Martin boundary for subordinate Brownian motion

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1. Motivation

2. Description of the class of processes - subordinate BM

3. Boundary Harnack principle

4. Boundary Harnack principle at infinity

5. Martin boundary of unbounded sets

6. Minimal thinness
Representation of harmonic functions on the halfspace

\[ \mathbb{H} = \{ x = (\tilde{x}, x_d) : \tilde{x} \in \mathbb{R}^{d-1}, x_d > 0\} - \text{halfspace in } \mathbb{R}^d. \]
 Representation of harmonic functions on the halfspace

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If \( h : \mathbb{H} \to [0, \infty) \) harmonic in \( \mathbb{H} \), then

$$h(x) = cx_d + \int_{\partial \mathbb{H}} \frac{x_d}{|x - z|^d} \, \mu(dz) = cx_d + \int_{\partial \mathbb{H}} \frac{x_d}{|x - z|^d} (1 + |z|^2)^{d/2} \, \nu(dz),$$

and a measure \( \mu \) on \( \partial \mathbb{H} \).
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\( c \geq 0 \) and a measure \( \mu \) on \( \partial \mathbb{H} \).

1–1 correspondence: \( \partial \mathbb{H} \ni z \leftrightarrow M(x, z) := \frac{x_d}{|x - z|} (1 + |z|^2)^{d/2} \),

\( \infty \leftrightarrow M(x, \infty) := x_d. \)
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If \( h : H \to [0, \infty) \) harmonic in \( H \), then

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Martin boundary of \( H \): \( \partial H \cup \{ \infty \} \):

\[
h(x) = \int_{\partial H \cup \{ \infty \}} M(x, z) \nu(dz).
\]
Motivation

Representation of $\alpha/2$-harmonic functions

$h : \mathbb{R}^d \to [0, \infty)$ is $\alpha/2$-harmonic in $\mathbb{H}$, $0 < \alpha < 2$, if “$\Delta^{\alpha/2} h = 0$ in $\mathbb{H}$”.

Probabilistic interpretation: $X = (X_t, P_x)$ isotropic $\alpha$-stable process. Then $h$ is harmonic in $\mathbb{H}$ (wrt to $X$) if for every relatively compact open $G \subset \mathbb{H}$

$h(x) = E_x h(X_{\tau_G}), \forall x \in G$,

and regular harmonic in $\mathbb{H}$ if

$h(x) = E_x h(X_{\tau_H}), \forall x \in \mathbb{H}$.

If $P_x(z) dz = P_x(X_{\tau_H} \in dz)$ - Poisson kernel, then regular harmonic $h$ has a representation

$h(x) = \int_{\mathbb{H}} c P(x, z) h(z) dz$.

$x \mapsto P(x, z)$ is not harmonic in $\mathbb{H}$.
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$$h(x) = \int_{\mathbb{H}^c} P(x, z) h(z) \, dz .$$
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Representation of singular $\alpha/2$-harmonic functions

Suppose $h : \mathbb{R}^d \to [0, \infty)$ is $\alpha/2$-harmonic and $h \equiv 0$ on $\mathbb{H}^c$. Then $h$ is called singular harmonic. This corresponds to harmonic functions of the killed process $X^\mathbb{H}$.
Suppose \( h : \mathbb{R}^d \to [0, \infty) \) is \( \alpha/2 \)-harmonic and \( h \equiv 0 \) on \( \mathbb{H}^c \). Then \( h \) is called **singular harmonic**. This corresponds to harmonic functions of the killed process \( X^\mathbb{H} \).

Representation of harmonic functions for \( X^\mathbb{H} \). Let

\[
M(x, z) := \frac{x^\alpha/2}{|x - z|^d} (1 + |z|^2)^{d/2}, \quad M(x, \infty) := x^\alpha/2.
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If $h : \mathbb{H} \rightarrow [0, \infty)$ is harmonic wrt $X^\mathbb{H}$ (singular $\alpha/2$-harmonic), then there exists a unique measure $\nu$ on $\partial \mathbb{H} \cup \{\infty\}$ such that

$$h(x) = \int_{\partial \mathbb{H} \cup \{\infty\}} M(x, z) \nu(dz) = c x^{\alpha/2}_d + \int_{\partial \mathbb{H}} \frac{x^{\alpha/2}_d}{|x - z|^d} (1 + |z|^2)^{d/2} \nu(dz).$$
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Martin boundary of $\mathbb{H}$ with respect to $X$: $\partial\mathbb{H} \cup \{\infty\}$. 
Let $X = (X_t, \mathbb{P}_x)$ be rotationally invariant Lévy process in $\mathbb{R}^d$, $D \subset \mathbb{R}^d$ open, $X^D$ the killed process, $G_D(x, y)$ the Green function of $X^D$. 

$D$ has a Martin boundary $\partial M_D$ with respect to $X^D$ satisfying the following properties:

1. $D \cup \partial M_D$ is compact metric space;
2. $D$ is open and dense in $D \cup \partial M_D$, and its relative topology coincides with its original topology;
3. $M_D(x, \cdot)$ can be uniquely extended to $\partial M_D$ in such a way that, $M_D(x, y)$ converges to $M_D(x, z)$ as $y \to z \in \partial M_D$, the function $x \to M_D(x, z)$ is excessive with respect to $X^D$, the function $(x, z) \to M_D(x, z)$ is jointly continuous on $D \times \partial M_D$ and $M_D(\cdot, z_1) \neq M_D(\cdot, z_2)$ if $z_1 \neq z_2$. 

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Martin boundary

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A harmonic function $h : D \to [0, \infty)$ is minimal (with respect to $X^D$), if $g \leq h$, $g$ harmonic, implies that $g = ch$.

The minimal Martin boundary of $X^D$ is defined as

$$\partial m^D = \{ z \in \partial M^D : M^D(\cdot, z) \text{ is minimal harmonic with respect to } X^D \}.$$
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A function $h : D \to [0, \infty)$ is harmonic if and only if there exists a finite measure $\nu$ on $\partial_m D$ such that

$$h(x) = \int_{\partial_m D} M_D(x, z) \nu(dz), \quad \text{Martin integral representation.}$$
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$X$ is Brownian motion, $D$ bounded Lipschitz domain $D$: Hunt and Wheeden (1970) proved that the (minimal) Martin boundary can be identified with the Euclidean boundary.
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(1) Bounded Lipschitz domain: Chen and Song (1998) and Bogdan (1999);

(2) Bounded $\kappa$-fat open set: Song and Wu (1999).
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Certain subordinate BM, $D$ bounded $\kappa$-fat open set: Kim, Song, V. (2009).
Martin boundary for unbounded sets?

In all mentioned results $D$ is bounded. The reason: Proofs depend on the boundary Harnack principle for non-negative harmonic functions which implies the existence of the limit $\lim_{y \to z \in \partial D} M_D(x, y)$.
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Known results for unbounded sets. Complete description of the Martin boundary only for Brownian motion and isotropic stable processes. Explicit formulae for the Martin kernel in case of the half-space $\mathbb{H}$. 
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In case of unbounded open $D$, inversion through the sphere implies the existence of $M_D(x, \infty) := \lim_{|y| \to \infty, y \in D} M_D(x, y)$: Bogdan, Kulczycki, Kwaśnicki (2008)
Finite part and infinite part of Martin boundary

Partial results for some subordinate Brownian motion – description of the finite part of the Martin boundary.
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Partial results for some subordinate Brownian motion – description of the finite part of the Martin boundary.

A point $z \in \partial_M D$ is called a **finite Martin boundary point** if there exists a bounded sequence $(y_n)_{n \geq 1}$ converging to $z$ in the Martin topology.
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A point \( z \) is called an **infinite Martin boundary point** if every sequence \( (y_n)_{n \geq 1} \) converging to \( z \) in the Martin topology is unbounded.
**Finite part and infinite part of Martin boundary**

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A point \( z \) is called an **infinite Martin boundary point** if every sequence \( (y_n)_{n \geq 1} \) converging to \( z \) in the Martin topology is unbounded.

In case \( X \) is a subordinate Brownian motion satisfying certain condition, the finite part of the Martin boundary of \( \mathbb{H} \) can be identified with the Euclidean boundary \( \partial \mathbb{H} \), Kim, Song, V. (2011).
Describe under what conditions on the process $X$ and the unbounded open set $D$ one can identify the (minimal) Martin boundary of $D$ with $\partial D \cup \{\infty\}$. 

Two types of assumptions for the process: small time-small scale, and large time-large scale.

Assumptions on $D$: $\kappa$-fat at each boundary point, and $\kappa$-fat at infinity.
Goal of the talk

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Subordinators

$S = (S_t)_{t \geq 0}$ a subordinator with the Laplace exponent $\phi$:

$$\mathbb{E}[e^{-\lambda S_t}] = e^{-t\phi(\lambda)}, \quad \phi(t) = \int_{(0,\infty)} (1 - e^{-\lambda t}) \mu(dt)$$
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Assumptions on \( \phi \): \( \phi \) is \( \mathcal{CBF} - \mu(dt) = \mu(t) dt \) where \( \mu \) is \( \mathcal{CM} \).

Consequence: the renewal measure has a \( \mathcal{CM} \) density \( u \). WLOG \( \phi(1) = 1 \).
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Upper and lower scaling conditions at infinity and at zero:

(H1): There exist constants \( 0 < \delta_1 \leq \delta_2 < 1 \) and \( a_1, a_2 > 0 \) such that

\[
a_1 \lambda^{\delta_1} \phi(t) \leq \phi(\lambda t) \leq a_2 \lambda^{\delta_2} \phi(t), \quad \lambda \geq 1, \ t \geq 1.
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Upper and lower scaling conditions at infinity and at zero:
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a_1 \lambda^{\delta_1} \phi(t) \leq \phi(\lambda t) \leq a_2 \lambda^{\delta_2} \phi(t), \quad \lambda \geq 1, t \geq 1.
\]

(H2): There exist constants \( 0 < \delta_3 \leq \delta_4 < 1 \) and \( a_3, a_4 > 0 \) such that

\[
a_3 \lambda^{\delta_4} \phi(t) \leq \phi(\lambda t) \leq a_4 \lambda^{\delta_3} \phi(t), \quad \lambda \leq 1, t \leq 1.
\]
Examples

If $0 < \alpha < 2$ and $\tilde{\ell}$ slowly varying at infinity, then

$$\phi(\lambda) \asymp \lambda^{\alpha/2} \tilde{\ell}(\lambda), \quad \lambda \to \infty,$$

implies (H1). Assumption on the behavior of the subordinator (hence SBM) for small time, small space.
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If $0 < \beta < 2$ and $\ell$ slowly varying at infinity, then

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(H1) (resp. (H2)) is equivalent to $\phi$ is an $O$-regularly varying functions at $\infty$ (resp. at 0) with Matuszewska indices in $(0,1)$. 
Properties of the potential and the Lévy density

There exists a constant $C = C(\phi) > 0$ such that

$$u(t) \leq Ct^{-1} \phi(t^{-1})^{-1}, \quad \mu(t) \leq Ct^{-1} \phi(t^{-1}), \quad \forall t \in (0, \infty).$$

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Subordinate Brownian motion

$W = (W_t, \mathbb{P}_x)$ $d$-dimensional Brownian motion, $S = (S_t)$ and independent subordinator with the Laplace exponent $\phi$ satisfying (H1), (H2) and $CBF$. 
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$X$ is a Lévy process with characteristic exponent $\Phi(x) = \phi(|x|^2)$, infinitesimal generator $A = \phi(-\Delta)$, and Lévy measure with density $J(x) = j(|x|)$ where

$$j(r) = \int_0^{\infty} (4\pi t)^{-d/2} e^{-r^2/4t} \mu(t) \, dt, \quad r > 0.$$
**Subordinate Brownian motion**

\[ W = (W_t, \mathbb{P}_x) \] is the \( d \)-dimensional Brownian motion, \( S = (S_t) \) and independent subordinator with the Laplace exponent \( \phi \) satisfying (H1), (H2) and CBF. The SBM is the process \( X = (X_t)_{t \geq 0} \) defined as \( X_t := W_{S_t} \).

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j(r) = \int_0^\infty (4\pi t)^{-d/2} e^{-r^2/4t} \mu(t) \, dt, \quad r > 0.
\]

Assume \( X \) is transient; then \( X \) has the Green function

\[
G(x, y) = G(x - y) = g(|x - y|) \quad \text{where}
\]

\[
g(r) = \int_0^\infty (4\pi t)^{-d/2} e^{-r^2/4t} u(t) \, dt, \quad r > 0.
\]
**Theorem:** Assume (H1) and (H2).

\[ J(x) \approx |x| - d \phi(|x| - 2), \quad x \neq 0. \]

If \( d > 2(\delta^2 \lor \delta^4) \), then \( X \) is transient and
\[ G(x) \approx |x| - d \phi(|x| - 2) - 1, \quad x \neq 0. \]
Theorem: Assume (H1) and (H2).
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\[ J(x) \asymp |x|^{-d} \phi(|x|^{-2}), \quad x \neq 0. \]
(b) If \( d > 2(\delta_2 \vee \delta_4) \), then \( X \) is transient and
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**Theorem:** Assume (H1) and (H2).

(a) Then

\[ J(x) \asymp |x|^{-d} \phi(|x|^{-2}), \quad x \neq 0. \]

(b) If \( d > 2(\delta_2 \lor \delta_4) \), then \( X \) is transient and

\[ G(x) \asymp |x|^{-d} \phi(|x|^{-2})^{-1}, \quad x \neq 0. \]

**Corollary:** (Doubling property) \( J(2x) \asymp J(x), \ G(2x) \asymp G(x), \ x \neq 0. \)
1 Motivation

2 Description of the class of processes - subordinate BM

3 Boundary Harnack principle

4 Boundary Harnack principle at infinity

5 Martin boundary of unbounded sets

6 Minimal thinness
Recall that $u : \mathbb{R}^d \to [0, \infty)$ is regular harmonic in open $D \subset \mathbb{R}^d$ with respect to $X$ if

$$u(x) = \mathbb{E}_x [u(X_{\tau_D}) : \tau_D < \infty],$$

for all $x \in D$. 

Theorem: There exists a constant $c = c(\phi, d) > 0$ such that for every $z_0 \in \mathbb{R}^d$, every open set $D \subset \mathbb{R}^d$, every $r > 0$ and for any nonnegative functions $u, v$ in $\mathbb{R}^d$ which are regular harmonic in $D \cap B(z_0, r)$ with respect to $X$ and vanish in $D^c \cap B(z_0, r)$, we have

$$u(x) v(x) \leq c u(y) v(y)$$

for all $x, y \in D \cap B(z_0, r/2)$. 

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\frac{u(x)}{v(x)} \leq c \frac{u(y)}{v(y)} \quad \text{for all } x, y \in D \cap B(z_0, r/2).
\]
Lemma: (Approximate factorization) For every $z_0 \in \mathbb{R}^d$, every open set $U \subset B(z_0, r)$ and for any nonnegative function $u$ in $\mathbb{R}^d$ which is regular harmonic in $U$ with respect to $X$ and vanishes a.e. in $U^c \cap B(z_0, r)$,

$$u(x) \asymp \mathbb{E}_x[\tau_U] \int_{B(z_0, r/2)^c} j(|z - z_0|)u(z)dz, \quad x \in U \cap B(z_0, r/2).$$
Lemma: (Approximate factorization) For every $z_0 \in \mathbb{R}^d$, every open set $U \subset B(z_0, r)$ and for any nonnegative function $u$ in $\mathbb{R}^d$ which is regular harmonic in $U$ with respect to $X$ and vanishes a.e. in $U^c \cap B(z_0, r)$,

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Lemma: (Approximate factorization) For every \( z_0 \in \mathbb{R}^d \), every open set \( U \subset B(z_0, r) \) and for any nonnegative function \( u \) in \( \mathbb{R}^d \) which is regular harmonic in \( U \) with respect to \( X \) and vanishes a.e. in \( U^c \cap B(z_0, r) \),

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\]

For all \( r \in (0, 1] \) under (H1) (Kim, Song, V. (2011)), for all \( r \in (0, \infty) \) under (H1) and (H2) (Kim, Song, V (2012)).
Take $z_0 = 0$. Then the above reads:

$$u(x) \asymp \int_U G_U(x, y) \, dy \int_{B(0, r/2)^c} j(|y|) u(y) \, dy, \quad x \in U \cap B(0, r/2).$$
**Theorem:** Assume (H1) and (H2). There exists $c = c(\phi) > 0$ such that for every open set $D$ satisfying the interior and exterior ball conditions with radius $R > 0$, every $r \in (0, R]$, every $Q \in \partial D$ and every nonnegative function $u$ in $\mathbb{R}^d$ which is harmonic in $D \cap B(Q, r)$ with respect to $X$ and vanishes continuously on $D^c \cap B(Q, r)$, we have

$$\frac{u(x)}{(\phi(\delta_D(x)^{-2}))^{-1/2}} \leq c \frac{u(y)}{(\phi(\delta_D(y)^{-2}))^{-1/2}}$$

for all $x, y \in D \cap B(Q, \frac{r}{2})$. 

Global: it holds for all $R > 0$ with the comparison constant not depending on $D$.

Uniform: it holds for all balls with radii $r \leq R$ and the comparison constant depends neither on $D$ nor on $r$. 

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Global and uniform BHP in smooth sets with explicit decay rate

**Theorem:** Assume (H1) and (H2). There exists $c = c(\phi) > 0$ such that for every open set $D$ satisfying the interior and exterior ball conditions with radius $R > 0$, every $r \in (0, R]$, every $Q \in \partial D$ and every nonnegative function $u$ in $\mathbb{R}^d$ which is harmonic in $D \cap B(Q, r)$ with respect to $X$ and vanishes continuously on $D^c \cap B(Q, r)$, we have

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In case of rotationally invariant $\alpha$-stable process, M. Kwaśnicki (2009) used the inversion through the sphere $B(0, \sqrt{r})$ to obtain a BHP at infinity.
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Recall that the Poisson kernel $K_U(x, z)$ is the exit density from an open set $U$: $\mathbb{P}_x(X_{\tau_U} \in B) = \int_B K_U(x, z) \, dy$, $B \subset \overline{U}^c$,

$$K_U(x, z) = \int_U G_U(x, y) j(|y - z|) \, dy, \quad x \in U, z \in \overline{U}^c.$$
In case of rotationally invariant $\alpha$-stable process, M. Kwaśnicki (2009) used the inversion through the sphere $B(0, \sqrt{r})$ to obtain a BHP at infinity.

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$x \in U, z \in U^c$.

If $u$ regular harmonic in $U$, then $u(x) = \int_{U^c} K_U(x, z) u(z) \, dz$. 

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**Theorem:** There exists $C = C(\phi) > 1$ such that for all $r \geq 1$, for all open sets $U \subset \overline{B}(0, r)^c$ and all nonnegative functions $u$ on $\mathbb{R}^d$ that are regular harmonic in $U$ and vanish on $\overline{B}(0, r)^c \setminus U$, it holds that
**Theorem:** There exists $C = C(\phi) > 1$ such that for all $r \geq 1$, for all open sets $U \subset \overline{B}(0, r)^c$ and all nonnegative functions $u$ on $\mathbb{R}^d$ that are regular harmonic in $U$ and vanish on $\overline{B}(0, r)^c \setminus U$, it holds that

$$u(x) \approx \int_{U} G_U(x, y) j(|y|) \, dy \int_{B(0, 2r)} u(z) \, dz, \quad x \in U \cap B(0, 2r)^c.$$
Theorem: There exists $C = C(\phi) > 1$ such that for all $r \geq 1$, for all open sets $U \subset \overline{B}(0, r)^c$ and all nonnegative functions $u$ on $\mathbb{R}^d$ that are regular harmonic in $U$ and vanish on $\overline{B}(0, r)^c \setminus U$, it holds that

$$\frac{1}{C} \leq \frac{u(x)}{K_U(x, 0) \int_{B(0,2r)} u(z) \, dz} \leq C, \quad \text{for all } x \in U \cap \overline{B}(0, 2r)^c.$$
BHP at infinity – continuation

**Theorem:** There exists $C = C(\phi) > 1$ such that for all $r \geq 1$, for all open sets $U \subset \overline{B}(0, r)^c$ and all nonnegative functions $u$ on $\mathbb{R}^d$ that are regular harmonic in $U$ and vanish on $\overline{B}(0, r)^c \setminus U$, it holds that

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$$u(x) \preceq \int_U G_U(x, y) j(|y|) \, dy \int_{B(0, 2r)} u(z) \, dz, \quad x \in U \cap \overline{B}(0, 2r)^c.$$
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$$u(x) \asymp \int_U G_U(x, y) j(|y|) \, dy \int_{B(0,2r)} u(z) \, dz, \quad x \in U \cap \overline{B}(0, 2r)^c.$$

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**Zoran Vondraček (University of Zagreb)**

Martin boundary for SBM

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Corollary: There exists $C = C(\phi) > 1$ such that for all $r \geq 1$, for all open sets $U \subset \overline{B}(0, r)^c$ and all nonnegative functions $u$ and $v$ on $\mathbb{R}^d$ that are regular harmonic in $U$ and vanish on $\overline{B}(0, r)^c \setminus U$, it holds that

$$C^{-1} \frac{u(y)}{v(y)} \leq \frac{u(x)}{v(x)} \leq C \frac{u(y)}{v(y)},$$

for all $x, y \in U \cap \overline{B}(0, 2r)^c$. 

Not true if regular harmonic is replaced by harmonic: $w(x) := (x + \alpha/2)^{\alpha/2}$ is harmonic in the upper half-space $H \subset \mathbb{R}^{\alpha/2}$, vanishes on $\overline{B}(\tilde{0}, -1)^c \setminus H$, but $\lim_{x \rightarrow \infty} w(x) = \infty$. 
Corollary: There exists $C = C(\phi) > 1$ such that for all $r \geq 1$, for all open sets $U \subset \overline{B}(0, r)^c$ and all nonnegative functions $u$ and $v$ on $\mathbb{R}^d$ that are regular harmonic in $U$ and vanish on $\overline{B}(0, r)^c \setminus U$, it holds that

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Corollary: Let $r \geq 1$ and $U \subset \overline{B}(0, r)^c$. If $u$ is a non-negative function on $\mathbb{R}^d$ which is regular harmonic in $U$ and vanishes on $\overline{B}(0, r)^c \setminus U$, then

$$\lim_{|x| \to \infty} u(x) = 0.$$
Corollaries

**Corollary:** There exists $C = C(\phi) > 1$ such that for all $r \geq 1$, for all open sets $U \subset \overline{B}(0, r)^c$ and all nonnegative functions $u$ and $v$ on $\mathbb{R}^d$ that are regular harmonic in $U$ and vanish on $\overline{B}(0, r)^c \setminus U$, it holds that

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**Corollary:** Let $r \geq 1$ and $U \subset \overline{B}(0, r)^c$. If $u$ is a non-negative function on $\mathbb{R}^d$ which is regular harmonic in $U$ and vanishes on $\overline{B}(0, r)^c \setminus U$, then

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Zoran Vondraček (University of Zagreb)
Upper bound on the Green function $\overline{B}(0, r)^c$, $r \geq 1$: Let $1 < p < q < 4$ and $b > 0$. There exist a constant $C = C(\phi, p, q, b) > 0$ such that for all $r \geq 1$, all $x \in A(0, pr, qr)$ and all $y \in A(0, r, 2qr)$ such that $br < |x - y|$ it holds that

$$G_{\overline{B}(0,r)^c}(x, y) \leq c \frac{\phi(r^{-2})^{1/2}}{\phi(\delta_{\overline{B}(0,r)^c}(y)^{-2})^{1/2}} g(r)$$
Ingredients of the proof

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G_{\overline{B}(0,r)^c}(x, y) \leq c \frac{\phi(r^{-2})^{1/2}}{\phi(\delta_{\overline{B}(0,r)^c}(y)^{-2})^{1/2}} g(r) \\
\leq C \phi(r^{-2})^{-1/2} \phi(\delta_{\overline{B}(0,r)^c}(y)^{-2})^{-1/2} r^{-d}.
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Upper bound on the Green function $\overline{B}(0, r)^c$, $r \geq 1$: Let $1 < p < q < 4$ and $b > 0$. There exist a constant $C = C(\phi, p, q, b) > 0$ such that for all $r \geq 1$, all $x \in A(0, pr, qr)$ and all $y \in A(0, r, 2qr)$ such that $br < |x - y|$ it holds that

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$$\leq C \phi(r^{-2})^{-1/2} \phi(\delta_{\overline{B}(0,r)^c}(y)^{-2})^{-1/2} r^{-d}.$$

The proof uses the global, uniform BHP with explicit decay rate.
Upper bound for the Poisson kernel of $\overline{B}(0, r)^c$, $r \geq 1$: Let $1 < p < q < 4$. There exists $C = C(\phi, p, q) > 1$ such that for all $r \geq 1$, all $x \in A(0, pr, qr)$ and $z \in B(0, r)$ it holds that

$$K_{\overline{B}(0, r)^c}(x, z) \leq Cr^{-d} \phi(r^{-2})^{-1/2} \phi((r - |z|)^{-2})^{1/2}.$$

This bound is crucial for the analysis of the boundary behavior of harmonic functions in the context of the Martin boundary for symmetric Hunt processes.
Ingredients of the proof – continuation

Upper bound for the Poisson kernel of $\overline{B}(0, r)^c$, $r \geq 1$: Let $1 < p < q < 4$. There exists $C = C(\phi, p, q) > 1$ such that for all $r \geq 1$, all $x \in A(0, pr, qr)$ and $z \in B(0, r)$ it holds that

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For $\alpha$-stable process

$$K_{\overline{B}(0, r)^c}(x, z) = c(\alpha, d) \frac{(|x|^2 - r^2)^{\alpha/2}}{(r^2 - |z|^2)^{\alpha/2}} |x - z|^{-d}$$
Upper bound for the Poisson kernel of $\overline{B}(0, r)^c$, $r \geq 1$: Let $1 < p < q < 4$. There exists $C = C(\phi, p, q) > 1$ such that for all $r \geq 1$, all $x \in A(0, pr, qr)$ and $z \in B(0, r)$ it holds that

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For $\alpha$-stable process

$$K_{\overline{B}(0,r)^c}(x, z) = c(\alpha, d)\frac{|x|^2 - r^2}{(r^2 - |z|^2)^{\alpha/2}} |x - z|^{-d} \lesssim r^{-d}\frac{r^{\alpha/2}}{(r - |z|)^{\alpha/2}}.$$
Exit probability estimate: For every $a \in (1, \infty)$, there exists a positive constant $C = C(\phi, a) > 0$ such that for any $r \in (0, \infty)$ and any open set $U \subset \overline{B}(0, r)^c$ we have

$$\mathbb{P}_x \left( X_{\tau_U} \in \overline{B}(0, r) \right) \leq Cr^d K_U(x, 0), \quad x \in U \cap \overline{B}(0, ar)^c.$$
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Ingredients of the proof – continuation

Regularization of the Poisson kernel in the spirit of Bogdan, Kulczycki and Kwaśnicki (2008) leading to

\[ K_U(x, z) \approx K_U(x, 0) \left( \int_{U \cap B(0, 2r)} K_U(y, z) \, dy + 1 \right). \]
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$\kappa$-fat sets

Let $\kappa \in (0, 1/2]$. An open set $D$ is said to be $\kappa$-fat open at $Q \in \partial D$, if there exists $R > 0$ such that for each $r \in (0, R)$ there exists a point $A_r(Q)$ satisfying $B(A_r(Q), \kappa r) \subset D \cap B(Q, r)$. 
Let $\kappa \in (0, 1/2]$. An open set $D$ is said to be $\kappa$-fat open at $Q \in \partial D$, if there exists $R > 0$ such that for each $r \in (0, R)$ there exists a point $A_r(Q)$ satisfying $B(A_r(Q), \kappa r) \subset D \cap B(Q, r)$. If $D$ is $\kappa$-fat at each boundary point $Q \in \partial D$ with the same $R > 0$, $D$ is called $\kappa$-fat with characteristics $(R, \kappa)$. 

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$\kappa$-fat sets

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If $D$ is $\kappa$-fat at each boundary point $Q \in \partial D$ with the same $R > 0$, $D$ is called $\kappa$-fat with characteristics $(R, \kappa)$. 
\( \kappa \)-fat sets at infinity

An open set \( D \) in \( \mathbb{R}^d \) is \( \kappa \)-fat at infinity if there exists \( R > 0 \) such that for every \( r \in [R, \infty) \) there exists \( A_r \in \mathbb{R}^d \) such that \( B(A_r, \kappa r) \subset D \cap \overline{B}(0, r)^c \) and \( |A_r| < \kappa^{-1} r \). The pair \( (R, \kappa) \) will be called the characteristics of the \( \kappa \)-fat open set \( D \) at infinity.
An open set \( D \) in \( \mathbb{R}^d \) is \( \kappa \)-fat at infinity if there exists \( R > 0 \) such that for every \( r \in [R, \infty) \) there exists \( A_r \in \mathbb{R}^d \) such that \( B(A_r, \kappa r) \subset D \cap \overline{B}(0, r)^c \) and \( |A_r| < \kappa^{-1} r \). The pair \( (R, \kappa) \) will be called the characteristics of the \( \kappa \)-fat open set \( D \) at infinity.
$\kappa$-fat sets at infinity

An open set $D$ in $\mathbb{R}^d$ is $\kappa$-fat at infinity if there exists $R > 0$ such that for every $r \in [R, \infty)$ there exists $A_r \in \mathbb{R}^d$ such that $B(A_r, \kappa r) \subset D \cap \overline{B}(0, r)^c$ and $|A_r| < \kappa^{-1} r$. The pair $(R, \kappa)$ will be called the characteristics of the $\kappa$-fat open set $D$ at infinity.

All half-space-like open sets, all exterior open sets and all infinite cones are $\kappa$-fat at infinity.
**Oscillation reduction**

**Lemma:** Let $D \subset \mathbb{R}^d$ be an open set which is $\kappa$-fat at infinity with characteristics $(R, \kappa)$. There exist $C = C(\phi, d) > 0$ and $\nu = \nu(d, \phi) > 0$ such that for all $r \geq 1$ and all non-negative functions $u$ and $v$ on $\mathbb{R}^d$ which are regular harmonic in $D \cap \overline{B}(0, r/2)^c$, vanish in $D^c \cap \overline{B}(0, r/2)^c$ and satisfy $u(A_r) = v(A_r)$, there exists the limit

$$g = \lim_{|x| \to \infty, x \in D} \frac{u(x)}{v(x)},$$

and we have

$$\left| \frac{u(x)}{v(x)} - g \right| \leq C \left( \frac{|x|}{r} \right)^{-\nu}, \quad x \in D \cap \overline{B}(0, r)^c.$$
Fix $x_0 \in D \cap \overline{B}(0, R)^c$ and recall that

$$M_D(x, y) = \frac{G_D(x, y)}{G_D(x_0, y)}, \quad x, y \in D \cap \overline{B}(0, R)^c.$$ 

For $r > (2|x| \lor R)$, both functions $y \mapsto G_D(x, y)$ and $y \mapsto G_D(x_0, y)$ are regular harmonic in $D \cap \overline{B}(0, r/2)^c$ and vanish on $D^c \cap \overline{B}(0, r/2)^c$. 

Theorem: (Kim, Song, V 2012) For each $x \in D$ there exists the limit

$$M_D(x, \infty) := \lim_{y \in D, |y| \to \infty} M_D(x, y).$$

This implies that every infinite Martin boundary point can be mapped to $\{\infty\}$. Since Martin kernels for different Martin boundary points are different, the infinite part of the Martin boundary is exactly $\{\infty\}$. 

Zoran Vondraček (University of Zagreb)
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Harmonicity and minimality

**Lemma:** For each $x \in D$ and $\rho \in (0, \frac{1}{3} \delta_D(x)]$,

$$M_D(x, \infty) = \mathbb{E}_x[M_D(X_{T_{B(x, \rho)}}, \infty)].$$
Harmonicity and minimality

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**Theorem:** The function $M_D(\cdot, \infty)$ is minimal harmonic in $D$ with respect to $X$. 
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By use of the lemma above, one shows that $M_D(\cdot, \infty)$ is harmonic (exit time from a relatively compact $D_1 \subset D$ is an increasing limit of exit times from balls with radii comparable to the distance to the boundary).
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Minimality: \( h \) positive harmonic, \( h \leq M_D(\cdot, \infty) \). Write

\[
h(x) = \int_{\partial^f_M D} M_D(x, w) \mu(dw) + M_D(x, \partial_\infty)\mu(\{\partial_\infty\}),
\]

and show that \( \mu(\partial^f_M D) = 0 \).
Finite part of Martin boundary: If $D$ is $\kappa$-fat open set, then the finite part of the Martin boundary can be identified with the Euclidean boundary.
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Define $w(x) = w(\tilde{x}, x_d) := V(x_d)$ where $V$ is the renewal functional of the last component of $X$. 

**Corollary:** The Martin boundary and the minimal Martin boundary of the half space $H$ with respect to $X$ can be identified with $\partial H \cup \{\infty\}$ and $M_H(x, \infty) = w(x)/w(x_0)$ for $x \in H$. 
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Corollary: The Martin boundary and the minimal Martin boundary of the half space $\mathbb{H}$ with respect to $X$ can be identified with $\partial \mathbb{H} \cup \{\infty\}$ and $M_{\mathbb{H}}(x, \infty) = w(x)/w(x_0)$ for $x \in \mathbb{H}$.
1 Motivation

2 Description of the class of processes - subordinate BM

3 Boundary Harnack principle

4 Boundary Harnack principle at infinity

5 Martin boundary of unbounded sets

6 Minimal thinness
Minimal thinness

Let $A \subset D$, $T_A = \inf\{t > 0 : X^D_t \in A\}$ the hitting time to $A$, $P_A f(x) := \mathbb{E}_x[f(X^D_{T_A})]$ the hitting operator to $A$. 
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Let $z \in \partial_m D$. Then $A \subset D$ is minimally thin at $z$ if $P_A M_D(\cdot, z) \neq M(\cdot, z)$. 
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Let $z \in \partial_m D$. Then $A \subset D$ is **minimally thin** at $z$ if $P_A M_D(\cdot, z) \neq M(\cdot, z)$. Equivalently, there exists $x \in D$ such that the $M_D(\cdot, z)$-conditioned process $X^D,z$ will not hit $A$ with positive probability.
Minimal thinness in the halfspace

**Theorem:** Assume that $X$ is SBM satisfying (H1) and $z \in \partial \mathbb{H}$. If $A \subset \mathbb{H}$ is minimally thin in $\mathbb{H}$ at $z$, then

$$
\int_{A \cap B(z,1)} |x - z|^{-d} \, dx < \infty.
$$

Conversely, suppose that $A$ is a union of Whitney cubes. If $A$ is not minimally thin at $z$, then the above integral is infinite.
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**Theorem:** Assume that $X$ is SBM satisfying (H1) and (H2). If $A \subset \mathbb{H}$ is minimally thin in $\mathbb{H}$ at $\infty$, then

$$\int_{A \cap B(0,1)^c} |x - z|^{-d} \, dx < \infty.$$ 

Conversely, suppose that $A$ is a union of Whitney cubes. If $A$ is not minimally thin at $\infty$, then the above integral is infinite.