The Lamperti representation of multitype branching processes.

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CSBP in $\mathbb{R}_+$

Definition

A càdlàg Markov process $(Z, \mathbb{P}_x)$ on $\mathbb{R}_+$ is called a continuous state branching process (CSBP) if it satisfies :

$$\mathbb{E}_x(e^{-\lambda Z_t}) = \mathbb{E}_1(e^{-\lambda Z_t})^x ,$$

for all $\lambda \in \mathbb{R}_+$ and $x \in \mathbb{R}_+$.

Branching mechanism Ψ : $\mathbb{E}_x(e^{-\lambda Z_t}) = e^{-x u_t(\lambda)}$.

$$u_{t+s}(\lambda) = u_t(u_s(\lambda)) , \quad \frac{\partial u_t(\lambda)}{\partial t} = \Psi(\lambda) .$$

$$\Psi(\lambda) = a\lambda + \frac{q\lambda^2}{2} + \int_{\mathbb{R}_+} (e^{-\lambda x} - 1 + \lambda x 1_{\{x<1\}}) \, \Pi(dx) .$$

CSBP in $\mathbb{R}_+$

Theorem (Lamperti 1967)

Let X be a Lévy process with characteristic exponent Ψ and absorbed at 0, then

$$Z_t = X_{\int_0^t Z_s\, ds}$$

has a unique solution. The later is a CSBP with branching mechanism Ψ. Moreover,

$$\int_0^t Z_s\, ds = \inf\left\{u : \int_0^u \frac{dv}{X_v} > t\right\}.$$

Proofs in Caballero, Lambert, Uribe (2009)

CSBP with immigration in Caballero, Lambert, Pérez (2013)

Multitype CSBP

$$Z_t = (Z_t^{(1)}, \ldots, Z_t^{(d)}) \in \mathbb{R}_+^d$$

Definition

A Markov process $(Z, \mathbb{P}_x)$ on $\mathbb{R}_+^d$ is called a *multitype* continuous state branching process (MCSBP) if it satisfies :

$$\mathbb{E}_x(e^{-\lambda Z_t}) = \prod_{i=1}^{d} \mathbb{E}_{e_i}(e^{-\lambda Z_t})^{x_i} \; ,$$

for all $\lambda = (\lambda_1, \ldots, \lambda_d) \in \mathbb{R}_+^d$ and $x = (x_1, \ldots, x_d) \in \mathbb{R}_+^d$.
Here $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$, is the i-th unit vector of $\mathbb{R}_+^d$.

Branching mechanism ?

Lamperti representation ?

Theorem (Watanabe, 1969)

There is a function $t \mapsto u_t(\lambda) = (u_t^{(i)}(\lambda),\, i = 1, \ldots, d)$ *satisfying,*

$$u_{t+s}(\lambda) = u_t(u_s(\lambda))$$

$$\frac{\partial}{\partial t} u_t^{(i)}(\lambda) = \Psi_i(u_t^{(i)}(\lambda))\,, \quad \lambda \in \mathbb{R}_+^d\,,$$

and such that

$$\mathbb{E}_x(e^{-\lambda Z_t}) = \exp\left(-\sum_{i=1}^{d} x_i u_t^{(i)}(\lambda)\right)\,,$$

where Ψ_i *is the characteristic exponent of a* d*-dimensional Lévy process* $X^{(i)} = (X^{i,1}, \ldots, X^{i,d})$*, such that*

$X^{i,j}$*,* $i \neq j$ *are subordinators and*

$X^{j,j}$ *is a spectrally positive Lévy process.*

The d-dimensional Lévy process

$$X^{(i)} = (X^{i,1}, X^{i,2}, \dots, X^{i,d}), \quad i = 1, \dots, d,$$

has characteristic exponent

$$\Psi_i(\lambda) = a_i\lambda + \frac{1}{2}q_i\lambda_i^2 + \int_{\mathbb{R}_+^d} (e^{-\lambda x} - 1 + \lambda x 1_{\{|x|<1\}})\,\Pi_i(dx),$$

where $a_i \in \mathbb{R}^d$, $q_i \in \mathbb{R}_+$ and Π_i is a measure on $\mathbb{R}_+^d$ such that $\Pi_i(\{0\}) = 0$ and

$$\int_{\mathbb{R}_+^d} [(1 \wedge |x|^2) + \sum_{i \neq j} (1 \wedge x_j)]\Pi_i(dx) < \infty.$$

Theorem

The MCSBP Z, issued from $x = (x_1, \dots, x_d)$ admits the following representation,

$$(Z_t^{(1)}, \dots, Z_t^{(d)}) = \left(\sum_{i=1}^{d} X^{i,1}_{\int_0^t Z_s^{(i)}\, ds}, \dots, \sum_{i=1}^{d} X^{i,d}_{\int_0^t Z_s^{(i)}\, ds} \right), \quad t \geq 0\,,$$

where the processes,

$$X^{(i)} = (X^{i,1}, X^{i,2}, \dots, X^{i,d}), \quad i = 1, \dots, d\,,$$

are independent $\mathbb{R}_+^d$ valued Lévy processes with respective characteristic exponent Ψ_i, and stopped at the first time $(t_1, \dots, t_d)$, where

$$x_i + \sum_{k=1}^{d} X^{k,i}_{t_k} = 0\,.$$

The discrete space case

The same representation holds for $\mathbb{Z}_+^d$ valued (continuous time) multitype branching processes :

$$(Z_t^{(1)}, \dots, Z_t^{(d)}) = \left(\sum_{i=1}^d X^{i,1}_{\int_0^t Z_s^{(i)}\, ds}, \dots, \sum_{i=1}^d X^{i,d}_{\int_0^t Z_s^{(i)}\, ds} \right), \quad t \geq 0\,,$$

where the processes,

$$X^{(i)} = (X^{i,1}, X^{i,2}, \dots, X^{i,d})\,, \quad i = 1, \dots, d\,,$$

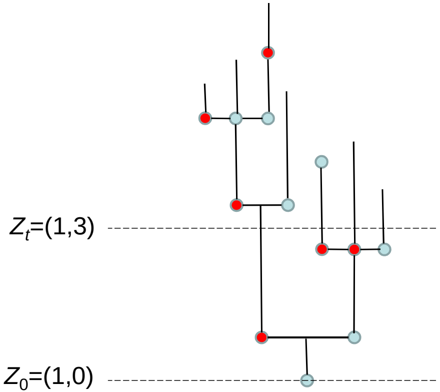
are independent $\mathbb{Z}^d$ valued Lévy processes.

$X^{i,j}$, $i \neq j$ are increasing compound Poisson processes.

$X^{j,j}$ is a compound Poisson process such that $X_t^{j,j} - X_{t-}^{j,j} \geq -1$.

The discrete space case : $d = 2$

$$(Z_t^{(1)}, Z_t^{(2)}) = \left(X^{1,1}_{\int_0^t Z_s^{(1)}\,ds} + X^{2,1}_{\int_0^t Z_s^{(2)}\,ds}, X^{2,2}_{\int_0^t Z_s^{(2)}\,ds} + X^{1,2}_{\int_0^t Z_s^{(1)}\,ds} \right)$$



The discrete space case : idea of the proof

Let $\nu = (\nu_1, \nu_2)$ be the offspring distribution of $Z = (Z^{(1)}, Z^{(2)})$ and let (λ_1, λ_2) be its birth rates.

- For $i \neq j$, let $Z_t^{i,j}$ be the total number of individuals of type j, whose parent has type i, who were born before time t.

  For $i = j$, the definition is the same except that $Z_t^{i,i}$ has a jump of size -1 when an individual of type i dies.

$$Z_t^{(1)} = Z_t^{1,1} + Z_t^{2,1}, \qquad Z_t^{(2)} = Z_t^{2,2} + Z_t^{1,2}$$

The discrete space case : idea of the proof

$$Y_t^{(1)} = (Z_t^{1,1}, Z_t^{1,2})$$

Let $T_k,\ k \geq 1$ be the jump times of the process $Y^{(1)}$.

- $(Y^{(1)}_{T_k} - Y^{(1)}_{T_{k-1}},\ k \geq 1)$ are i.i.d. with law ν_1.
- $(Y_t^{(1)},\ t \geq 0)$ and $\left(Y^{(1)}_{(\int_0^t Z_s^{(1)}\, ds)^{-1}},\ t \geq 0\right)$ have jumps of the same size.

The jumps of the process $X^{(1)} := \left(Y^{(1)}_{(\int_0^t Z_s^{(1)}\, ds)^{-1}},\ t \geq 0\right)$ are i.i.d. with law ν_1.

The discrete space case : idea of the proof

$$Y_t^{(1)} = (Z_t^{1,1}, Z_t^{1,2})$$

- The process $(Y_t^{(1)}, t \geq 0)$ increases with rate $\lambda_1 Z_t^{(1)}$:

$$\mathbb{P}(T_k - T_{k-1} \in dt \,|\, Y^{(1)}) = \lambda_1 Z^{(1)}_{T_{k-1}+t} \exp\left(- \int_0^t \lambda_1 Z^{(1)}_{T_{k-1}+s} \, ds \right) dt \, .$$

Consequence :

- The r.v.'s of the sequence $\int_{T_{k-1}}^{T_k} Z_s^{(1)} \, ds$ of jump times of $X^{(1)} = \left(Y^{(1)}_{(\int_0^t Z_s^{(1)} \, ds)^{-1}}, \, t \geq 0 \right)$ are independent and exponentially distributed with rate λ_1.

The discrete space case : idea of the proof

Through the same arguments, the sequence of jump times $\int_{T_{k-1}}^{T_k} Z_s^{(1)}\,ds$, $k \geq 1$ is independent of the sequence of jumps $(Y_{T_k}^{(1)} - Y_{T_{k-1}}^{(1)},\, k \geq 1)$ of $X^{(1)}$.

We have proved that the process

$$X^{(1)} = \left(Y^{(1)}_{(\int_0^t Z_s^{(1)}\,ds)^{-1}},\, t \geq 0 \right)$$

is a compound Poisson process with rate λ_1 and jumps measure ν_1.

The same holds for $X^{(2)} = \left(Y^{(2)}_{(\int_0^t Z_s^{(2)}\,ds)^{-1}},\, t \geq 0 \right)$ which is independent of $X^{(1)}$ since both processes do not jump at the same time.

The discrete space case : idea of the proof

Then it remains to reconstruct Z. With :

$$\begin{cases} Z^{(1)} = Z^{1,1} + Z^{2,1}\,, & Z^{(2)} = Z^{2,2} + Z^{1,2} \\ Y_t^{(1)} = (Z_t^{1,1}, Z_t^{1,2})\,, & Y_t^{(2)} = (Z_t^{2,2}, Z_t^{2,1}) \end{cases}$$

and

$$X^{(1)} = \left(Y^{(1)}_{(\int_0^t Z_s^{(1)}\, ds)^{-1}}, \, t \geq 0 \right)\,, \quad X^{(2)} = \left(Y^{(2)}_{(\int_0^t Z_s^{(2)}\, ds)^{-1}}, \, t \geq 0 \right)$$

we obtain

$$(Z_t^{(1)}, Z_t^{(2)}) = \left(X^{1,1}_{\int_0^t Z_s^{(1)}\, ds} + X^{2,1}_{\int_0^t Z_s^{(2)}\, ds}, X^{2,2}_{\int_0^t Z_s^{(2)}\, ds} + X^{1,2}_{\int_0^t Z_s^{(1)}\, ds} \right)$$

In the representation,

$$(Z_t^{(1)}, \ldots, Z_t^{(d)}) = \left(\sum_{i=1}^{d} X^{i,1}_{\int_0^t Z_s^{(i)}\, ds}, \ldots, \sum_{i=1}^{d} X^{i,d}_{\int_0^t Z_s^{(i)}\, ds} \right), \quad t \geq 0\,,$$

the Lévy processes

$$X^{(i)} = (X^{i,1}, X^{i,2}, \ldots, X^{i,d})\,, \quad i = 1, \ldots, d\,,$$

contains more information than Z.

Indeed,

$$Z_t^{i,j} = X^{i,j}_{\int_0^t Z_s^{(i)}\, ds}\,,$$

but $Z^{i,j}$ cannot be recovered from $(Z_t^{(1)}, \ldots, Z_t^{(d)})$.

First passage time of additive Lévy processes

Recall that in the representation :

$$(Z_t^{(1)},\dots,Z_t^{(d)})=\left(\sum_{i=1}^d X^{i,1}_{\int_0^t Z_s^{(i)}\,ds},\dots,\sum_{i=1}^d X^{i,d}_{\int_0^t Z_s^{(i)}\,ds}\right),\quad t\ge 0\,,$$

the processes,

$$X^{(i)}=(X^{i,1},X^{i,2},\dots,X^{i,d})\,,\quad i=1,\dots,d\,,$$

are stopped at the 'first time' $(t_1,\dots,t_d)$, where

$$\sum_{i=1}^d X^{i,j}_{t_i}=-x_j\,,\quad j=1,\dots,d\,.$$

First passage time of additive Lévy processes

Define the random set

$$A = \left\{ \exists (s_1, \dots, s_d) \in \mathbb{R}_+^d : \sum_{i=1}^{d} X_{s_i}^{i,j} = -x_j,\ j = 1, \dots, d \right\}$$

Assume that $X^{i,i}$, $1 \le i \le d$ are not subordinators, then

$$P(A) > 0\,,$$

and for any $\omega \in A$, there is a smallest solution (in $s = (s_1, \dots, s_d)$) to the system $\sum_{i=1}^{d} X_{s_i}^{i,j}(\omega) = -x_j$:

$$T_x = (T_x^{(1)}, \dots, T_x^{(d)}) = \inf\left\{ (s_1, \dots, s_d) : \sum_{i=1}^{d} X_{s_i}^{i,j} = -x_j \right\}.$$

First passage time of additive Lévy processes

Note that

$$\sum_{i=1}^{d} X_{t_i}^{i,j} = X_{t_1}^{(1)} + \dots X_{t_d}^{(d)} .$$

So that the first time

$$T_x = \inf\{(s_1, \dots, s_d) : \sum_{i=1}^{d} X_{s_i}^{i,j} = -x_j\} .$$

can be seen as a 'first passage time' of the 'spectrally positive' additive Lévy process

$$t = (t_1, \dots, t_d) \mapsto X_t := X_{t_1}^{(1)} + \dots X_{t_d}^{(d)}$$

whose Laplace exponent is

$$\mathbb{E}(\exp(-\lambda X_t)) = e^{t\Psi(\lambda)} , \quad \Psi(\lambda) = (\Psi_1(\lambda), \dots, \Psi_d(\lambda)) .$$

The d-dimensional random field

$$x = (x_1, \dots x_d) \mapsto T_x = (T_x^{(1)}, \dots, T_x^{(d)})$$

satisfies

$$T_{x+x'} \stackrel{(d)}{=} T_x + \tilde{T}_{x'}, \quad x, x' \in \mathbb{R}_+^d,$$

where $\tilde{T}$ is an independent copy of T.

Proposition

There is a map $\Phi : \mathbb{R}_+^d \to \mathbb{R}_+^d$, such that $\Phi(\Psi(\lambda)) = \lambda$, and

$$\mathbb{E}(e^{-\lambda T_x}) = e^{-\Phi(\lambda)x}, \quad \lambda \in \mathbb{R}_+^d.$$

The process Z starting at x dies with probability $e^{-\Phi(0)x}$.

Theorem

The law of $(T_x,\, x \in \mathbb{R}_+^d)$ *is given by*

$$\mathbb{P}(T_x \in dt,\, X_{t_i}^{i,j} \in dx_{i,j},\, 1 \le i,j \le d) =$$

$$\frac{\text{det}(-x_{i,j})}{t_1 t_2 \dots t_d} \prod_{i=1}^{d} \mathbb{P}(X_{t_i}^{i,j} \in dx_{i,j},\, 1 \le j \le d) dt_1 dt_2 \dots dt_d \,.$$

(*Here* $x_{i,i} := x_i$ *and* $\sum_{i=1}^d x_{i,j} = 0$.)

Extension of the real case $(d = 1)$:

$$\mathbb{P}(T_x \in dt) = -\frac{x}{t}\mathbb{P}(X_t \in dx)\,, \quad x \ge 0\,.$$

Discrete time equivalent for cyclically exchangeable random sequences : Chaumont, Liu (2013).