p > 2$ and $C > 0$ such that for any $\xi \in \mathbf{R}^d$

$$|E[e^{i\xi \cdot F} G]| \le C(1 + |\xi|^2)^{-q_0 n/2} |G|_{k,l,p}, \tag{1.7}$$

where $0 < q_0 < 1 - \frac{\alpha}{2}$ is a constant independent of n. Putting $G = 1$, this inequality shows that the characteristic function of F satisfies

$$E[e^{i\xi\cdot F}] = O((1 + |\xi|^2)^{-q_0 n/2}) \tag{1.8}$$

as $|\xi| \to \infty$ for any n. Here we used the notation $e^{i\xi\cdot F} = e_\xi(F) = e_\xi \circ F$.

This implies that F has a C^∞-density function, since

$$\partial_x^l p_F(x) = (\frac{1}{2\pi})^d \int e^{-ix\cdot\xi}(-i\xi)^l E[e^{i\xi\cdot F}]d\xi, l = 0, 1, 2, ...$$

1.3 The nondegeneracy condition

We say that F satisfies the **(ND)** condition if for all $p \geq 1, k \geq 0$ there exists $\beta \in (\frac{\alpha}{2}, 1]$ such that

$$\sup_{\rho\in(0,1)} \sup_{\substack{v\in\mathbf{R}^d, \\ |v|=1}} \sup_{\tau\in A^k(\rho)} E[\left|\left((v, \Sigma v) + \varphi(\rho)^{-1}\int_{A(1)} |(v, \tilde{D}_u F)|^2 1_{\{|\tilde{D}_u F| \leq \rho^\beta\}} \hat{N}(du)\right)^{-1} \circ \epsilon_\tau^+\right|^p] < \infty,$$

where Σ is the Malliavin's covariance matrix $\Sigma = (\Sigma_{i,j})$, where $\Sigma_{i,j} = \int_T (D_t F_i, D_t F_j)dt$.

More precisely we can prove the following result.

Proposition 1 *Let $F \in \mathbf{D}_\infty$ satisfy the* **(ND)** *condition. For any m there exist k, l, $p > 2$ and $C_m > 0$ such that*

$$|\varphi \circ F|'_{k,l,p} \leq C_m(\sum_{\beta\leq m} |(1 + |F|^2)^\beta|_{k,l,p})\|\varphi\|_{-2m} \tag{1.9}$$

for $\varphi \in \mathcal{S}$.

Here $|F|_{k,l,p}$ is a 3-parameters Sobolev norm on the Wiener-Poisson space, and $|F|'_{k,l,p}$ is its dual norm given above. $\|\varphi\|_{-2m}$ is a norm introduced on $\mathcal{S}$ as follows.

Let $\mathcal{S}$ be the set of all rapidly decreasing C^∞-functions and let $\mathcal{S}'$ be the set of tempered distributions. For $\varphi \in \mathcal{S}$ we introduce a norm

$$\|\varphi\|_{2m} = (\int \sum_{|\alpha|+|\beta|\leq m} \{|(1 - \Delta)^\beta (1 + |y|^2)^\alpha \varphi|^2\}dy)^{\frac{1}{2}} \tag{1.10}$$

for $m = 1, 2, ...$ We let $\mathcal{S}_{2m}$ to be the completion of $\mathcal{S}$ with respect to this norm. We remark $\mathcal{S} \subset \mathcal{S}_{2m}, m = 1, 2, ...$ We introduce the dual norm $\|\psi\|_{-2m}$ of $\|\varphi\|_{2m}$ by

$$\|\psi\|_{-2m} = \sup_{\varphi\in\mathcal{S}_{2m}, \|\varphi\|_{2m}=1} |(\varphi, \psi)|, \tag{1.11}$$

where $(\varphi, \psi) = \int \varphi(x)\bar{\psi}(x)dx$. We denote by $\mathcal{S}_{-2m}$ the completion of $\mathcal{S}$ with respect to the norm $\|\psi\|_{-2m}$. Further, put

$$\mathcal{S}_\infty = \cap_{m\geq1}\mathcal{S}_{2m}, \quad \mathcal{S}_{-\infty} = \cup_{m\geq1}\mathcal{S}_{-2m}$$

By (1.9) we can extend φ to $T \in \mathcal{S}_{-2m}$. Since $\cup_{m\geq 1}\mathcal{S}_{-2m} = \mathcal{S}'$, we can define the composition $T \circ F$ for $T \in \mathcal{S}'$ as an element in $\mathbf{D}'_\infty$.

A sufficient condition for the composition using the Fourier method is the **(ND)** condition stated above.

2 Application to the asymptotic expansion

2.1 The (UND) condition

We state a sufficient condition for the asymptotic expansion so that $\Phi \circ F(\epsilon)$ can be expanded as

$$\Phi \circ F(\epsilon) \sim \sum_{m=0}^{\infty} \sum_{|\mathbf{n}|=\mathbf{m}} \frac{1}{\mathbf{n}!} (\partial^{\mathbf{n}} \Phi) \circ f_0 \cdot (F(\epsilon) - f_0)^{\mathbf{n}}$$

$$\sim \Phi_0 + \epsilon \Phi_1 + \epsilon^2 \Phi_2 + ... \ \text{ in } \ \mathbf{D}'_\infty$$

for $\Phi \in \mathcal{S}'$, under the assumption that

$$F(\epsilon) \sim \sum_{m=0}^{\infty} \epsilon^j f_j \sim f_0 + \epsilon f_1 + \epsilon^2 f_2 + ... \ \text{ in } \ \mathbf{D}_\infty.$$

Definition 1 *We say $F(\epsilon) = (F^1, \ldots, F^d)$ satisfies the* **(UND)** *condition (uniformly non-degenerate) if for any integer k it holds that*

$$\limsup_{\epsilon \to 0} \sup_{\rho \in (0,1)} \sup_{\substack{v \in \mathbf{R}^d \\ |v|=1}} \operatorname{ess\,sup}_{\tau \in A^k(\rho)} E[|((v, \Sigma(\epsilon) v)$$

$$+ \varphi(\rho)^{-1} \int_{A(1)} |(v, \tilde{D}_u F(\epsilon))|^2 1_{\{|\tilde{D}_u F(\epsilon)| \le \rho^\beta\}} \hat{N}(du))^{-1} \circ \epsilon_\tau^+ |^p] < +\infty,$$

where $\Sigma(\epsilon) = (\Sigma_{i,j}(\epsilon)), \Sigma_{i,j}(\epsilon) = \int_{\mathrm{T}} D_t F^i(\epsilon) D_t F^j(\epsilon)\, dt$.

Definition 2 *Let $k, l \ge 0, p \ge 2$, and $m \ge 1$ an integer. We write*

$$F(\epsilon) \sim O(\epsilon^m) \quad \text{in} \quad \mathbf{D}_{k,l,p}$$

if $F(\epsilon) \in \mathbf{D}_{k,l,p}$ for $\epsilon \in (0,1]$, and it holds

$$\limsup_{\epsilon \to 0} |\frac{F(\epsilon)}{\epsilon^m}|_{k,l,p} < \infty.$$

Definition 3 *(1). We say that $F(\epsilon)$ has the asymptotic expansion*

$$F(\epsilon) \sim \sum_{j=0}^{\infty} \epsilon^j f_j \quad \text{in} \quad \mathbf{D}_\infty,$$

if it hold that

(i) $F(\epsilon)$, $f_0, f_1, \ldots \in \mathbf{D}_\infty$ for any ϵ ;

(ii) For all $k, l \ge 0, p \ge 2$ and any non-negative integer m,

$$F(\epsilon) - \sum_{\nu=0}^{m} \epsilon^\nu f_\nu \sim O(\epsilon^{m+1}) \quad \text{in} \quad \mathbf{D}_{k,l,p}.$$

(2) We say that $\Phi(\epsilon) \in \mathbf{D}'_\infty$ has an asymptotic expansion

$$\Phi(\epsilon) \sim \sum_{j=0}^{\infty} \epsilon^j \Phi_\nu \quad in\ \mathbf{D}'_\infty$$

if for any $m \geq 0$ there exist $k = k(m) \geq 0, l = l(m) \geq 0$ and $p \geq 2$ such that for some $\Phi(\epsilon), \Phi_0, \Phi_1, \Phi_2, \ldots, \Phi_m \in (\mathbf{D}_{k,l,p})'$ it holds that

$$\limsup_{\epsilon \to 0} \frac{1}{\epsilon^{m+1}} |\Phi(\epsilon) - \sum_{\nu=0}^{m} \epsilon^j \Phi_\nu|_{k,l,p}{}' < +\infty.$$

Note that these are not *formal* expansions, but they are asymptotic expansions with respect to the norms $|.|_{k,l,p}$ and $|.|'_{k,l,p}$ respectively.

Proposition 2 *Suppose $F(\epsilon)$ satisfies the* **(UND)** *condition, and that $F(\epsilon) \sim \sum_{j=0}^{\infty} \epsilon^j f_j$ in $\mathbf{D}_\infty$. Then, for all $\Phi \in \mathcal{S}'$, we have $\Phi \circ F(\epsilon) \in \mathbf{D}'_\infty$ has an asymptotic expansion in $\mathbf{D}'_\infty$:*

$$\Phi \circ F(\epsilon) \sim \sum_{m=0}^{\infty} \sum_{|\mathbf{n}|=m} \frac{1}{\mathbf{n}!} (\partial^{\mathbf{n}} \Phi) \circ f_0 \cdot (F(\epsilon) - f_0)^{\mathbf{n}}$$

$$\sim \Phi_0 + \epsilon \Phi_1 + \epsilon^2 \Phi_2 + \ldots \quad in \quad \mathbf{D}'_\infty.$$

Here Φ_0, Φ_1, Φ_2 are given by the formal expansion

$$\Phi_0 = \Phi \circ f_0, \qquad \Phi_1 = \sum_{i=1}^{d} f_1^i (\partial_{x_i} \Phi) \circ f_0,$$

$$\Phi_2 = \sum_{i=1}^{d} f_2^i (\partial_{x_i} \Phi) \circ f_0 + \frac{1}{2} \sum_{i,j=1}^{d} f_1^i f_1^j (\partial^2_{x_i x_j} \Phi) \circ f_0,$$

$$\Phi_3 = \sum_{i=1}^{d} f_3^i (\partial_{x_i} \Phi) \circ f_0 + \frac{2}{2!} \sum_{i,j=1}^{d} f_1^i f_2^j (\partial^2_{x_i x_j} \Phi) \circ f_0 + \frac{1}{3!} \sum_{i,j,k=1}^{d} f_1^i f_1^j f_1^k (\partial^3_{x_i x_j x_k} \Phi) \circ f_0$$

$$\ldots$$

2.2 Neuron cell model

2.2.1 H-H (*Hodgkin-Huxley*) model

A H-H model is a biological model of an active nerve cell (neuron) which produces spikes (electric bursts) according to the input from other neurons.

The simplified H-H model, called *Fitzhugh-Nagumo* model, is described in terms of $(V(t), n(t))$ by

$$\frac{\partial V}{\partial t}(t, V(t)) = -g \cdot h(V(t) - E_{N_a}) - n^4(t, V(t) - E_K) + I(t),$$

$$\frac{\partial n}{\partial t}(t, V(t)) = \alpha(V(t))(1 - n(t)) - \beta(V(t))n(t). \tag{$*$}$$

Here $h(.)$ denotes "conductance" of the natrium (sodium) ion channel, g is a coefficient, $I(.)$ is an input current, and $n(t) = n(t, v(t))$ denotes the depolarization rate (permeability) of potasium ion chanel. Constants E_{Na}, E_K correspond to standstill electric potential due to natrium, potasium ions respectively, and $\alpha(.)$ and $\beta(.)$ are some functions describing the transition rate from closed potasium channel to open potasium channel (open potasium channel to closed potasium channel), respectively.

In case we study the chain of nerve cells, we have to take into consideration the effect of transmission of external signals and noise throgh synapse. To this end we introduce a jump-diffusion process $z(t)$ to model the signal and noise. A stochastic model $V(t, \epsilon)$ in this case is described by the SDE

$$dV(t, \epsilon) = -g \cdot h(V(t) - E_{N_a})dt - n^4(t, V(t) - E_K)dt + \sum_{t_i \le t} A_i 1_{\{t_i\}}(t) + I(t)dt + \epsilon dz(t),$$

$$dn(t) = \alpha(V(t))(1 - n(t))dt - \beta(V(t))n(t)dt. \tag{$**$}$$

Here (t_i) denotes the arrival times of external spikes, $\epsilon > 0$ is a parameter, and $dz(t)$ denotes a stochastic integral with respect to the noise process $z(t)$.

In this model we take $z(t)$ to be a jump-diffusion; the diffusion part corresponds to the continuous noise (i.e. white noise), and the jump part corresponds to the discontinuous noise. The reason for taking such kind of noise is that the transmission of information among nerve cells are due to chemical particles (synaptic vesicles) which may induce discontinuous random effects.

We construct the following model; the pair $(V(t), n(t))$ is denoted by $(X_t^\epsilon, Y_t^\epsilon)$.

$$X_t^\epsilon = x_0 + \int_0^t c_1(X_s^\epsilon, Y_s^\epsilon)ds + \sum_{t_i \le t} A_i 1_{\{t_i\}}(t) + \int_0^t I(s)ds + Z_t^\epsilon,$$

$$Y_t^\epsilon = y_0 + \int_0^t c_2(X_s^\epsilon, Y_s^\epsilon)ds. \tag{2.1}$$

Here

$$c_1(x, y) = k(1 - \sqrt{x^2 + y^2})x - y,$$

$$c_2(x, y) = k(1 - \sqrt{x^2 + y^2})y + x,$$

with $k > 0$, and $x_0^2 + y_0^2 < 1$, x_0, y_0 not depending on ϵ. The process Z_t^ϵ denotes a one-dimensional Lévy process depending on a small parameter $\epsilon > 0$, $A_i > 0$ denotes the amplitude of the shift at i-th spike, and (t_i) denotes the arrival times of the exterior spikes (deterministic).

In case $A_i \equiv 0, I(t) \equiv 0$ and $\epsilon = 0$, this equation denotes the deterministic motion in the unit disk, described by

$$\frac{dr}{dt} = kr(1-r), \quad \frac{d\phi}{dt} = 1$$

in the polar coordinate. Since $k > 0$, starting from any point inside the disk ($r \neq 0$), the particle moves to grow up to it holds that $r = 1$, and stay on the circle thereafter. This corresponds to a stationary state of the axial fiber cell.

On the other hand, the noise Z_t^ϵ corresponds to the effect due to the input from other neurons via synapses. We take Z_t^ϵ to be a jump-diffusion in this model; the diffusion part corresponds to the continuous noise (i.e. white noise), and the jump part corresponds to the discontinuous noise.

The reason for taking such kind of noise is that the transmission of information among nerve cells are due to chemical particles (synaptic vesicles) which may induce discontinuous random effects.

More precisely, we assume the noise process Z_t^ϵ satisfies the SDE

$$dZ_t^\epsilon = \epsilon(\sigma(t, Z_{t-}^\epsilon)dW(t) + dJ_t) = \epsilon\ \sigma(t, Z_{t-}^\epsilon)dW(t) + \epsilon\ dJ_t,$$

$$Z_0^\epsilon = z_0 = 0. \tag{2.2}$$

Here $W(t)$ is the Wiener process (standard Brownian motion) on $\mathrm{T} = [0, 1]$. J_t is a compound Poisson process

$$J_t = \sum_{i=1}^{N(t)} Y_i, \tag{2.3}$$

where Y_i are i.i.d. random variables obeying the normal law $N(0, 1)$ represented by $Y_j = \int_{\mathrm{T}} 1dW(t) = W(1)$, and $N(t)$ is a Poisson process with intensity $\lambda > 0$. We assume (Y_i), $W(t)$ and $N(t)$ are mutually independent.

More precisely, we can write

$$J_t = (\int_0^t \int zN(dsdz)) \circ Y. \tag{2.4}$$

Here $N(dsdz)$ denotes the Poisson random measure with the mean measure $\lambda ds \times \delta_{\{e\}}$ where $\lambda > 0$, and $Y = (Y_i)$ is a series of i.i.d. random variables given above.

The function $\sigma(t, x)$ in (2.2) is a function which is strictly positive, infinitely times continuously diffrentiable with bounded derivatives of all orders for all variables, and is assumed to satisfy

$$|\sigma(t, x)| \leq K(1 + |x|),\ |\sigma(t, x) - \sigma(t, y)| \leq L|x - y|,\ K > 0, L > 0 \tag{2.5}$$

for all x, t. By this assumption, $F(\epsilon) = Z_t^\epsilon$ satisfies the **(UND)** condition.

Under these assumptions the SDE (2.2) has a unique solution. We sometimes write $\sigma(t,x)$ as $\sigma_t(x)$ in what follows. We also write $\sigma_t = \sigma(t, z_0)$.

Let $(x_0(t), y_0(t))$ be the solution of the ODE

$$dx_0(t) = c_1(x^0(t), y_0(t))dt,\ x_0(0) = x_0.$$

$$dy_0(t) = c_2(x_0(t), y_0(t))dt,\ y_0(0) = y_0.$$

2.2.2 Comparison under the expectation

We assume $A_i \equiv 0, I(t) = 0$ in what follows. We then extend (2.1) to

$$dX_t^\epsilon = c_1(X_t^\epsilon, Y_t^\epsilon)dt + \epsilon\ \sigma(t, X_t^\epsilon, Y_t^\epsilon)dW(t) + \epsilon\ dJ_t,$$

$$dY_t^\epsilon = c_2(X_s^\epsilon, Y_s^\epsilon)dt. \tag{2.6}$$

Here the function $\sigma(t,x,y)$ satisfies the similar properties as (2.5).

Let $T = 1$ and let $h \in \mathcal{S}$. Our aim is to obtain the approximate expression of

$$E[h(X_T^\epsilon)], \tag{2.7}$$

where $X_T^\epsilon = X_t^\epsilon|_{t=T}$ and $(X_t^\epsilon, Y_t^\epsilon)$ is given by (2.6).

We introduce another process

$$d\tilde{X}_t^\epsilon = c_1(x_0(t), y_0(t))dt + \epsilon\sigma(t, \tilde{X}_t^\epsilon, y_0(t))dW(t) + \epsilon dJ_t, \tilde{X}_0^\epsilon = x_0,$$

$$d\tilde{Y}_t^\epsilon = c_2(x_0(t), y_0(t))dt, \tilde{Y}_0^\epsilon = y_0. \tag{2.8}$$

Below we shall compare (2.6) with

$$E[h(\tilde{X}_T^\epsilon)]. \tag{2.9}$$

Namely

$$E[h(X_T^\epsilon)] = E[h(\tilde{X}_T^\epsilon)] + (\text{approximation error}).$$

Fortunately, the second order coeffcient in the expansion of the first term in R.H.S. in (2.8) can be given explicitly by using Malliavin calculus. In the way $0 < \epsilon \leq 1$ may be regarded as small, however it is not expected that $\epsilon \to 0$; since our aim is the comparison (not the convergence).

2.2.3 Expansion of $\tilde{X}_t^\epsilon$

Recall that $\tilde{X}_t^\epsilon$ is given by the SDE

$$d\tilde{X}_t^\epsilon = c_1(x_0(t), y_0(t))dt + \epsilon\sigma(t, \tilde{X}_t^\epsilon, y_0(t))dW(t) + \epsilon dJ_t, \tilde{X}_0^\epsilon = x_0. \tag{2.10}$$

We have a similar lemma as Lemma 3 in [5], and the mapping $\epsilon \mapsto \tilde{X}_t^\epsilon$ is n times differentiable a.s. for $\epsilon \in (0,1]$ for $n = 1, 2, ...$

We put

$$\tilde{X}_t^{(n)}(\epsilon) = \frac{d^n \tilde{X}_t^\epsilon}{d\epsilon^n}, n = 1, 2, ...$$

as above.

We then have that $\tilde{X}_t^{(1)}(0), \tilde{X}_t^{(2)}(0)$ satisfy

$$d\tilde{X}_t^{(1)}(0) = \sigma(t, x_0(t), y_0(t))dW(t) + dJ_t, \tilde{X}_0^{(1)}(0) = 0. \tag{2.11}$$

$$d\tilde{X}_t^{(2)}(0) = 2\nabla_x \sigma(t, x_0(t), y_0(t))\tilde{X}_t^{(1)}(0)dW(t), \tilde{X}_0^{(2)}(0) = 0, \tag{2.12}$$

respectively. For the precise proof of the form of the SDE, see [1] Theorem 6-24.

Then we have an expansion

$$\tilde{X}_T(\epsilon) \sim \tilde{X}_T(0) + \epsilon \tilde{X}_T^{(1)}(0) + \frac{1}{2}\epsilon^2 \tilde{X}_t^{(2)}(0) + ... \text{ in } \mathbf{D}_\infty. \tag{2.13}$$

Let $\tilde{X}_t^\epsilon$ denote the process given by SDE (2.10). We will make a composition $h \circ \tilde{X}_t^\epsilon$ with $h \in \mathcal{S}$. In this case we have to take care of $\tilde{X}_t^\epsilon$ and $\tilde{Y}_t^\epsilon$ simultaneously, since $\tilde{Y}_T^\epsilon$ also depends on $\tilde{X}_T^\epsilon$. We will expand it as follows :

$$h(\tilde{X}_T^\epsilon) \sim h(x_0(T) + \epsilon \tilde{X}_T^{(1)}(0)) + \frac{1}{2}\epsilon^2 h'(x_0(T) + \epsilon \tilde{X}_T^{(1)}(0))\tilde{X}_T^{(2)}(0) + ... \text{ in } \mathbf{D}'_\infty. \tag{2.14}$$

Hence it follows

$$E[h(\tilde{X}_T^\epsilon)] = E[h(x_0 + \epsilon \tilde{X}_T^{(1)}(0) + \frac{1}{2}\epsilon^2 \tilde{X}_T^{(2)}(0) + ...)]$$

$$= E[h(x_0(T) + \epsilon \tilde{X}_T^{(1)}(0))] + \frac{1}{2}\epsilon^2 E[h'(x_0(T) + \epsilon \tilde{X}_T^{(1)}(0))\tilde{X}_T^{(2)}(0)] + ... \quad . \tag{2.15}$$

Here again, the first term in R.H.S. of (2.15) corresponds to the value of h at the first order noise with respect to the observation of $\tilde{X}_T(\epsilon)$.

In order to calculate (2.15), we use the SDE (2.12). Using this property, we can lead the asymptotic expansion for $h(\tilde{X}_T^\epsilon)$ as follows.

Theorem 2.1 *Let $T = 1$. For a smooth function h,*

$$E[h(\tilde{X}_T^\epsilon)] = E[h(x_0(T) + \epsilon \tilde{X}_T^{(1)}(0))] +$$

$$\epsilon^2 \Big\{ \int_{\mathrm{T}} c_1(x_0(t), y_0(t)) \int_t^T \sigma(s, x_0(s), y_0(s)) \nabla_x \sigma(s, x_0(s), y_0(s)) ds dt$$

$$\times E[h''(x_0(T) + \int_{\mathrm{T}} c_1(x_0(t), y_0(t))dt + \int_{\mathrm{T}} \sigma(t, x_0(t), y_0(t))dW(t) + J_T)]$$

$$+ (\int_{\mathrm{T}} \sigma^2(t, x_0(t), y_0(t)) \int_t^T \sigma(s, x_0(s), y_0(t)) \nabla \sigma(s, x_0(s), y_0(t)) ds dt)$$

$$\times E[h'''\Big(x_0(T) + \int_{\mathrm{T}} c_1(x_0(t), y_0(t))dt + \int_{\mathrm{T}} \sigma(t, x_0(t), y_0(t))dW(t) + J_T\Big)]$$

$$+\lambda T(\int_{\mathrm{T}} t\sigma(t,x_0(t),y_0(t))\nabla\sigma(t,x_0(t),y_0(t))dt)E[h'''\Big(x_0(T)+\int_{\mathrm{T}} c_1(x_0(t),y_0(t))dt$$

$$+\int_{\mathrm{T}}\sigma(t,x_0(t),y_0(t))dW(t)+J_T+Y'\Big)]\Big\}+O(\epsilon^3). \tag{2.16}$$

Here Y' is an independent copy of Y_1.

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