

Reflected spectrally negative stable processes and fractional Cauchy problems

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Abstract

In this talk we will show how to explicitly compute the transition densities of a spectrally negative stable process with index greater than one, reflected at its infimum. First we derive the forward equation using the theory of sun-dual semigroups. The resulting forward equation is a boundary value problem on the positive half-line that involves a negative Riemann-Liouville fractional derivative in space, and a fractional reflecting boundary condition at the origin. Then we apply numerical methods to explicitly compute the transition density of this space-inhomogeneous Markov process, for any starting point, to any desired degree of accuracy. Finally, we discuss an application to fractional Cauchy problems, which involve a positive Caputo fractional derivative in time.

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New Books

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Reflected stable process

Take a spectrally negative stable Lévy process Y_t with

$$\mathbb{E}[e^{ikY_t}] = e^{t(ik)^\alpha}$$

for some $1 < \alpha \leq 2$, and define the reflected process

$$Z_t = Y_t - \inf\{Y_s : 0 \leq s \leq t\}.$$

Then $Z_t \geq 0$ is a Markov process on the real line. The semigroup

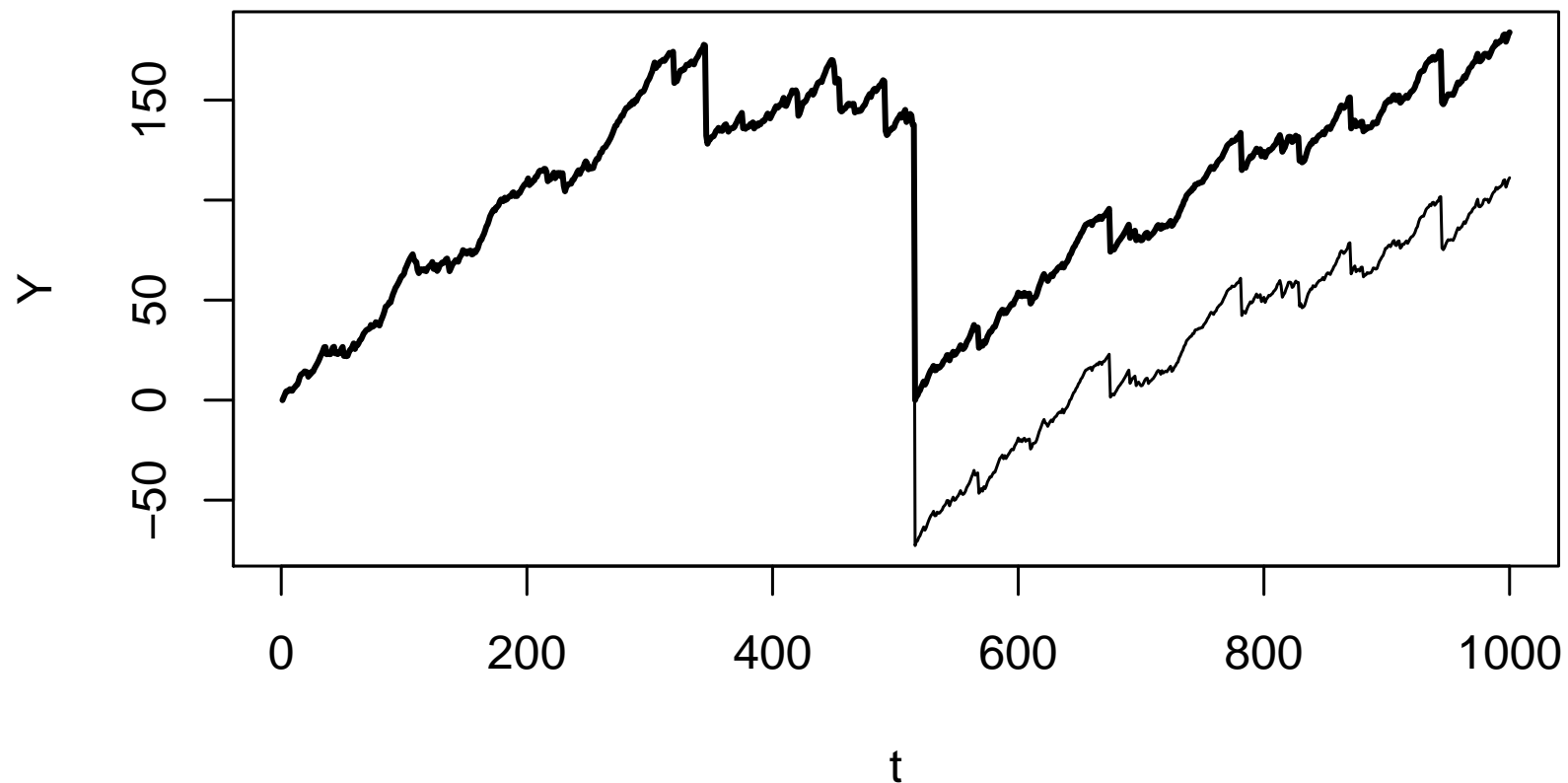
$$T_t f(x) = \mathbb{E}[f(Z_{t+s}) | Z_s = x]$$

on the Banach space $C_\infty(\mathbb{R})$ satisfies the Feller property

$$\|T_t f - f\| := \sup\{|T_t f(x) - f(x)| : x \in \mathbb{R}\} \rightarrow 0 \quad \text{as } t \rightarrow 0.$$

Reflected stable sample path

Stable process Y_t (thin line) with $\alpha = 1.3$ and reflected stable Z_t (thick line). See Appendix for R code.



Fractional Calculus

Positive and negative Riemann-Liouville fractional integrals

$$I_x^\alpha f(x) := \frac{1}{\Gamma(\alpha)} \int_0^\infty f(x-y)y^{\alpha-1} dy = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^x f(y)(x-y)^{\alpha-1} dy$$

$$I_{-x}^\alpha f(x) := \frac{1}{\Gamma(\alpha)} \int_0^\infty f(x+y)y^{\alpha-1} dy = \frac{1}{\Gamma(\alpha)} \int_x^\infty f(y)(y-x)^{\alpha-1} dy$$

Positive and negative Riemann-Liouville fractional derivatives

$$D_x^\alpha f(x) := \frac{d^n}{dx^n} I_x^{n-\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_{-\infty}^x f(y)(x-y)^{n-\alpha-1} dy$$

$$D_{-x}^\alpha f(x) := \frac{d^n}{d(-x)^n} I_{-x}^{n-\alpha} f(x) = \frac{(-1)^n}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_x^\infty f(y)(y-x)^{n-\alpha-1} dy$$

Positive Caputo fractional derivative

$$\partial_x^\alpha f(x) := I_x^{n-\alpha} \frac{d^n}{dx^n} f(x) = \frac{1}{\Gamma(n-\alpha)} \int_0^x (x-y)^{n-1-\alpha} f^{(n)}(y) dy$$

where $\alpha > 0$ and $n-1 < \alpha < n$. If $\alpha \in (1, 2)$, then $n = 2$.

Forward equation for Y_t

The stable Lévy process Y_t has pdf $p(x, t)$ with Fourier transform

$$\hat{p}(k, t) = \int e^{ikx} p(x, t) dx = e^{t(ik)^\alpha}$$

so that

$$\frac{\partial}{\partial t} \hat{p}(k, t) = (ik)^\alpha \hat{p}(k, t).$$

Since $(ik)^\alpha \hat{f}(k)$ is the Fourier transform of $D_{-x}^\alpha f(x)$, we have

$$\frac{\partial}{\partial t} p(x, t) = D_{-x}^\alpha p(x, t).$$

This is the forward equation of the Markov process Y_t .

It describes the forward evolution of the pdf.

Forward semigroup and its generator

The forward semigroup is defined by

$$T_t^* f(x) := \mathbb{E}[f(x - Y_t)] = \int f(x - y)p(y, t) dy.$$

If X has pdf $f(x)$ independent of Y_t , $X + Y_t$ has density $T_t^* f(x)$.

The generator of the forward semigroup T_t^* is

$$A^* f(x) := \lim_{t \downarrow 0} \frac{T_t^* f(x) - f(x)}{t} = D_{-x}^\alpha f(x)$$

since this expression has Fourier transform

$$\lim_{t \downarrow 0} \frac{e^{t(ik)^\alpha} \hat{f}(k) - \hat{f}(k)}{t} = (ik)^\alpha \hat{f}(k).$$

Then $p(x, t) = T_t^* f(x)$ solves the Cauchy problem

$$\frac{\partial}{\partial t} p(x, t) = D_{-x}^\alpha p(x, t); \quad p(x, 0) = f(x)$$

for any $f \in D(A^*)$.

Backward semigroup

The backward semigroup $T_t f(x) := \mathbb{E}[f(x+Y_t)] = \int f(x-y)p(-y, t) dy$.

If $f(x) = I_B(x)$ then $T_t f(x) = \mathbb{P}[x + Y_t \in B]$ with initial state x .

Note $p(-y, t)$ has Fourier transform $\hat{p}(-k, t) = e^{t(-ik)^\alpha}$

Can show $(-ik)^\alpha \hat{f}(k)$ is the Fourier transform of $D_x^\alpha f(x)$

Then $A = D_x^\alpha$ is the backward generator of Y_t

Adjoint relation: For suitably nice functions

$$\int_{-\infty}^{\infty} D_x^\alpha f(x)g(x) dx = \int_{-\infty}^{\infty} f(x)D_{-x}^\alpha g(x) dx.$$

Easy to check using integration by parts.

Dual semigroups for Z_t

Backward semigroup $T_t f(x) := \mathbb{E}[f(x + Z_t)]$ on $\mathbb{B} := C_\infty[0, \infty)$.

Dual semigroup T_t^* on the dual space $\mathbb{B}^* := \mathcal{M}_b[0, \infty)$ of (signed) bounded Borel measures with the total variation norm.

Then $\int T_t f(x) \mu(dx) = \int f(x) [T_t^* \mu](dx)$ for $f \in \mathbb{B}$ and $\mu \in \mathbb{B}^*$

and $\int A f(x) \mu(dx) = \int f(x) [A^* \mu](dx)$ for $f \in D(A)$ and $\mu \in D(A^*)$.

Sun-dual semigroup: $\mathbb{B}^\odot := \{\mu \in X^* : \lim_{t \downarrow 0} \|T_t^* \mu - \mu\| = 0\}$

Needed because T_t^* is not strongly continuous on \mathbb{B}^* .

Backward generator for Z_t

Fourier transform methods are not useful on $C_\infty[0, \infty)$.

Bernyk, Dalang and Peskir (2011) compute $Af(x) = \partial_x^\alpha f(x)$ for $f \in C_A := \{f \in S_b : \partial_x^\alpha f \in C_\infty[0, \infty)\}$, where

$$S_b := \left\{ f \in C_\infty[0, \infty) : f'' \in C(0, \infty), f''(x) = O(1) \text{ as } x \rightarrow \infty, \right. \\ \left. f''(x) = O(x^{\alpha-2}) \text{ as } x \rightarrow 0, f' \in C_b(0, \infty), f'(0+) = 0 \right\}$$

Their proof uses the Lévy-Itô formula.

Patie and Simon (2012) extend to $f'(0+) \neq 0$, compute $D(A)$.

We show T_t is analytic, and C_A is a core.

Proof: Show that $\|\lambda R(\lambda, A)g\| \leq M\|g\|$ for $g \in C_\infty[0, \infty)$ and $\operatorname{Re} \lambda > 0$, where $R(\lambda, A)$ is the resolvent.

Forward generator for Z_t

The sun-dual space $\mathbb{B}^\odot = \mathcal{M}_{ac}[0, \infty)$, all μ with Lebesgue density

Sun-dual generator $A^\odot f(x) = D_{-x}^\alpha f(x)$ similar to Y_t , but here

$$\mathbb{D}(A^\odot) = \{f \in L^1[0, \infty) : D_{-x}^\alpha f(x) \in L^1[0, \infty), D_{-x}^{\alpha-1} f(0) = 0\}$$

Then the forward equation for the reflected stable process Z_t is

$$\frac{\partial}{\partial t} p(x, t) = D_{-y}^\alpha p(x, t); \quad D_{-x}^{\alpha-1} p(0, t) = 0.$$

Similar to Y_t except for the reflecting boundary condition.

This solves an open problem from Baeumer, M and Nane (2009) [What is the Markov process with this forward equation?].

For $\mu_n \in \mathbb{B}^\odot$ and $\mu \in \mathbb{B}^*$ if $\mu_n \rightarrow \mu$ then $T_t^\odot \mu_n \rightarrow T_t^* \mu$ in \mathbb{B}^* (vague convergence). Use to prove numerics converge when $\mu = \delta_x$.

Adjoint calculations

The backward/forward generators for Y_t are adjoints:

$$Y_t : \int_{-\infty}^{\infty} D_{-x}^{\alpha} f(x) g(x) dx = \int_{-\infty}^{\infty} f(x) D_x^{\alpha} g(x) dx.$$

Integrate by parts, use the definitions.

The backward/forward generators for Z_t are also adjoints:

$$Z_t : \int_0^{\infty} D_{-x}^{\alpha} f(x) g(x) dx = \int_0^{\infty} f(x) \partial_x^{\alpha} g(x) dx.$$

Integrate by parts, use the boundary condition $D_{-x}^{\alpha-1} g(x) = 0$.

Lesson: Adjoint of D_{-x}^{α} depends on the space.

No-Flux Boundary Condition

Since $1 = \mathbb{P}(Z_t \geq 0)$ is constant for all $t > 0$,

$$\begin{aligned} 0 &= \frac{\partial}{\partial t} \mathbb{P}(Z_t \geq 0) = \int_0^\infty \frac{\partial}{\partial t} p(x, t) dx \\ &= \int_0^\infty D_{-x}^\alpha p(x, t) dx \\ &= \int_0^\infty \frac{d}{dx} \left(\frac{1}{\Gamma(2 - \alpha)} \frac{d}{dx} \int_x^\infty p(y, t) (y - x)^{1 - \alpha} dy \right) dx \\ &= - \int_0^\infty \frac{d}{dx} D_{-x}^{\alpha - 1} p(x, t) dx \\ &= D_{-x}^{\alpha - 1} p(0, t). \end{aligned}$$

The boundary condition keeps all probability mass in $[0, \infty)$.

Lévy-like Markov processes

Patie and Simon (2012) show that Z_t has backward generator

$$Af(x) = f'(0) \frac{x^{1-\alpha}}{\Gamma(2-\alpha)} + \int_0^x f''(x-y) \frac{y^{1-\alpha}}{\Gamma(2-\alpha)} dy.$$

Integrate by parts to get the pseudo-differential operator form

$$Af(x) = b(x)f'(x) + \int [f(x+y) - f(x) - yf'(x)] \phi(x, dy).$$

The drift coefficient $b(x) = x^{1-\alpha}/\Gamma(2-\alpha)$ blows up as $x \downarrow 0$.

The jump intensity

$$\phi(x, dy) = \frac{\alpha(\alpha-1)}{\Gamma(2-\alpha)} |y|^{-1-\alpha} dy I(-x < y < 0) + \frac{\alpha-1}{\Gamma(2-\alpha)} x^{-\alpha} \delta_{-x}(dy)$$

keeps negative jumps inside the state space $[0, \infty)$.

Compare Y_t Lévy measure $\phi(dy) = \frac{\alpha(\alpha-1)}{\Gamma(2-\alpha)} |y|^{-1-\alpha} dy I(y < 0)$.

Reflected Brownian motion

We show Z_{t+s} given $Z_s = x$ has a smooth density $y \mapsto p(x, y, t)$.

Reflected Brownian motion transition density solves the diffusion equation with a reflecting boundary condition

$$\frac{\partial}{\partial y} p(x, y, t) \Big|_{y=0+} := \lim_{h \rightarrow 0+} \frac{p(x, y+h, t) - p(x, y, t)}{h} \Big|_{y=0} = 0$$

In the stable case, the reflecting boundary condition is

$$D_{-y}^{\alpha-1} p(x, y, t) \Big|_{y=0+} := \lim_{h \rightarrow 0+} \frac{1}{h^{\alpha-1}} \sum_{k=0}^{\infty} w_k^{\alpha-1} p(x, y+kh, t) \Big|_{y=0} = 0$$

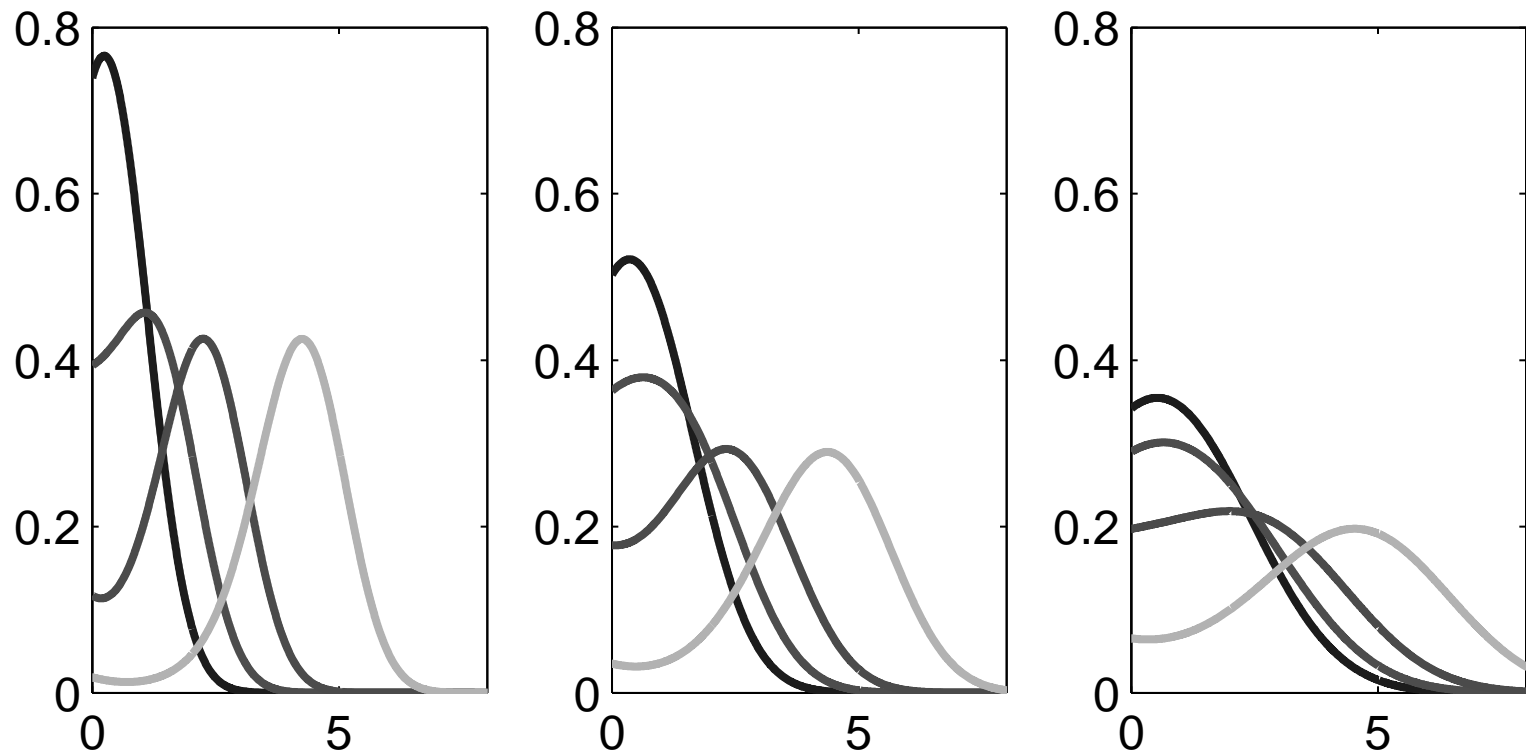
where

$$w_k^\alpha := (-1)^k \frac{\Gamma(\alpha+1)}{k! \Gamma(\alpha-k+1)}$$

When $\alpha = 2$, $w_0^{\alpha-1} = 1$, $w_1^{\alpha-1} = -1$, and $w_k^{\alpha-1} = 0$ for $k > 1$.

Reflected stable transition density

Transition densities $y \mapsto p(x, y, t)$ with index $\alpha = 1.8$ and initial state $x = 0, 1, 2, 4$ (left to right) at times $t = 0.5$ (left), $t = 1$ (middle), and $t = 2$ (right). See Appendix for MATLAB code.



Fractional Cauchy problems

Suppose $T_t f(x) = \mathbb{E}[f(X_t)|X_0 = x]$ is a Feller semigroup

Then $p(x, t) = \mathbb{E}[f(X_{Z_t})|X_0 = x]$ solves the fractional Cauchy problem

$$\partial_t^\beta p(x, t) = Lp(x, t); \quad p(x, 0) = f(x)$$

for any $f \in \mathcal{D}(L)$, when $\beta = 1/\alpha$ for some $\alpha \in (1, 2)$.

Proof: $D_t = \inf\{r > 0; Y_r > t\}$ is a β -stable subordinator whose inverse $E_t = \inf\{u > 0; D_u > t\}$ is also the supremum process (inverse of the inverse) of Y_t .

The result is known for the inverse stable subordinator E_t .

But Prop VI.3 in Bertoin (1996) implies $E_t = Z_t$ in distribution.

Open problems

- General reflected stable processes (w/ drift)
- Compute forward generator
- Compute transition density
- Understand fractional boundary conditions
- Particle tracking codes

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Duality and adjoints

For a Markov process Z_t with transition density $p(x, y, t)$, let

$$u(t, x) = T_t^* u_0(x) = \int p(x, y, t) u_0(y) dy$$

$$v(t, y) = T_t v_0(x) = \int p(x, y, t) v_0(x) dx$$

$$\begin{aligned} H(s, t) &= \int v(s, y) u(t, y) dy \\ &= \int \int p(x, y, s) v_0(x) dx \int p(y, z, t) u_0(z) dz dy \\ &= \int \int p(x, z, s + t) v_0(x) u_0(z) dx dz \end{aligned}$$

Then $\partial H / \partial s = \partial H / \partial t$ so

$$\begin{aligned} \int A v(s, y) u(t, y) dy &= \int \frac{\partial}{\partial s} v(s, y) u(t, y) dy \\ &= \int v(s, y) \frac{\partial}{\partial t} u(t, y) dy = \int v(s, y) A^* u(t, y) dy \end{aligned}$$

R code for reflected stable sample path

```
# Plot stable  $Y_t$  with characteristic function  $\exp(t(ik)^a)$ 
# and the reflected stable process  $Z_t=Y_t-\inf\{Y_u:0\leq u\leq t\}$ 
#
# You need to install the fBasics package on your R platform.
# Try Packages > Load package to see if fBasics is available.
# If not then use Packages > Install package(s)
library(fBasics)
t=seq(1:1000)
a=1.3
g=(abs(cos(pi*a/2)))^(1/a)
y=rstable(t,alpha=a,beta=-1.0,gamma=g,delta=0.0,pm=1)
Y=cumsum(y)
Z=Y-cummin(Y)
plot(t,Y,type="l",ylim=c(min(Y),max(Z)))
lines(t,Z,lwd=2)
```

MATLAB code for reflected stable pdf

```
%%% Matlab script to compute p(x,y,t)
%% enter variables
    alpha=1.2; ymax=12; N=1200; t=[0,.5,1,2]; x=2;
%% initialise parameters
    h=ymax/N; y=(h:h:ymax)';
    u0=zeros(N,1);u0(floor(x/h)+1)=1/h; % initial condition
%% Make Grunwald matrix
    w=ones(1,N+1);
    for k=1:N
        w(k+1)=w(k)*(k-alpha-1)/k;
    end
    w=w/h^alpha;
    M=spdiags(repmat(w,N,1),-1:1:N-1,N,N); %enter w's along diagonal
    M(1,:)=-cumsum(w(1:N))'; %change first row for BC
%% Solve ODE system
    [~,p]=ode113(@(t,u) M*u,t,u0);
```