Martin boundary for subordinate Brownian motion

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Representation of harmonic functions on the halfspace

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$$h(x) = cx_d + \int_{\partial\mathbb{H}} \frac{x_d}{|x-z|^d}\,\mu(dz) = cx_d + \int_{\partial\mathbb{H}} \frac{x_d}{|x-z|^d}(1+|z|^2)^{d/2}\,\nu(dz)\,,$$

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$1-1$ correspondence: $\partial\mathbb{H} \ni z \longleftrightarrow M(x,z) := \dfrac{x_d}{|x-z|^d}(1+|z|^2)^{d/2}$,
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Martin boundary of $\mathbb{H}$: $\partial\mathbb{H} \cup \{\infty\}$:

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$x \mapsto P(x, z)$ is not harmonic in $\mathbb{H}$.

Representation of singular $\alpha/2$-harmonic functions

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If $h : \mathbb{H} \to [0, \infty)$ is harmonic wrt $X^{\mathbb{H}}$ (singular $\alpha/2$-harmonic), then there exists a unique measure ν on $\partial\mathbb{H} \cup \{\infty\}$ such that

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Martin boundary of $\mathbb{H}$ with respect to X: $\partial\mathbb{H} \cup \{\infty\}$.

Martin boundary

Let $X = (X_t, \mathbb{P}_x)$ be rotationally invariant Lévy process in $\mathbb{R}^d$, $D \subset \mathbb{R}^d$ open, X^D the killed process, $G_D(x, y)$ the Green function of X^D.

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D has a Martin boundary $\partial_M D$ with respect to X^D satisfying the following properties:

(1) $D \cup \partial_M D$ is compact metric space;

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(3) $M_D(x, \cdot)$ can be uniquely extended to $\partial_M D$ in such a way that, $M_D(x,y)$ converges to $M_D(x,z)$ as $y \to z \in \partial_M D$, the function $x \to M_D(x,z)$ is excessive with respect to X^D, the function $(x,z) \to M_D(x,z)$ is jointly continuous on $D \times \partial_M D$ and $M_D(\cdot, z_1) \neq M_D(\cdot, z_2)$ if $z_1 \neq z_2$.

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A function $h : D \to [0, \infty)$ is harmonic if and only if there exists a finite measure ν on $\partial_m D$ such that

$$h(x) = \int_{\partial_m D} M_D(x, z)\, \nu(dz)\,, \quad \text{Martin integral representation.}$$

Some history

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Certain subordinate BM, D bounded κ-fat open set: Kim, Song, V. (2009).

Martin boundary for unbounded sets?

In all mentioned results D is bounded. The reason: Proofs depend on the boundary Harnack principle for non-negative harmonic functions which implies the existence of the limit $\lim_{y \to z \in \partial D} M_D(x, y)$.

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In case of unbounded open D, inversion through the sphere implies the existence of $M_D(x, \infty) := \lim_{|y| \to \infty,\, y \in D} M_D(x, y)$:
Bogdan, Kulczycki, Kwaśnicki (2008)

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In case X is a subordinate Brownian motion satisfying certain condition, the finite part of the Martin boundary of $\mathbb{H}$ can be identified with the Euclidean boundary $\partial\mathbb{H}$, Kim, Song, V. (2011).

Goal of the talk

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Two types of assumptions for the process: small time-small scale, and large time-large scale.
Assumptions on D: κ-fat at each boundary point, and κ-fat at infinity.

Subordinators

$S = (S_t)_{t \geq 0}$ a subordinator with the Laplace exponent ϕ:

$$\mathbb{E}[e^{-\lambda S_t}] = e^{-t\phi(\lambda)}, \qquad \phi(t) = \int_{(0,\infty)} (1 - e^{-\lambda t})\, \mu(dt)$$

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Assumptions on ϕ: ϕ is $\mathcal{CBF}$ – $\mu(dt) = \mu(t)\, dt$ where μ is $\mathcal{CM}$.
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Upper and lower scaling conditions at infinity and at zero:
(H1): There exist constants $0 < \delta_1 \leq \delta_2 < 1$ and $a_1, a_2 > 0$ such that

$$a_1 \lambda^{\delta_1} \phi(t) \leq \phi(\lambda t) \leq a_2 \lambda^{\delta_2} \phi(t), \quad \lambda \geq 1, t \geq 1\,.$$

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(H2): There exist constants $0 < \delta_3 \leq \delta_4 < 1$ and $a_3, a_4 > 0$ such that

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Examples

If $0 < \alpha < 2$ and $\widetilde{\ell}$ slowly varying at infinity, then

$$\phi(\lambda) \asymp \lambda^{\alpha/2}\widetilde{\ell}(\lambda)\,, \quad \lambda \to \infty\,,$$

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(H1) (resp. (H2)) is equivalent to ϕ is an O-regularly varying functions at ∞ (resp. at 0) with Matuszewska indices in $(0, 1)$.

Properties of the potential and the Lévy density

There exists a constant $C = C(\phi) > 0$ such that

$$u(t) \le Ct^{-1}\phi(t^{-1})^{-1}, \qquad \mu(t) \le Ct^{-1}\phi(t^{-1}), \qquad \forall\, t \in (0,\infty).$$

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X is a Lévy process with characteristic exponent $\Phi(x) = \phi(|x|^2)$, infinitesimal generator $\mathcal{A} = \phi(-\Delta)$, and Lévy measure with density $J(x) = j(|x|)$ where

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Assume X is transient; then X has the Green function $G(x, y) = G(x - y) = g(|x - y|)$ where

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Corollary: (Doubling property) $J(2x) \asymp J(x)$, $G(2x) \asymp G(x)$, $x \neq 0$.

Uniform BHP

Recall that $u : \mathbb{R}^d \to [0, \infty)$ is regular harmonic in open $D \subset \mathbb{R}^d$ with respect to X if

$$u(x) = \mathbb{E}_x \left[u(X_{\tau_D}) : \tau_D < \infty \right] , \quad \text{for all } x \in D .$$

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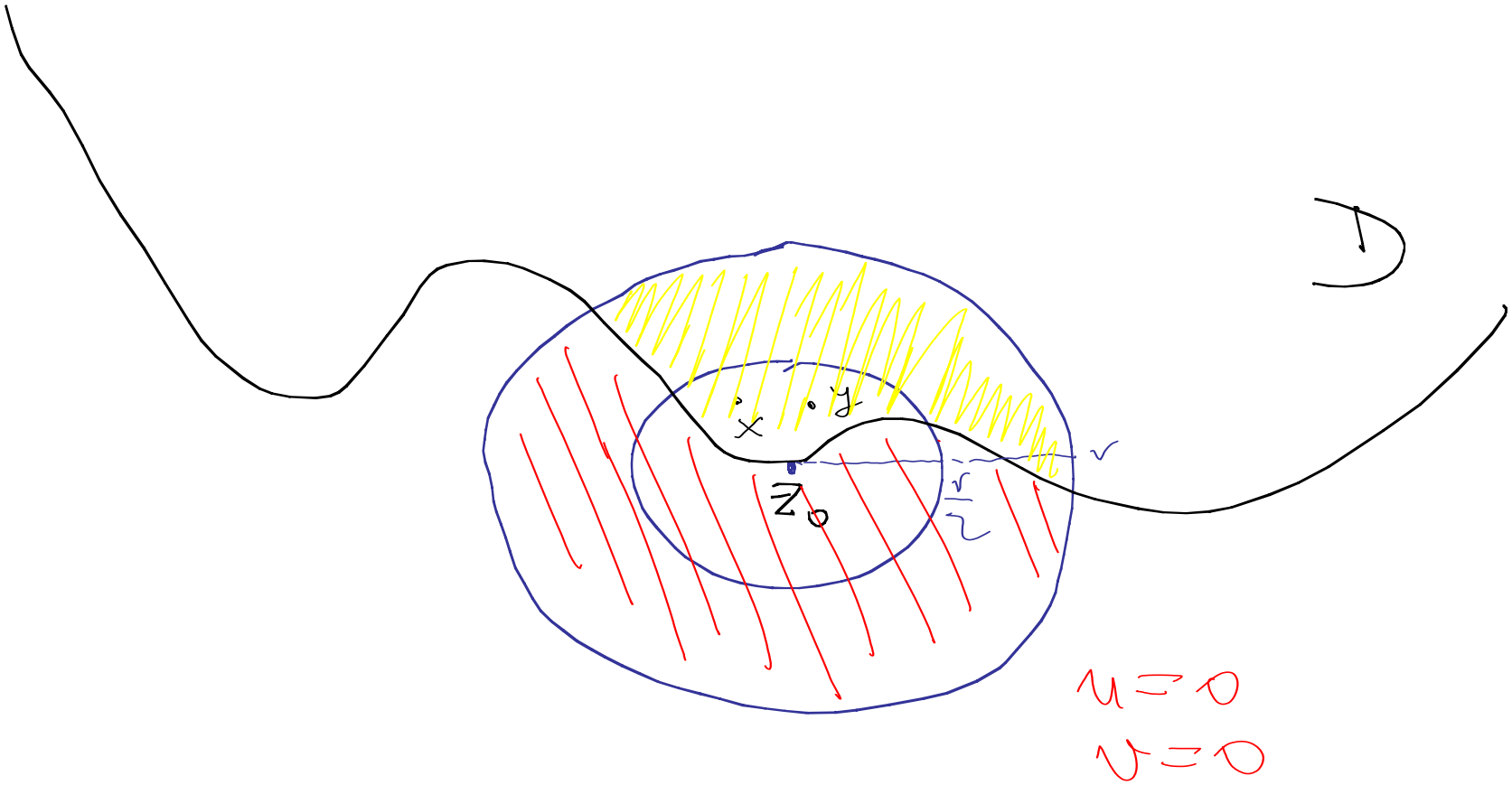
Theorem: There exists a constant $c = c(\phi, d) > 0$ such that for every $z_0 \in \mathbb{R}^d$, every open set $D \subset \mathbb{R}^d$, every $r > 0$ and for any nonnegative functions u, v in $\mathbb{R}^d$ which are regular harmonic in $D \cap B(z_0, r)$ with respect to X and vanish in $D^c \cap B(z_0, r)$, we have

$$\frac{u(x)}{v(x)} \le c \, \frac{u(y)}{v(y)} \quad \text{for all } x, y \in D \cap B(z_0, r/2) .$$


D
x
y
z_0
$\frac{r}{2}$
r
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Lemma: (Approximate factorization) For every $z_0 \in \mathbb{R}^d$, every open set $U \subset B(z_0, r)$ and for any nonnegative function u in $\mathbb{R}^d$ which is regular harmonic in U with respect to X and vanishes a.e. in $U^c \cap B(z_0, r)$,

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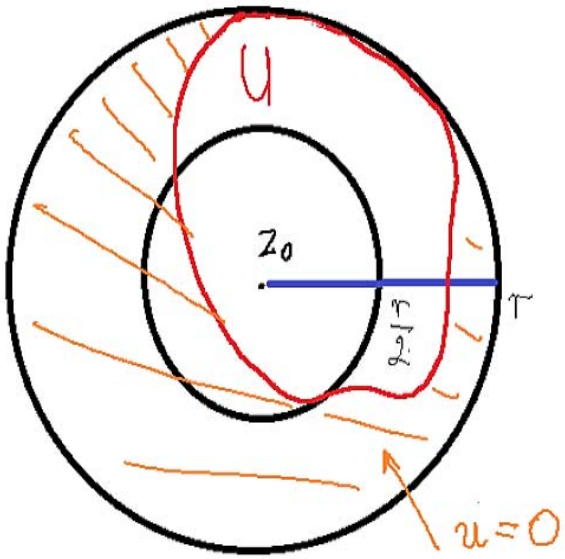

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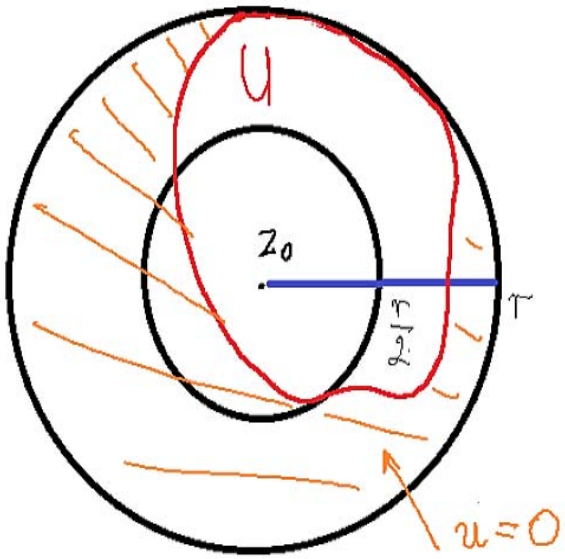
For all $r \in (0, 1]$ under (H1) (Kim, Song, V. (2011)), for all $r \in (0, \infty)$ under (H1) and (H2) (Kim, Song, V (2012)).



Take $z_0 = 0$. Then the above reads:

$$u(x) \asymp \int_U G_U(x,y)\,dy \int_{B(0,r/2)^c} j(|y|)u(y)dy\,, \quad x \in U \cap B(0,r/2)\,.$$

Global and uniform BHP in smooth sets with explicit decay rate

Theorem: Assume (H1) and (H2). There exists $c = c(\phi) > 0$ such that for every open set D satisfying the interior and exterior ball conditions with radius $R > 0$, every $r \in (0, R]$, every $Q \in \partial D$ and every nonnegative function u in $\mathbb{R}^d$ which is harmonic in $D \cap B(Q, r)$ with respect to X and vanishes continuously on $D^c \cap B(Q, r)$, we have

$$\frac{u(x)}{(\phi(\delta_D(x)^{-2}))^{-1/2}} \leq c \, \frac{u(y)}{(\phi(\delta_D(y)^{-2}))^{-1/2}} \qquad \text{for all } x, y \in D \cap B(Q, \frac{r}{2}).$$

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Global: it holds for all $R > 0$ with the comparison constant *not* depending on R.
Uniform: it holds for all balls with radii $r \leq R$ and the comparison constant depends neither on D nor on r.

Boundary Harnack principle at infinity

In case of rotationally invariant α-stable process, M. Kwaśnicki (2009) used the inversion through the sphere $B(0, \sqrt{r})$ to obtain a BHP at infinity.

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If u regular harmonic in U, then $u(x) = \int_{\overline{U}^c} K_U(x, z) u(z)\, dz$.

BHP at infinity – continuation

Theorem: There exists $C = C(\phi) > 1$ such that for all $r \geq 1$, for all open sets $U \subset \overline{B}(0,r)^c$ and all nonnegative functions u on $\mathbb{R}^d$ that are regular harmonic in U and vanish on $\overline{B}(0,r)^c \setminus U$, it holds that

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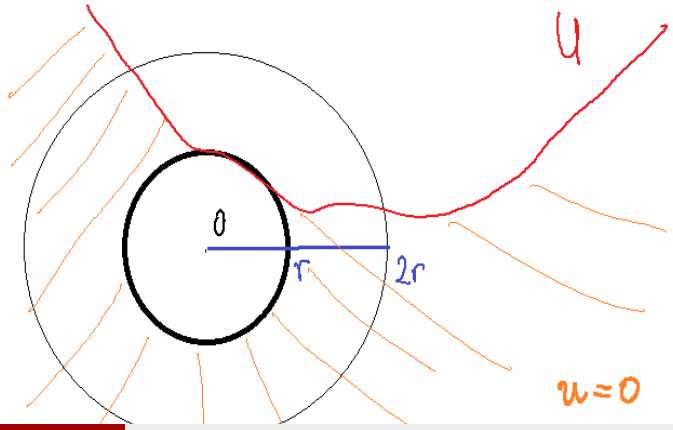

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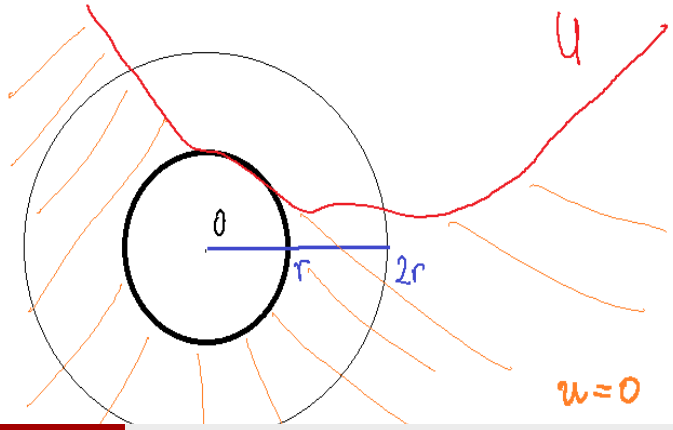

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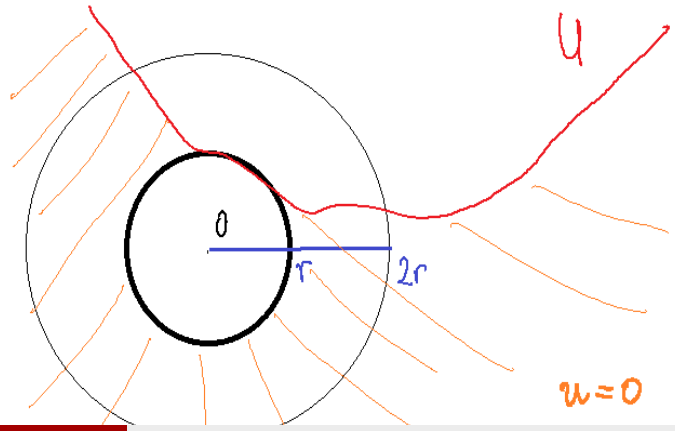

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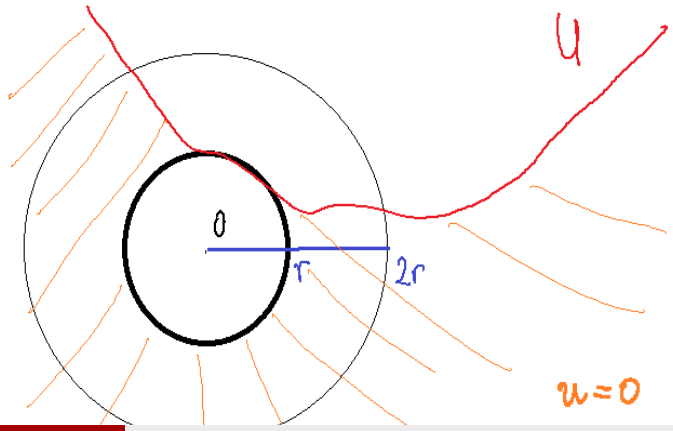

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Corollary: There exists $C = C(\phi) > 1$ such that for all $r \geq 1$, for all open sets $U \subset \overline{B}(0,r)^c$ and all nonnegative functions u and v on $\mathbb{R}^d$ that are regular harmonic in U and vanish on $\overline{B}(0,r)^c \setminus U$, it holds that

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Not true if *regular harmonic* is replaced by *harmonic*: $w(x) := (x_d^+)^{\alpha/2}$ is harmonic in the upper half-space $\mathbb{H} \subset B((\widetilde{0},-1),1)^c$, vanishes on $\overline{B}((\widetilde{0},-1),1)^c \setminus \mathbb{H}$, but $\lim_{x_d\to\infty} w(x) = \infty$.

Ingredients of the proof

Upper bound on the Green function $\overline{B}(0,r)^c$, $r \geq 1$: Let $1 < p < q < 4$ and $b > 0$. There exist a constant $C = C(\phi, p, q, b) > 0$ such that for all $r \geq 1$, all $x \in A(0, pr, qr)$ and all $y \in A(0, r, 2qr)$ such that $br < |x - y|$ it holds that

$$G_{\overline{B}(0,r)^c}(x,y) \quad \leq \quad c \frac{\phi(r^{-2})^{1/2}}{\phi(\delta_{\overline{B}(0,r)^c}(y)^{-2})^{1/2}} g(r)$$

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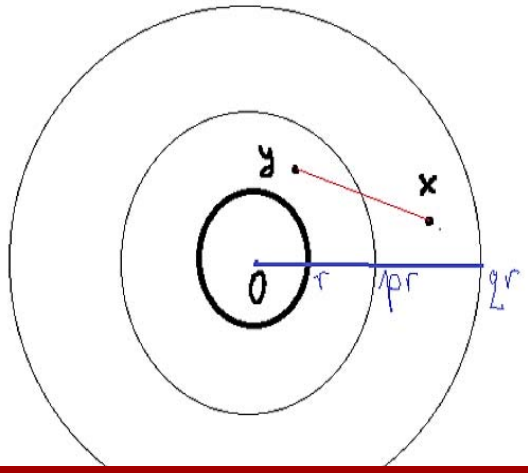

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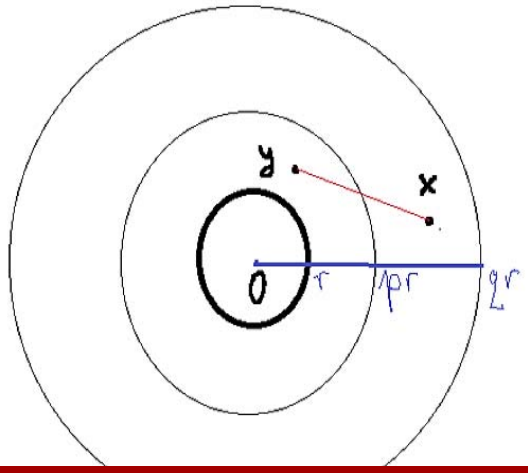

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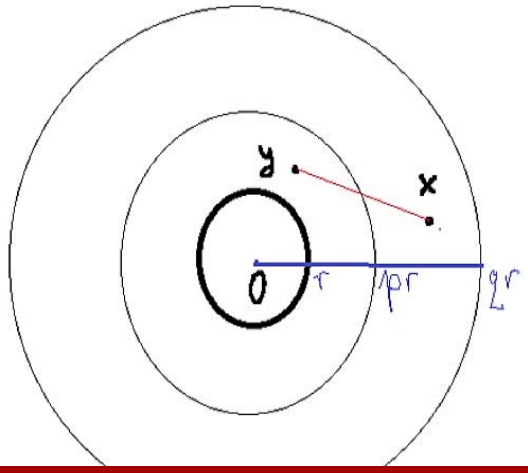

The proof uses the global, uniform BHP with explicit decay rate.

Ingredients of the proof – continuation

Upper bound for the Poisson kernel of $\overline{B}(0,r)^c$, $r \geq 1$: Let $1 < p < q < 4$. There exists $C = C(\phi, p, q) > 1$ such that for all $r \geq 1$, all $x \in A(0, pr, qr)$ and $z \in B(0, r)$ it holds that

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Upper bound for the Poisson kernel of $\overline{B}(0,r)^c$, $r \geq 1$: Let $1 < p < q < 4$. There exists $C = C(\phi, p, q) > 1$ such that for all $r \geq 1$, all $x \in A(0, pr, qr)$ and $z \in B(0, r)$ it holds that

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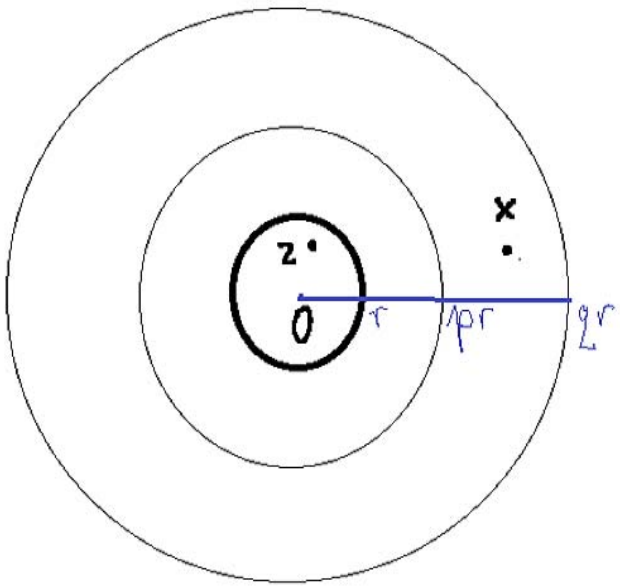

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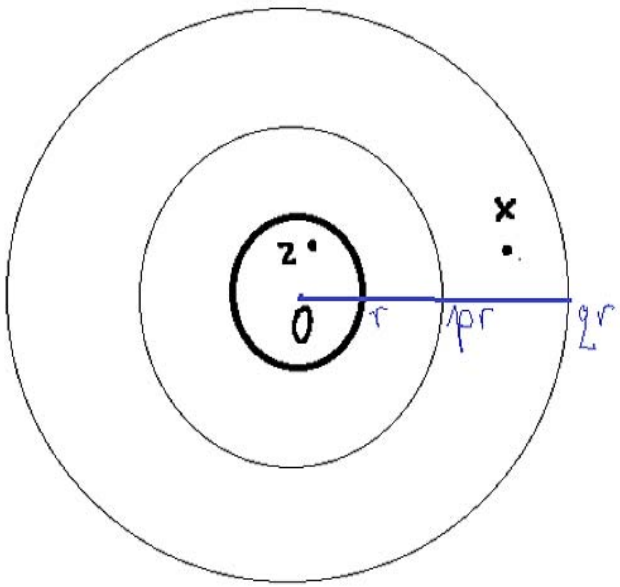

For α-stable process

$$K_{\overline{B}(0,r)^c}(x,z) = c(\alpha,d)\frac{(|x|^2-r^2)^{\alpha/2}}{(r^2-|z|^2)^{\alpha/2}}\,|x-z|^{-d}$$

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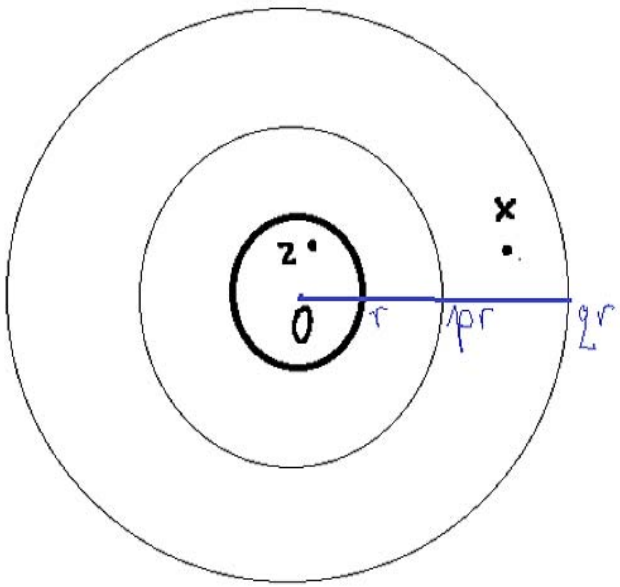

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Ingredients of the proof – continuation

Exit probability estimate: For every $a \in (1, \infty)$, there exists a positive constant $C = C(\phi, a) > 0$ such that for any $r \in (0, \infty)$ and any open set $U \subset \overline{B}(0, r)^c$ we have

$$\mathbb{P}_x \left(X_{\tau_U} \in \overline{B(0, r)} \right) \leq C r^d K_U(x, 0) \,, \qquad x \in U \cap \overline{B}(0, ar)^c \,.$$

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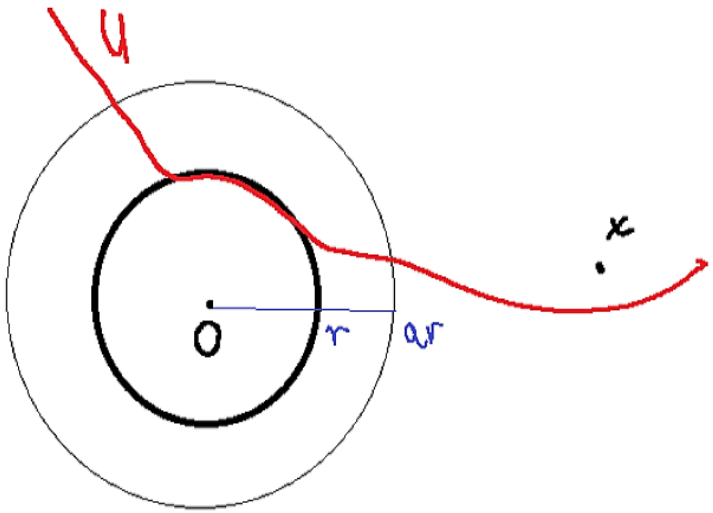

Ingredients of the proof – continuation

Regularization of the Poisson kernel in the spirit of Bogdan, Kulczycki and Kwaśnicki (2008) leading to

$$K_U(x,z) \asymp K_U(x,0)\left(\int_{U\cap B(0,2r)} K_U(y,z)\,dy + 1\right).$$

κ-fat sets

Let $\kappa \in (0, 1/2]$. An open set D is said to be κ-fat open at $Q \in \partial D$, if there exists $R > 0$ such that for each $r \in (0, R)$ there exists a point $A_r(Q)$ satisfying $B(A_r(Q), \kappa r) \subset D \cap B(Q, r)$.

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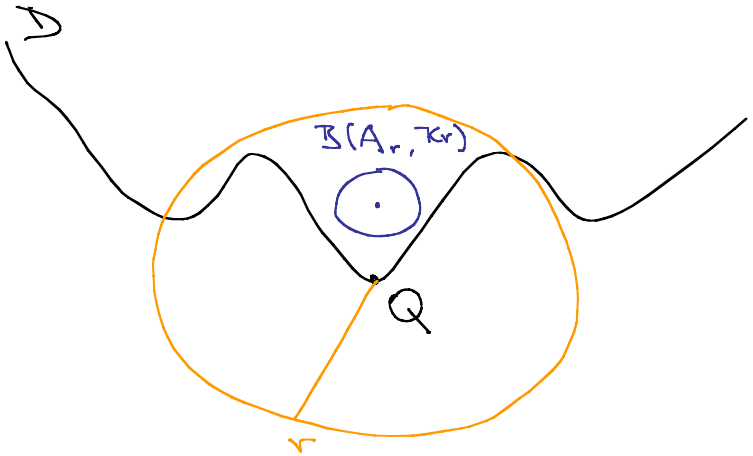

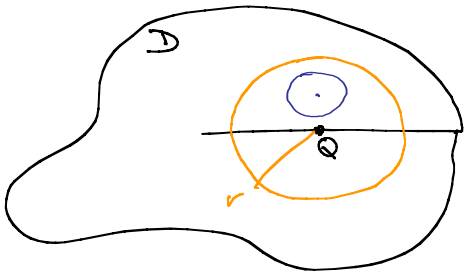

κ-fat sets at infinity

An open set D in $\mathbb{R}^d$ is κ-fat at infinity if there exists $R > 0$ such that for every $r \in [R, \infty)$ there exists $A_r \in \mathbb{R}^d$ such that $B(A_r, \kappa r) \subset D \cap \overline{B}(0, r)^c$ and $|A_r| < \kappa^{-1} r$. The pair (R, κ) will be called the characteristics of the κ-fat open set D at infinity.

κ-fat sets at infinity

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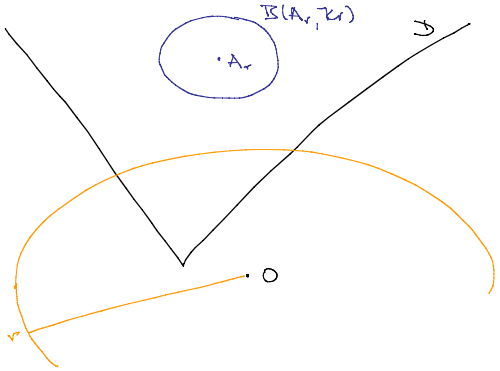

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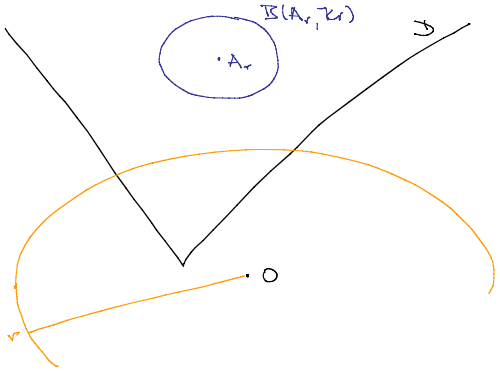


All half-space-like open sets, all exterior open sets and all infinite cones are κ-fat at infinity.

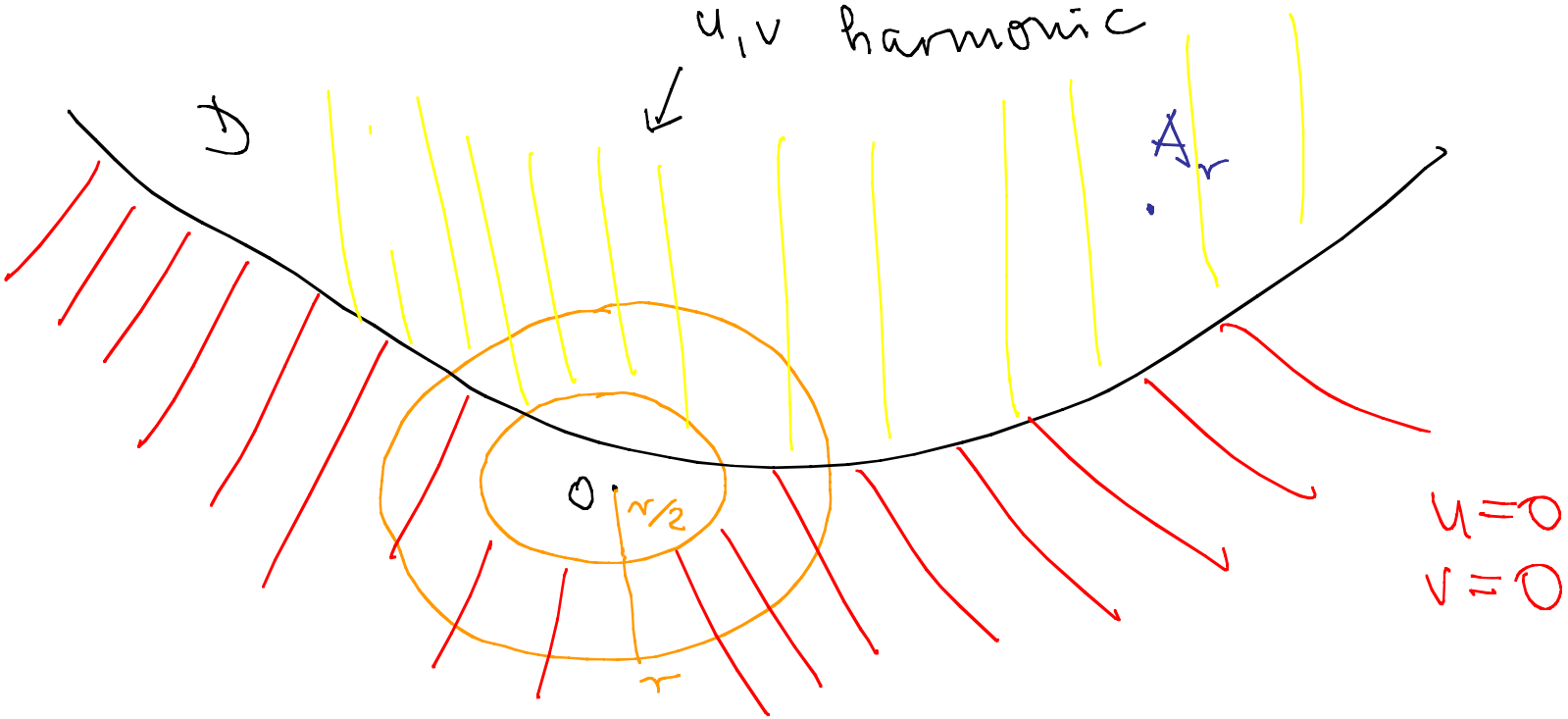
Oscillation reduction

Lemma: Let $D \subset \mathbb{R}^d$ be an open set which is κ-fat at infinity with characteristics (R, κ). There exist $C = C(\phi, d) > 0$ and $\nu = \nu(d, \phi) > 0$ such that for all $r \geq 1$ and all non-negative functions u and v on $\mathbb{R}^d$ which are regular harmonic in $D \cap \overline{B}(0, r/2)^c$, vanish in $D^c \cap \overline{B}(0, r/2)^c$ and satisfy $u(A_r) = v(A_r)$, there exists the limit

$$g = \lim_{|x| \to \infty,\, x \in D} \frac{u(x)}{v(x)},$$

and we have

$$\left| \frac{u(x)}{v(x)} - g \right| \leq C \left(\frac{|x|}{r} \right)^{-\nu}, \quad x \in D \cap \overline{B}(0, r)^c .$$


u, v harmonic
D
A_r
0
r/2
r
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Martin kernel at infinity

Fix $x_0 \in D \cap \overline{B}(0,R)^c$ and recall that

$$M_D(x,y) = \frac{G_D(x,y)}{G_D(x_0,y)}, \quad x,y \in D \cap \overline{B}(0,R)^c.$$

For $r > (2|x| \vee R)$, both functions $y \mapsto G_D(x,y)$ and $y \mapsto G_D(x_0,y)$ are regular harmonic in $D \cap \overline{B}(0,r/2)^c$ and vanish on $D^c \cap \overline{B}(0,r/2)^c$.

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This implies that every infinite Martin boundary point can be mapped to $\{\infty\}$. Since Martin kernels for different Martin boundary points are different, the infinite part of the Martin boundary is exactly $\{\infty\}$.

Harmonicity and minimality

Lemma: For each $x \in D$ and $\rho \in (0, \frac{1}{3}\delta_D(x)]$,

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Minimality: h positive harmonic, $h \leq M_D(\cdot, \infty)$. Write

$$h(x) = \int_{\partial_M^f D} M_D(x, w)\, \mu(dw) + M_D(x, \partial_\infty)\mu(\{\partial_\infty\}) \,,$$

and show that $\mu(\partial_M^f D) = 0$.

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Corollary: The Martin boundary and the minimal Martin boundary of the half space $\mathbb{H}$ with respect to X can be identified with $\partial\mathbb{H} \cup \{\infty\}$ and $M_{\mathbb{H}}(x, \infty) = w(x)/w(x_0)$ for $x \in \mathbb{H}$.

Minimal thinness

Let $A \subset D$, $T_A = \inf\{t > 0 : X_t^D \in A\}$ the hitting time to A,
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Let $z \in \partial_m D$. Then $A \subset D$ is minimally thin at z if $P_A M_D(\cdot, z) \neq M(\cdot, z)$.
Equivalently, there exists $x \in D$ such that the $M_D(\cdot, z)$-conditioned process $X^{D,z}$ will not hit A with positive probability.

Minimal thinness in the halfspace

Theorem: Assume that X is SBM satisfying (H1) and $z \in \partial\mathbb{H}$. If $A \subset \mathbb{H}$ is minimally thin in $\mathbb{H}$ at z, then

$$\int_{A\cap B(z,1)} |x-z|^{-d}\, dx < \infty\,.$$

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Theorem: Assume that X is SBM satisfying (H1) and (H2). If $A \subset \mathbb{H}$ is minimally thin in $\mathbb{H}$ at ∞, then

$$\int_{A\cap B(0,1)^c} |x-z|^{-d}\, dx < \infty\,.$$

Conversely, suppose that A is a union of Whitney cubes. If A is not minimally thin at ∞, then the above integral is infinite.