Brownian Motion and Thermal Capacity

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Outline

- Intersection of the Brownian images and thermal capacity
- Hausdorff dimension of $W(E) \cap F$
- Open problems

1. Intersection of the Brownian images and thermal capacity

Let $W := \{W(t)\}_{t\geq 0}$ denote standard d-dimensional Brownian motion where $d \geq 1$, and let E and F be compact subsets of $(0\,,\infty)$ and $\mathbb{R}^d$, respectively.

The following problems are of interest:

1. When is $\mathbb{P}(W(E) \cap F \neq \varnothing) > 0$?
2. What is $\dim_{_\mathrm{H}}(W(E) \cap F)$?

Note that

$$\{W(E) \cap F \neq \varnothing\} = \{(t, W(t)) \in E \times F \text{ for some } \ t > 0\}.$$

Problem 1 is an interesting problem in probabilistic potential theory.

Conditions for $\mathbb{P}(W(E) \cap F \neq \emptyset) > 0$

Necessary and sufficient condition in terms of "thermal capacity" for $\mathbb{P}(W(E) \cap F \neq \emptyset) > 0$ were proved by Waston (1978) and Doob (1984). See Waston (2012) for more information.

Waston and Taylor (1985) provided a simple-to-use condition:

$$\mathbb{P}(W(E) \cap F \neq \emptyset) \begin{cases} > 0, & \text{if } \dim_{\mathrm{H}}(E \times F\,; \varrho) > d, \\ = 0, & \text{if } \dim_{\mathrm{H}}(E \times F\,; \varrho) < d. \end{cases}$$

In the above, $\dim_{\mathrm{H}}(E \times F\,; \varrho)$ is the Hausdorff dimension of $E \times F$ using the metric

$$\varrho\left((s\,, x)\,; (t\,, y)\right) := \max\left(|t - s|^{1/2},\ \|x - y\|\right).$$

Recall that the Hausdorff dimension of the compact set $E \times F$ in the metric ϱ is defined by

$$\dim_{\mathrm{H}}(E \times F\,;\varrho) = \inf \Bigg\{ s \geq 0 : \\ \lim_{\varepsilon \to 0} \inf \left\{ \sum_{j=1}^{\infty} \left|\operatorname{diam}_{\varrho}(E_j \times F_j)\right|^s \right\} < \infty \Bigg\},$$

where the infimum is taken over all closed covers $\{E_j \times F_j\}_{j=1}^{\infty}$ of $E \times F$ with $\operatorname{diam}_{\varrho}(E_j \times F_j) < \varepsilon$, and "$\operatorname{diam}_{\varrho}(\Lambda)$" denotes the diameter of the space-time set Λ, as measured by the metric ϱ.

As a by-product of our main result, we obtain an improved version of the result of Waston (1978) and Doob (1984).

Theorem 1.1

Suppose $F \subset \mathbb{R}^d$ is compact and has Lebesgue measure 0. Then

$$\mathbb{P}\{W(E) \cap F \neq \emptyset\} > 0 \iff \exists\, \mu \in \mathcal{P}_d(E \times F) \text{ such that } \mathcal{E}_0(\mu) < \infty,$$

where $\mathcal{P}_d(E \times F)$ is the collection of all probability measures μ on $E \times F$ such that $\mu(\{t\} \times F) = 0$ for all $t > 0$ and the energy $\mathcal{E}_0(\mu)$ is defined by

$$\mathcal{E}_0(\mu) := \iint \frac{e^{-\|x-y\|^2/(2|t-s|)}}{|t-s|^{d/2}}\, \mu(ds\, dx)\, \mu(dt\, dy).$$

2. Hausdorff dimension of $\dim_{\mathrm{H}}(W(E) \cap F)$

- If $F = \mathbb{R}^d$, then $\dim_{\mathrm{H}} W(E) = \min\{d, 2\dim_{\mathrm{H}} E\}$ a.s.
- If $E = \mathbb{R}_+$, then

$$\dim_{\mathrm{H}}(W(\mathbb{R}_+) \cap F) = \begin{cases} \dim_{\mathrm{H}} F & \text{if } d = 1; \\ 2 + \dim_{\mathrm{H}} F - d & \text{if } d \geq 2. \end{cases}$$

- For compact sets $E \subset (0, \infty)$ and $F \subset \mathbb{R}$ ($d = 1$), Kaufman (1972) obtained $\|\dim_{\mathrm{H}}(W^{-1}(F) \cap E)\|_{L^\infty(\mathbb{P})}$, where $\|\cdot\|_{L^\infty(\mathbb{P})}$ denotes the $L^\infty(\mathbb{P})$-norm. However, this does not provide information on $\dim_{\mathrm{H}}(W(E) \cap F)$.
- Hawkes (1978) considered the problem for an α-stable Lévy process in $\mathbb{R}$ with $0 < \alpha < 1$.
- We solve this problem completely for Brownian motion (and Lévy stable processes).

The co-dimension argument

Recall that the two common ways to compute the Hausdorff dimension of a set are

- Use a covering argument for obtaining an upper bound and a capacity argument for lower bound;
- The co-dimension argument.

The "co-dimension argument" was initiated by S.J. Taylor (1966) for computing the Hausdorff dimension of the multiple points of a stable Lévy process in $\mathbb{R}^d$. His method was based on potential theory of Lévy processes.

Let $Z_\alpha = \{Z_\alpha(t), t \in \mathbb{R}_+\}$ be a (symmetric) stable Lévy process in $\mathbb{R}^d$ of index $\alpha \in (0, 2]$ and let $F \subset \mathbb{R}^d$ be a Borel set. Then

$$\mathbb{P}(Z_\alpha((0,\infty)) \cap F \neq \varnothing) > 0 \Longleftrightarrow \mathrm{Cap}_{d-\alpha}(F) > 0,$$

where $\mathrm{Cap}_{d-\alpha}$ is the Riesz-Bessel capacity of order $d - \alpha$.

The co-dimension argument

The above result and Frostman's theorem lead to the *stochastic co-dimension argument*: If $\dim_{_H} F \geq d-2$, then

$$\begin{aligned}\dim_{_H} F &= \sup\{d-\alpha : Z_\alpha((0,\infty)) \cap F \neq \varnothing\}\\ &= d - \inf\left\{\alpha > 0 :\ F \text{ is not polar for } Z_\alpha\right\}.\end{aligned}$$

[The restriction $\dim_{_H} F \geq d-2$ is caused by the fact that $Z_\alpha((0,\infty)) \cap F = \emptyset$ if $\dim_{_H} F < d-2$.]

This method determines $\dim_{_H} F$ by intersecting F using a family of testing random sets.

Hawkes (1971) applied the co-dimension method for computing $\dim_{_H} X^{-1}(F)$ of a stable Lévy process.

The co-dimension argument

Families of testing random sets:

- ranges of symmetric stable Lévy processes;
- fractal percolation sets [Peres (1996, 1999)];
- ranges of additive Lévy processes [Khoshnevisan and X. (2003, 2005, 2009), Khoshnevisan, Shieh and X. (2008)].

Main results

To compute $\|\dim_{_{\rm H}}(W(E)\cap F)\|_{L^\infty(\mathbb{P})}$, we distinguish two cases: $|F|>0$ and $|F|=0$, where $|\cdot|$ denotes the Lebesgue measure.

Theorem 2.1 [Khoshnevisan and X. (2012)]

If $F\subset\mathbb{R}^d$ $(d\geq 1)$ is compact and $|F|>0$, then

$$\|\dim_{_{\rm H}}(W(E)\cap F)\|_{L^\infty(\mathbb{P})}=\min\{d\,,2\dim_{_{\rm H}}E\}. \qquad (1)$$

If $\dim_{_{\rm H}}E>\frac{1}{2}$ and $d=1$, then $\mathbb{P}\{|W(E)\cap F|>0\}>0$.

Proof of Theorem 2.1

1. Thanks to the uniform Hölder continuity of $W(t)$ on bounded sets, we have

$$\dim_{_H}(W(E) \cap F) \le \min\{d\,, 2\dim_{_H} E\}, \qquad \text{a.s.}$$

   This implies the upper bound in (1).

2. For proving the lower bound in (1), we construct a random measure on $W(E) \cap F$ and use the capacity argument.
3. The last part is proved by showing that the constructed random measure on $W(E) \cap F$ has a density function almost surely.

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Proof of Theorem 2.1

1. Thanks to the uniform Hölder continuity of $W(t)$ on bounded sets, we have

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Theorem 2.2 [Khoshnevisan and X. (2012)]

If $F \subset \mathbb{R}^d$ ($d \geq 1$) is compact and $|F| = 0$, then

$$\left\| \dim_{\rm H} \left(W(E) \cap F \right) \right\|_{L^\infty(\mathbb{P})} = \sup\left\{ \gamma \geq 0 : \inf_{\mu \in \mathcal{P}_d(E\times F)} \mathcal{E}_\gamma(\mu) < \infty \right\}, \qquad (2)$$

where $\mathcal{P}_d(E \times F)$ is the collection of all probability measures μ on $E \times F$ such that $\mu(\{t\} \times F) = 0$ for all $t > 0$, and

$$\mathcal{E}_\gamma(\mu) := \iint \frac{e^{-\|x-y\|^2/(2|t-s|)}}{|t-s|^{d/2} \cdot \|y-x\|^\gamma}\, \mu(ds\, dx)\, \mu(dt\, dy). \qquad (3)$$

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Hitting probability of random fields

We prove Theorem 2.2 by checking whether or not $W(E) \cap F$ intersects the range of an additive Lévy stable process.

Let $X^{(1)}, \ldots, X^{(N)}$ be N isotropic stable processes with common stability index $\alpha \in (0\,, 2]$. We assume that the $X^{(j)}$'s are independent from one another, as well as from the process W, and all take their values in $\mathbb{R}^d$.

We assume also that $X^{(1)}, \ldots, X^{(N)}$ have right-continuous sample paths with left-limits and

$$\mathbb{E}\left[e^{i\langle \xi, X^{(k)}(1)\rangle}\right] = e^{-\|\xi\|^\alpha/2}, \quad \forall\ \xi \in \mathbb{R}^d.$$

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$$\mathbb{E}\left[e^{i\langle \xi, X^{(k)}(1)\rangle}\right] = e^{-\|\xi\|^\alpha/2}, \quad \forall\ \xi \in \mathbb{R}^d.$$

Define the corresponding additive stable process $X_\alpha := \{X_\alpha(\boldsymbol{t}), \boldsymbol{t} \in \mathbb{R}^N_+\}$ as

$$X_\alpha(\boldsymbol{t}) := \sum_{k=1}^{N} X^{(k)}(t_k), \quad \forall\, \boldsymbol{t} = (t_1, \ldots, t_N) \in \mathbb{R}^N_+. \tag{4}$$

Khoshnevisan, X. and Zhong (2003) showed that for any Borel set $G \subset \mathbb{R}^d$,

$$\mathbb{P}\Big(X_\alpha(\mathbb{R}^N_+) \cap G \neq \varnothing\Big) \begin{cases} = 0 & \text{if } \dim_{_\mathrm{H}} G < d - \alpha N, \\ > 0 & \text{if } \dim_{_\mathrm{H}} G > d - \alpha N. \end{cases} \tag{5}$$

The key ingredient for proving Theorem 2.2

Theorem 2.3

If $d > \alpha N$ and $F \subset \mathbb{R}^d$ has Lebesgue measure 0, then

$$\mathbb{P}\left\{W(E) \cap X_\alpha(\mathbb{R}_+^N) \cap F \neq \varnothing\right\} > 0$$
$$\iff \mathcal{C}_{d-\alpha N}(E \times F) > 0.$$

Here $\mathcal{C}_\gamma$ is the capacity corresponding to the energy form (3): for all compact sets $U \subset \mathbb{R}_+ \times \mathbb{R}^d$ and $\gamma \geq 0$,

$$\mathcal{C}_\gamma(U) := \left[\inf_{\mu \in \mathcal{P}_d(U)} \mathcal{E}_\gamma(\mu)\right]^{-1}. \tag{6}$$

Proof of Theorem 2.2

Lower bound: Denote

$$\Delta := \sup\left\{\gamma \geq 0 : \inf_{\mu \in \mathcal{P}_d(E\times F)} \mathcal{E}_\gamma(\mu) < \infty\right\}. \tag{7}$$

If $\Delta > 0$ and we choose $\alpha \in (0\,,2]$ and $N \in \mathbf{Z}_+$ $0 < d - \alpha N < \Delta$. Then $\mathcal{C}_{d-\alpha N}(E \times F) > 0$. It follows from Theorem 2.3 and (5) that

$$\mathbb{P}\left\{\dim_{_\mathrm{H}}(W(E) \cap F) \geq d - \alpha N\right\} > 0. \tag{8}$$

Because $d - \alpha N \in (0\,,\Delta)$ is arbitrary, we have

$$\|\dim_{_\mathrm{H}}(W(E) \cap F)\|_{L^\infty(\mathbb{P})} \geq \Delta.$$

Upper bound: Similarly, Theorem 2.3 and (5) imply that

$$d - \alpha N > \Delta \;\Rightarrow\; \dim_{\mathrm{H}} \left(W(E) \cap F\right) \le d - \alpha N \qquad \text{a. s.} \quad (9)$$

Hence $\|\dim_{\mathrm{H}}(W(E) \cap F)\|_{L^\infty(\mathbb{P})} \le \Delta$ whenever $\Delta \ge 0$. This proves Theorem 2.2.

Proof of Theorem 2.3

To prove the sufficiency

$$\mathcal{C}_{d-\alpha N}(E \times F) > 0 \implies \mathbb{P}\left\{W(E) \cap X_\alpha(\mathbb{R}_+^N) \cap F \neq \emptyset\right\} > 0,$$

we define, for every $\mu \in \mathcal{P}_d(E \times F)$ and $\varepsilon > 0$, the occupation measure $Z_\varepsilon(\mu)$ by

$$Z_\varepsilon(\mu) = \int_{[1,2]^N} d\boldsymbol{u} \int_{E\times F} \mu(ds\,dx)\, \phi_\varepsilon(W(s)-x)\phi_\varepsilon(X_\alpha(\boldsymbol{u})-x),$$

where

$$\phi_\varepsilon(y) = \frac{1}{\varepsilon^d}\mathcal{I}_{B(0,\varepsilon)}(y).$$

The proof is based on computing $\mathbb{E}[Z_\varepsilon(\mu)]$ and $\mathbb{E}[Z_\varepsilon(\mu)^2]$.

Proof of Theorem 2.3

For proving the necessity, we assume

$$\mathbb{P}\left\{W(E)\cap X_\alpha(\mathbb{R}_+^N)\cap F\neq\varnothing\right\}>0,$$

and construct a probability measure $\mu\in\mathcal{P}_d(E\times F)$ such that $\mathcal{E}_{d-\alpha N}(\mu)<\infty$.

If $W(E)\cap X_\alpha(\mathbb{R}_+^N)$ is replaced by the range of a Lévy process, then we can use a stoping time argument and the strong Markov property.

The current random field case is much harder. We omit the details.

An explicit formula

Theorem 2.4 [Khoshnevisan and X. (2012)]

If $d \geq 2$ and $\dim_{\mathrm{H}}(E \times F\,; \varrho) \geq d$, then

$$\left\| \dim_{\mathrm{H}}(W(E) \cap F) \right\|_{L^\infty(\mathbb{P})} = \dim_{\mathrm{H}}(E \times F\,; \varrho) - d. \quad (10)$$

Remarks

- Eq (10) does not always hold for $d = 1$: For $E := [0\,,1]$ and $F = \{0\}$, we have $\dim_{\mathrm{H}}(W(E) \cap F) = 0$ a.s., whereas $\dim_{\mathrm{H}}(E \times F\,; \varrho) - d = 1$.
- When $F \subset \mathbb{R}^d$ satisfies $|F| > 0$, it can be shown that

$$\dim_{\mathrm{H}}(E \times F\,; \varrho) = 2\dim_{\mathrm{H}} E + d.$$

Hence (1) coincides with (10) when $d \geq 2$.

Proof of Theorem 2.4

The proof replies on the following "uniform dimension result" of Kaufman (1968): If $\{W(t), t \in \mathbb{R}_+\}$ is a Brownian motion in $\mathbb{R}^d$ with $d \geq 2$, then

$$\mathbb{P}\{\dim_{_H} W(G) = 2\dim_{_H} G, \ \forall \text{ Borel sets } \ G \subset \mathbb{R}_+\} = 1.$$

It is sufficient to show that for all compact sets $E \subset (0, \infty)$ and $F \subset \mathbb{R}^d$,

$$\left\| \dim_{_H} \left(E \cap W^{-1}(F)\right) \right\|_{L^\infty(\mathbb{P})} = \frac{\dim_{_H}(E \times F\,;\varrho) - d}{2}. \tag{11}$$

When $d = 1$, the lower bound of (11) was found first by Kaufman (1972).

- As a consequence of Theorems 2.2 and 2.4, we have for $d \geq 2$,

$$\sup\left\{\gamma \geq 0 : \inf_{\mu \in \mathcal{P}_d(E\times F)} \mathcal{E}_\gamma(\mu) < \infty\right\}$$
$$= \dim_{_\mathrm{H}}(E \times F\,;\varrho) - d.$$

Is there a "geometric" proof of this identity?

Open problems

- If we replace $W(E)$ by the set of fast points of Brownian motion

$$G(\lambda) = \left\{ t \in [0,1] : \limsup_{h \to 0} \frac{|W(t+h) - W(t)|}{\sqrt{2h|\log h|}} \geq \lambda \right\},$$

then $\dim_{_H}\big(G(\lambda) \cap E\big)$ is unknown, except the following inequalities proved in Khoshnevisan, Peres and X. (2000): For any $\lambda \in (0,1)$ and $E \subset [0,1]$ with $\dim_{_P} E \geq \lambda^2$,

$$\dim_{_H} E - \lambda^2 \leq \dim_{_H}\big(G(\lambda) \cap E\big) \leq \dim_{_P} E - \lambda^2, \qquad \text{a.s.}$$

In the above, $\dim_{_P} E$ is the packing dimension of E.

Thank you