

Brownian Motion and Thermal Capacity

Davar Khoshnevisan and Yimin Xiao

Utah and MSU

7th Lévy Conference, July 15– 19, 2013

Outline

- Intersection of the Brownian images and thermal capacity
- Hausdorff dimension of $W(E) \cap F$
- Open problems

1. Intersection of the Brownian images and thermal capacity

Let $W := \{W(t)\}_{t \geq 0}$ denote standard d -dimensional Brownian motion where $d \geq 1$, and let E and F be compact subsets of $(0, \infty)$ and \mathbb{R}^d , respectively.

The following problems are of interest:

- 1 When is $\mathbb{P}(W(E) \cap F \neq \emptyset) > 0$?
- 2 What is $\dim_{\text{H}}(W(E) \cap F)$?

Note that

$$\{W(E) \cap F \neq \emptyset\} = \{(t, W(t)) \in E \times F \text{ for some } t > 0\}.$$

Problem 1 is an interesting problem in probabilistic potential theory.

Conditions for $\mathbb{P}(W(E) \cap F \neq \emptyset) > 0$

Necessary and sufficient condition in terms of “thermal capacity” for $\mathbb{P}(W(E) \cap F \neq \emptyset) > 0$ were proved by Weston (1978) and Doob (1984). See Weston (2012) for more information.

Weston and Taylor (1985) provided a simple-to-use condition:

$$\mathbb{P}(W(E) \cap F \neq \emptyset) \begin{cases} > 0, & \text{if } \dim_{\text{H}}(E \times F; \varrho) > d, \\ = 0, & \text{if } \dim_{\text{H}}(E \times F; \varrho) < d. \end{cases}$$

In the above, $\dim_{\text{H}}(E \times F; \varrho)$ is the Hausdorff dimension of $E \times F$ using the metric

$$\varrho((s, x); (t, y)) := \max(|t - s|^{1/2}, \|x - y\|).$$

Recall that the Hausdorff dimension of the compact set $E \times F$ in the metric ϱ is defined by

$$\dim_{\text{H}}(E \times F; \varrho) = \inf \left\{ s \geq 0 : \lim_{\varepsilon \rightarrow 0} \inf \left\{ \sum_{j=1}^{\infty} |\text{diam}_{\varrho}(E_j \times F_j)|^s \right\} < \infty \right\},$$

where the infimum is taken over all closed covers $\{E_j \times F_j\}_{j=1}^{\infty}$ of $E \times F$ with $\text{diam}_{\varrho}(E_j \times F_j) < \varepsilon$, and “ $\text{diam}_{\varrho}(\Lambda)$ ” denotes the diameter of the space-time set Λ , as measured by the metric ϱ .

As a by-product of our main result, we obtain an improved version of the result of Weston (1978) and Doob (1984).

Theorem 1.1

Suppose $F \subset \mathbb{R}^d$ is compact and has Lebesgue measure 0. Then

$$\mathbb{P}\{W(E) \cap F \neq \emptyset\} > 0 \iff \\ \exists \mu \in \mathcal{P}_d(E \times F) \text{ such that } \mathcal{E}_0(\mu) < \infty,$$

where $\mathcal{P}_d(E \times F)$ is the collection of all probability measures μ on $E \times F$ such that $\mu(\{t\} \times F) = 0$ for all $t > 0$ and the energy $\mathcal{E}_0(\mu)$ is defined by

$$\mathcal{E}_0(\mu) := \iint \frac{e^{-\|x-y\|^2/(2|t-s|)}}{|t-s|^{d/2}} \mu(ds dx) \mu(dt dy).$$

2. Hausdorff dimension of $\dim_{\mathbb{H}}(W(E) \cap F)$

- If $F = \mathbb{R}^d$, then $\dim_{\mathbb{H}} W(E) = \min\{d, 2\dim_{\mathbb{H}} E\}$ a.s.
- If $E = \mathbb{R}_+$, then

$$\dim_{\mathbb{H}}(W(\mathbb{R}_+) \cap F) = \begin{cases} \dim_{\mathbb{H}} F & \text{if } d = 1; \\ 2 + \dim_{\mathbb{H}} F - d & \text{if } d \geq 2. \end{cases}$$

- For compact sets $E \subset (0, \infty)$ and $F \subset \mathbb{R}$ ($d = 1$), Kaufman (1972) obtained $\|\dim_{\mathbb{H}}(W^{-1}(F) \cap E)\|_{L^\infty(\mathbb{P})}$, where $\|\cdot\|_{L^\infty(\mathbb{P})}$ denotes the $L^\infty(\mathbb{P})$ -norm. However, this **does not** provide information on $\dim_{\mathbb{H}}(W(E) \cap F)$.
- Hawkes (1978) considered the problem for an α -stable Lévy process in \mathbb{R} with $0 < \alpha < 1$.
- We solve this problem completely for Brownian motion (and Lévy stable processes).

The co-dimension argument

Recall that the two common ways to compute the Hausdorff dimension of a set are

- Use a covering argument for obtaining an upper bound and a capacity argument for lower bound;
- The co-dimension argument.

The “co-dimension argument” was initiated by S.J. Taylor (1966) for computing the Hausdorff dimension of the multiple points of a stable Lévy process in \mathbb{R}^d . His method was based on potential theory of Lévy processes.

Let $Z_\alpha = \{Z_\alpha(t), t \in \mathbb{R}_+\}$ be a (symmetric) stable Lévy process in \mathbb{R}^d of index $\alpha \in (0, 2]$ and let $F \subset \mathbb{R}^d$ be a Borel set. Then

$$\mathbb{P}(Z_\alpha((0, \infty)) \cap F \neq \emptyset) > 0 \iff \text{Cap}_{d-\alpha}(F) > 0,$$

where $\text{Cap}_{d-\alpha}$ is the Riesz-Bessel capacity of order $d - \alpha$.

The co-dimension argument

The above result and Frostman's theorem lead to the *stochastic co-dimension argument*: If $\dim_{\mathbb{H}} F \geq d - 2$, then

$$\begin{aligned}\dim_{\mathbb{H}} F &= \sup\{d - \alpha : Z_{\alpha}((0, \infty)) \cap F \neq \emptyset\} \\ &= d - \inf\{\alpha > 0 : F \text{ is not polar for } Z_{\alpha}\}.\end{aligned}$$

[The restriction $\dim_{\mathbb{H}} F \geq d - 2$ is caused by the fact that $Z_{\alpha}((0, \infty)) \cap F = \emptyset$ if $\dim_{\mathbb{H}} F < d - 2$.]

This method determines $\dim_{\mathbb{H}} F$ by intersecting F using a family of testing random sets.

Hawkes (1971) applied the co-dimension method for computing $\dim_{\mathbb{H}} X^{-1}(F)$ of a stable Lévy process.

The co-dimension argument

Families of testing random sets:

- ranges of symmetric stable Lévy processes;
- fractal percolation sets [Peres (1996, 1999)];
- ranges of additive Lévy processes [Khoshnevisan and X. (2003, 2005, 2009), Khoshnevisan, Shieh and X. (2008)].

Main results

To compute $\|\dim_{\mathbb{H}}(W(E) \cap F)\|_{L^\infty(\mathbb{P})}$, we distinguish two cases: $|F| > 0$ and $|F| = 0$, where $|\cdot|$ denotes the Lebesgue measure.

Theorem 2.1 [Khoshnevisan and X. (2012)]

If $F \subset \mathbb{R}^d$ ($d \geq 1$) is compact and $|F| > 0$, then

$$\|\dim_{\mathbb{H}}(W(E) \cap F)\|_{L^\infty(\mathbb{P})} = \min\{d, 2\dim_{\mathbb{H}}E\}. \quad (1)$$

If $\dim_{\mathbb{H}}E > \frac{1}{2}$ and $d = 1$, then $\mathbb{P}\{|W(E) \cap F| > 0\} > 0$.

Proof of Theorem 2.1

- 1 Thanks to the uniform Hölder continuity of $W(t)$ on bounded sets, we have

$$\dim_{\text{H}} (W(E) \cap F) \leq \min\{d, 2\dim_{\text{H}} E\}, \quad \text{a.s.}$$

This implies the upper bound in (1).

- 2 For proving the lower bound in (1), we construct a random measure on $W(E) \cap F$ and use the capacity argument.
- 3 The last part is proved by showing that the constructed random measure on $W(E) \cap F$ has a density function almost surely.

Proof of Theorem 2.1

- 1 Thanks to the uniform Hölder continuity of $W(t)$ on bounded sets, we have

$$\dim_{\text{H}} (W(E) \cap F) \leq \min\{d, 2\dim_{\text{H}} E\}, \quad \text{a.s.}$$

This implies the upper bound in (1).

- 2 For proving the lower bound in (1), we construct a random measure on $W(E) \cap F$ and use the capacity argument.
- 3 The last part is proved by showing that the constructed random measure on $W(E) \cap F$ has a density function almost surely.

Proof of Theorem 2.1

- 1 Thanks to the uniform Hölder continuity of $W(t)$ on bounded sets, we have

$$\dim_{\text{H}} (W(E) \cap F) \leq \min\{d, 2\dim_{\text{H}} E\}, \quad \text{a.s.}$$

This implies the upper bound in (1).

- 2 For proving the lower bound in (1), we construct a random measure on $W(E) \cap F$ and use the capacity argument.
- 3 The last part is proved by showing that the constructed random measure on $W(E) \cap F$ has a density function almost surely.

Theorem 2.2 [Khoshnevisan and X. (2012)]

If $F \subset \mathbb{R}^d$ ($d \geq 1$) is compact and $|F| = 0$, then

$$\begin{aligned} & \left\| \dim_{\text{H}} (W(E) \cap F) \right\|_{L^\infty(\mathbb{P})} \\ &= \sup \left\{ \gamma \geq 0 : \inf_{\mu \in \mathcal{P}_d(E \times F)} \mathcal{E}_\gamma(\mu) < \infty \right\}, \end{aligned} \quad (2)$$

where $\mathcal{P}_d(E \times F)$ is the collection of all probability measures μ on $E \times F$ such that $\mu(\{t\} \times F) = 0$ for all $t > 0$, and

$$\mathcal{E}_\gamma(\mu) := \iint \frac{e^{-\|x-y\|^2/(2|t-s|)}}{|t-s|^{d/2} \cdot \|y-x\|^\gamma} \mu(ds dx) \mu(dt dy). \quad (3)$$

Theorem 2.2 [Khoshnevisan and X. (2012)]

If $F \subset \mathbb{R}^d$ ($d \geq 1$) is compact and $|F| = 0$, then

$$\begin{aligned} & \left\| \dim_{\text{H}} (W(E) \cap F) \right\|_{L^\infty(\mathbb{P})} \\ &= \sup \left\{ \gamma \geq 0 : \inf_{\mu \in \mathcal{P}_d(E \times F)} \mathcal{E}_\gamma(\mu) < \infty \right\}, \end{aligned} \quad (2)$$

where $\mathcal{P}_d(E \times F)$ is the collection of all probability measures μ on $E \times F$ such that $\mu(\{t\} \times F) = 0$ for all $t > 0$, and

$$\mathcal{E}_\gamma(\mu) := \iint \frac{e^{-\|x-y\|^2/(2|t-s|)}}{|t-s|^{d/2} \cdot \|y-x\|^\gamma} \mu(ds dx) \mu(dt dy). \quad (3)$$

Hitting probability of random fields

We prove Theorem 2.2 by checking whether or not $W(E) \cap F$ intersects the range of an additive Lévy stable process.

Let $X^{(1)}, \dots, X^{(N)}$ be N isotropic stable processes with common stability index $\alpha \in (0, 2]$. We assume that the $X^{(j)}$'s are independent from one another, as well as from the process W , and all take their values in \mathbb{R}^d .

We assume also that $X^{(1)}, \dots, X^{(N)}$ have right-continuous sample paths with left-limits and

$$\mathbb{E} \left[e^{i \langle \xi, X^{(k)}(1) \rangle} \right] = e^{-\|\xi\|^\alpha / 2}, \quad \forall \xi \in \mathbb{R}^d.$$

Hitting probability of random fields

We prove Theorem 2.2 by checking whether or not $W(E) \cap F$ intersects the range of an additive Lévy stable process.

Let $X^{(1)}, \dots, X^{(N)}$ be N isotropic stable processes with common stability index $\alpha \in (0, 2]$. We assume that the $X^{(j)}$'s are independent from one another, as well as from the process W , and all take their values in \mathbb{R}^d .

We assume also that $X^{(1)}, \dots, X^{(N)}$ have right-continuous sample paths with left-limits and

$$\mathbb{E} \left[e^{i \langle \xi, X^{(k)}(1) \rangle} \right] = e^{-\|\xi\|^\alpha / 2}, \quad \forall \xi \in \mathbb{R}^d.$$

Hitting probability of random fields

We prove Theorem 2.2 by checking whether or not $W(E) \cap F$ intersects the range of an additive Lévy stable process.

Let $X^{(1)}, \dots, X^{(N)}$ be N isotropic stable processes with common stability index $\alpha \in (0, 2]$. We assume that the $X^{(j)}$'s are independent from one another, as well as from the process W , and all take their values in \mathbb{R}^d .

We assume also that $X^{(1)}, \dots, X^{(N)}$ have right-continuous sample paths with left-limits and

$$\mathbb{E} \left[e^{i \langle \xi, X^{(k)}(1) \rangle} \right] = e^{-\|\xi\|^\alpha / 2}, \quad \forall \xi \in \mathbb{R}^d.$$

Define the corresponding **additive stable process** $X_\alpha := \{X_\alpha(\mathbf{t}), \mathbf{t} \in \mathbb{R}_+^N\}$ as

$$X_\alpha(\mathbf{t}) := \sum_{k=1}^N X^{(k)}(t_k), \quad \forall \mathbf{t} = (t_1, \dots, t_N) \in \mathbb{R}_+^N. \quad (4)$$

Khoshnevisan, X. and Zhong (2003) showed that for any Borel set $G \subset \mathbb{R}^d$,

$$\mathbb{P}(X_\alpha(\mathbb{R}_+^N) \cap G \neq \emptyset) \begin{cases} = 0 & \text{if } \dim_{\text{H}} G < d - \alpha N, \\ > 0 & \text{if } \dim_{\text{H}} G > d - \alpha N. \end{cases} \quad (5)$$

The key ingredient for proving Theorem 2.2

Theorem 2.3

If $d > \alpha N$ and $F \subset \mathbb{R}^d$ has Lebesgue measure 0, then

$$\begin{aligned} \mathbb{P} \{ W(E) \cap X_\alpha(\mathbb{R}_+^N) \cap F \neq \emptyset \} &> 0 \\ \iff \mathcal{C}_{d-\alpha N}(E \times F) &> 0. \end{aligned}$$

Here \mathcal{C}_γ is the capacity corresponding to the energy form (3): for all compact sets $U \subset \mathbb{R}_+ \times \mathbb{R}^d$ and $\gamma \geq 0$,

$$\mathcal{C}_\gamma(U) := \left[\inf_{\mu \in \mathcal{P}_d(U)} \mathcal{E}_\gamma(\mu) \right]^{-1}. \quad (6)$$

Proof of Theorem 2.2

Lower bound: Denote

$$\Delta := \sup \left\{ \gamma \geq 0 : \inf_{\mu \in \mathcal{P}_d(E \times F)} \mathcal{E}_\gamma(\mu) < \infty \right\}. \quad (7)$$

If $\Delta > 0$ and we choose $\alpha \in (0, 2]$ and $N \in \mathbf{Z}_+$ $0 < d - \alpha N < \Delta$. Then $\mathcal{C}_{d-\alpha N}(E \times F) > 0$. It follows from Theorem 2.3 and (5) that

$$\mathbb{P} \{ \dim_{\mathbb{H}}(W(E) \cap F) \geq d - \alpha N \} > 0. \quad (8)$$

Because $d - \alpha N \in (0, \Delta)$ is arbitrary, we have

$$\| \dim_{\mathbb{H}}(W(E) \cap F) \|_{L^\infty(\mathbb{P})} \geq \Delta.$$

Upper bound: Similarly, Theorem 2.3 and (5) imply that

$$d - \alpha N > \Delta \Rightarrow \dim_{\text{H}}(W(E) \cap F) \leq d - \alpha N \quad \text{a. s. (9)}$$

Hence $\|\dim_{\text{H}}(W(E) \cap F)\|_{L^\infty(\mathbb{P})} \leq \Delta$ whenever $\Delta \geq 0$.
This proves Theorem 2.2.

Proof of Theorem 2.3

To prove the sufficiency

$$\begin{aligned} \mathcal{C}_{d-\alpha N}(E \times F) > 0 &\implies \\ \mathbb{P} \{ W(E) \cap X_\alpha(\mathbb{R}_+^N) \cap F \neq \emptyset \} &> 0, \end{aligned}$$

we define, for every $\mu \in \mathcal{P}_d(E \times F)$ and $\varepsilon > 0$, the occupation measure $Z_\varepsilon(\mu)$ by

$$Z_\varepsilon(\mu) = \int_{[1,2]^N} du \int_{E \times F} \mu(ds dx) \phi_\varepsilon(W(s) - x) \phi_\varepsilon(X_\alpha(u) - x),$$

where

$$\phi_\varepsilon(y) = \frac{1}{\varepsilon^d} \mathcal{I}_{B(0,\varepsilon)}(y).$$

The proof is based on computing $\mathbb{E}[Z_\varepsilon(\mu)]$ and $\mathbb{E}[Z_\varepsilon(\mu)^2]$.

Proof of Theorem 2.3

For proving the necessity, we assume

$$\mathbb{P} \{ W(E) \cap X_\alpha(\mathbb{R}_+^N) \cap F \neq \emptyset \} > 0,$$

and construct a probability measure $\mu \in \mathcal{P}_d(E \times F)$ such that $\mathcal{E}_{d-\alpha N}(\mu) < \infty$.

If $W(E) \cap X_\alpha(\mathbb{R}_+^N)$ is replaced by the range of a Lévy process, then we can use a stopping time argument and the strong Markov property.

The current random field case is much harder. We omit the details.

An explicit formula

Theorem 2.4 [Khoshnevisan and X. (2012)]

If $d \geq 2$ and $\dim_{\mathbb{H}}(E \times F; \varrho) \geq d$, then

$$\|\dim_{\mathbb{H}}(W(E) \cap F)\|_{L^\infty(\mathbb{P})} = \dim_{\mathbb{H}}(E \times F; \varrho) - d. \quad (10)$$

Remarks

- Eq (10) does not always hold for $d = 1$: For $E := [0, 1]$ and $F = \{0\}$, we have $\dim_{\mathbb{H}}(W(E) \cap F) = 0$ a.s., whereas $\dim_{\mathbb{H}}(E \times F; \varrho) - d = 1$.
- When $F \subset \mathbb{R}^d$ satisfies $|F| > 0$, it can be shown that

$$\dim_{\mathbb{H}}(E \times F; \varrho) = 2\dim_{\mathbb{H}}E + d.$$

Hence (1) coincides with (10) when $d \geq 2$.

Proof of Theorem 2.4

The proof relies on the following “uniform dimension result” of Kaufman (1968): If $\{W(t), t \in \mathbb{R}_+\}$ is a Brownian motion in \mathbb{R}^d with $d \geq 2$, then

$$\mathbb{P}\{\dim_{\text{H}} W(G) = 2\dim_{\text{H}} G, \forall \text{ Borel sets } G \subset \mathbb{R}_+\} = 1.$$

It is sufficient to show that for all compact sets $E \subset (0, \infty)$ and $F \subset \mathbb{R}^d$,

$$\|\dim_{\text{H}} (E \cap W^{-1}(F))\|_{L^\infty(\mathbb{P})} = \frac{\dim_{\text{H}} (E \times F; \varrho) - d}{2}. \quad (11)$$

When $d = 1$, the lower bound of (11) was found first by Kaufman (1972).

3. Further research and open problems

- As a consequence of Theorems 2.2 and 2.4, we have for $d \geq 2$,

$$\begin{aligned} & \sup \left\{ \gamma \geq 0 : \inf_{\mu \in \mathcal{P}_d(E \times F)} \mathcal{E}_\gamma(\mu) < \infty \right\} \\ &= \dim_{\mathbb{H}}(E \times F; \varrho) - d. \end{aligned}$$

Is there a “geometric” proof of this identity?

Open problems

- If we replace $W(E)$ by the set of fast points of Brownian motion

$$G(\lambda) = \left\{ t \in [0, 1] : \limsup_{h \rightarrow 0} \frac{|W(t+h) - W(t)|}{\sqrt{2h|\log h|}} \geq \lambda \right\},$$

then $\dim_{\text{H}}(G(\lambda) \cap E)$ is unknown, except the following inequalities proved in Khoshnevisan, Peres and X. (2000): For any $\lambda \in (0, 1)$ and $E \subset [0, 1]$ with $\dim_{\text{p}}E \geq \lambda^2$,

$$\dim_{\text{H}}E - \lambda^2 \leq \dim_{\text{H}}(G(\lambda) \cap E) \leq \dim_{\text{p}}E - \lambda^2, \quad \text{a.s.}$$

In the above, $\dim_{\text{p}}E$ is the packing dimension of E .

Thank you