Reflected spectrally negative stable processes and fractional Cauchy problems

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Abstract

In this talk we will show how to explicitly compute the transition densities of a spectrally negative stable process with index greater than one, reflected at its infimum. First we derive the forward equation using the theory of sun-dual semigroups. The resulting forward equation is a boundary value problem on the positive half-line that involves a negative Riemann-Liouville fractional derivative in space, and a fractional reflecting boundary condition at the origin. Then we apply numerical methods to explicitly compute the transition density of this space-inhomogeneous Markov process, for any starting point, to any desired degree of accuracy. Finally, we discuss an application to fractional Cauchy problems, which involve a positive Caputo fractional derivative in time.

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New Books

Stochastic Models for Fractional Calculus
Mark M. Meerschaert and Alla Sikorskii
De Gruyter Studies in Mathematics **43**, 2012
Advanced graduate textbook
(Introduction to Lévy processes and fractional calculus)
ISBN 978-3-11-025869-1

Mathematical Modeling, 4th Edition
Mark M. Meerschaert
Academic Press, Elsevier, 2013
Advanced undergraduate / beginning graduate textbook
(new sections on particle tracking and anomalous diffusion)
ISBN 978-0-12-386912-8

Reflected stable process

Take a spectrally negative stable Lévy process Y_t with

$$\mathbb{E}[e^{ikY_t}] = e^{t(ik)^\alpha}$$

for some $1 < \alpha \leqslant 2$, and define the reflected process

$$Z_t = Y_t - \inf\{Y_s : 0 \leqslant s \leqslant t\}.$$

Then $Z_t \geqslant 0$ is a Markov process on the real line. The semigroup

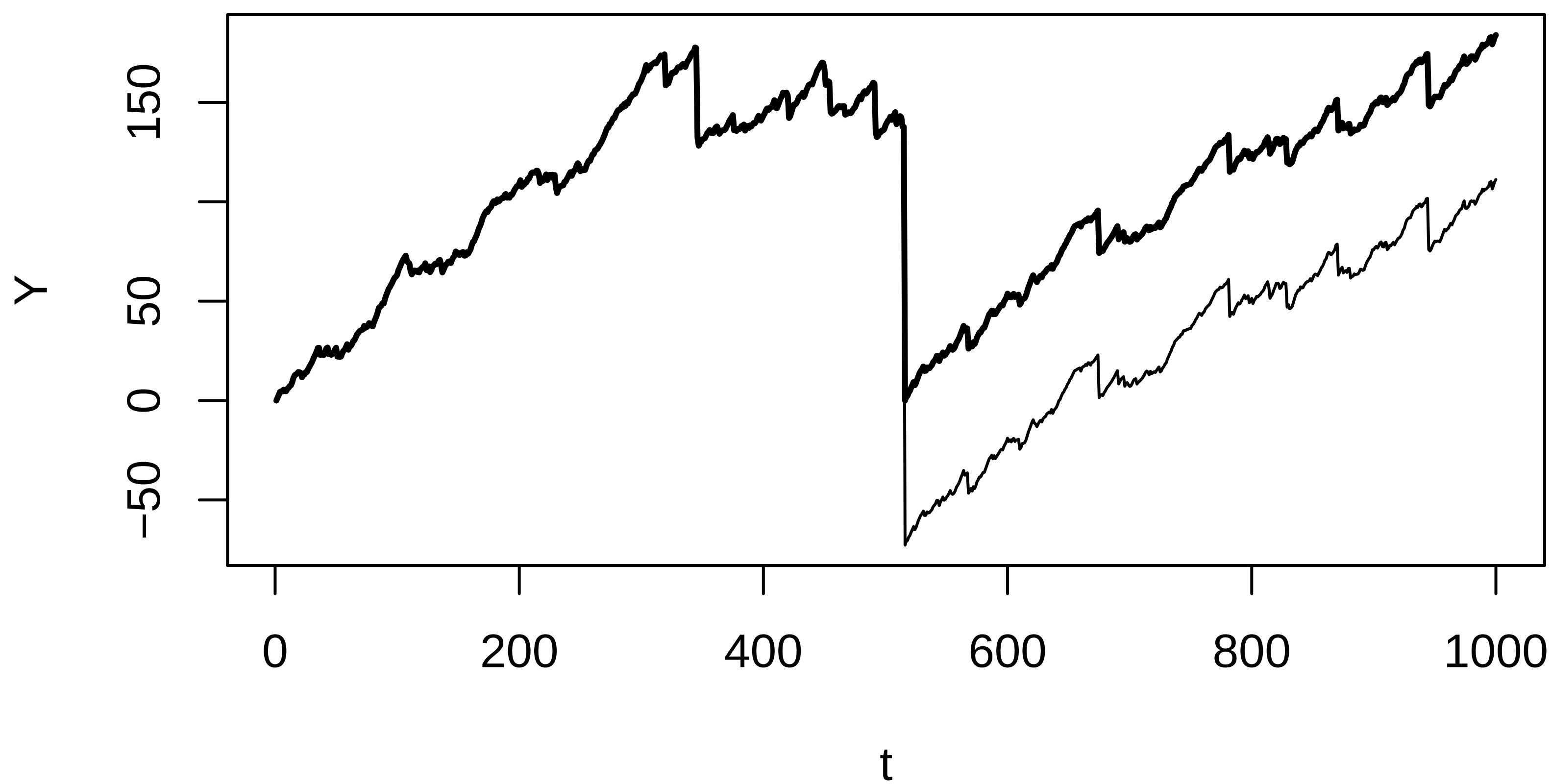
$$T_t f(x) = \mathbb{E}[f(Z_{t+s})|Z_s = x]$$

on the Banach space $C_\infty(\mathbb{R})$ satisfies the Feller property

$$\|T_t f - f\| := \sup\{|T_t f(x) - f(x)| : x \in \mathbb{R}\} \to 0 \quad \text{as } t \to 0.$$

Reflected stable sample path

Stable process Y_t (thin line) with $\alpha = 1.3$ and reflected stable Z_t (thick line). See Appendix for R code.



Fractional Calculus

Positive and negative Riemann-Liouville fractional integrals

$$I_x^\alpha f(x) := \frac{1}{\Gamma(\alpha)} \int_0^\infty f(x-y) y^{\alpha-1}\, dy = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^x f(y)(x-y)^{\alpha-1}\, dy$$

$$I_{-x}^\alpha f(x) := \frac{1}{\Gamma(\alpha)} \int_0^\infty f(x+y) y^{\alpha-1}\, dy = \frac{1}{\Gamma(\alpha)} \int_x^\infty f(y)(y-x)^{\alpha-1}\, dy$$

Positive and negative Riemann-Liouville fractional derivatives

$$D_x^\alpha f(x) := \frac{d^n}{dx^n} I_x^{n-\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_{-\infty}^x f(y)(x-y)^{n-\alpha-1}\, dy$$

$$D_{-x}^\alpha f(x) := \frac{d^n}{d(-x)^n} I_{-x}^{n-\alpha} f(x) = \frac{(-1)^n}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_x^\infty f(y)(y-x)^{n-\alpha-1}\, dy$$

Positive Caputo fractional derivative

$$\partial_x^\alpha f(x) := I_x^{n-\alpha} \frac{d^n}{dx^n} f(x) = \frac{1}{\Gamma(n-\alpha)} \int_0^x (x-y)^{n-1-\alpha} f^{(n)}(y)\, dy$$

where $\alpha > 0$ and $n-1 < \alpha < n$. If $\alpha \in (1,2)$, then $n = 2$.

Forward equation for Y_t

The stable Lévy process Y_t has pdf $p(x,t)$ with Fourier transform

$$\widehat{p}(k,t) = \int e^{ikx} p(x,t)\,dx = e^{t(ik)^\alpha}$$

so that

$$\frac{\partial}{\partial t}\widehat{p}(k,t) = (ik)^\alpha \widehat{p}(k,t).$$

Since $(ik)^\alpha \widehat{f}(k)$ is the Fourier transform of $D^\alpha_{-x} f(x)$, we have

$$\frac{\partial}{\partial t} p(x,t) = D^\alpha_{-x} p(x,t).$$

This is the forward equation of the Markov process Y_t.

It describes the forward evolution of the pdf.

Forward semigroup and its generator

The forward semigroup is defined by

$$T_t^* f(x) := \mathbb{E}[f(x - Y_t)] = \int f(x-y)p(y,t)\,dy.$$

If X has pdf $f(x)$ independent of Y_t, $X + Y_t$ has density $T_t^* f(x)$.

The generator of the forward semigroup T_t^* is

$$A^* f(x) := \lim_{t \downarrow 0} \frac{T_t^* f(x) - f(x)}{t} = D_{-x}^{\alpha} f(x)$$

since this expression has Fourier transform

$$\lim_{t \downarrow 0} \frac{e^{t(ik)^\alpha} \hat{f}(k) - \hat{f}(k)}{t} = (ik)^\alpha \hat{f}(k).$$

Then $p(x,t) = T_t^* f(x)$ solves the Cauchy problem

$$\frac{\partial}{\partial t} p(x,t) = D_{-x}^{\alpha} p(x,t); \quad p(x,0) = f(x)$$

for any $f \in D(A^*)$.

Backward semigroup

The backward semigroup $T_t f(x) := \mathbb{E}[f(x+Y_t)] = \int f(x-y)p(-y,t)\,dy$.

If $f(x) = I_B(x)$ then $T_t f(x) = \mathbb{P}[x + Y_t \in B]$ with initial state x.

Note $p(-y,t)$ has Fourier transform $\widehat{p}(-k,t) = e^{t(-ik)^\alpha}$

Can show $(-ik)^\alpha \widehat{f}(k)$ is the Fourier transform of $D_x^\alpha f(x)$

Then $A = D_x^\alpha$ is the backward generator of Y_t

Adjoint relation: For suitably nice functions

$$\int_{-\infty}^{\infty} D_x^\alpha f(x) g(x)\,dx = \int_{-\infty}^{\infty} f(x) D_{-x}^\alpha g(x)\,dx.$$

Easy to check using integration by parts.

Dual semigroups for Z_t

Backward semigroup $T_t f(x) := \mathbb{E}[f(x + Z_t)]$ on $\mathbb{B} := C_\infty[0, \infty)$.

Dual semigroup T_t^* on the dual space $\mathbb{B}^* := \mathcal{M}_b[0, \infty)$ of (signed) bounded Borel measures with the total variation norm.

Then $\int T_t f(x)\mu(dx) = \int f(x)[T_t^* \mu](dx)$ for $f \in \mathbb{B}$ and $\mu \in \mathbb{B}^*$

and $\int Af(x)\mu(dx) = \int f(x)[A^* \mu](dx)$ for $f \in \mathrm{D}(A)$ and $\mu \in \mathrm{D}(A^*)$.

Sun-dual semigroup: $\mathbb{B}^{\odot} := \{\mu \in X^* : \lim_{t \downarrow 0} \|T_t^* \mu - \mu\| = 0\}$

Needed because T_t^* is not strongly continuous on $\mathbb{B}^*$.

Backward generator for Z_t

Fourier transform methods are not useful on $C_\infty[0,\infty)$.

Bernyk, Dalang and Peskir (2011) compute $Af(x) = \partial_x^\alpha f(x)$ for $f \in C_A := \{f \in S_b : \partial_x^\alpha f \in C_\infty[0,\infty)\}$, where

$$S_b := \Big\{ f \in C_\infty[0,\infty) \, : \, f'' \in C(0,\infty),\ f''(x) = O(1) \text{ as } x \to \infty, \\ f''(x) = O(x^{\alpha-2}) \text{ as } x \to 0,\ f' \in C_b(0,\infty),\ f'(0+) = 0 \Big\}$$

Their proof uses the Lévy-Itô formula.

Patie and Simon (2012) extend to $f'(0+) \neq 0$, compute $D(A)$.

We show T_t is analytic, and C_A is a core.

Proof: Show that $\|\lambda R(\lambda, A) g\| \leqslant M\|g\|$ for $g \in C_\infty[0,\infty)$ and $\operatorname{Re}\lambda > 0$, where $R(\lambda, A)$ is the resolvent.

Forward generator for Z_t

The sun-dual space $\mathbb{B}^{\odot} = \mathcal{M}_{ac}[0,\infty)$, all μ with Lebesgue density

Sun-dual generator $A^{\odot} f(x) = D^{\alpha}_{-x} f(x)$ similar to Y_t, but here

$$\mathrm{D}(A^{\odot}) = \{f \in L^1[0,\infty) \,:\, D^{\alpha}_{-x} f(x) \in L^1[0,\infty), D^{\alpha-1}_{-x} f(0) = 0\}$$

Then the forward equation for the reflected stable process Z_t is

$$\frac{\partial}{\partial t} p(x,t) = D^{\alpha}_{-y} p(x,t); \quad D^{\alpha-1}_{-x} p(0,t) = 0.$$

Similar to Y_t except for the reflecting boundary condition.

This solves an open problem from Baeumer, M and Nane (2009) [What is the Markov process with this forward equation?].

For $\mu_n \in \mathbb{B}^{\odot}$ and $\mu \in \mathbb{B}^*$ if $\mu_n \to \mu$ then $T_t^{\odot} \mu_n \to T_t^* \mu$ in $\mathbb{B}^*$ (vague convergence). Use to prove numerics converge when $\mu = \delta_x$.

Adjoint calculations

The backward/forward generators for Y_t are adjoints:

$$Y_t: \quad \int_{-\infty}^{\infty} D_{-x}^{\alpha} f(x) g(x)\, dx = \int_{-\infty}^{\infty} f(x) D_x^{\alpha} g(x)\, dx.$$

Integrate by parts, use the definitions.

The backward/forward generators for Z_t are also adjoints:

$$Z_t: \quad \int_0^{\infty} D_{-x}^{\alpha} f(x) g(x)\, dx = \int_0^{\infty} f(x) \partial_x^{\alpha} g(x)\, dx.$$

Integrate by parts, use the boundary condition $D_{-x}^{\alpha-1} g(x) = 0$.

Lesson: Adjoint of D_{-x}^{α} depends on the space.

No-Flux Boundary Condition

Since $1 = \mathbb{P}(Z_t \geqslant 0)$ is constant for all $t > 0$,

$$
\begin{aligned}
0 &= \frac{\partial}{\partial t}\mathbb{P}(Z_t \geqslant 0) = \int_0^\infty \frac{\partial}{\partial t} p(x,t)\,dx \\
&= \int_0^\infty D_{-x}^{\alpha} p(x,t)\,dx \\
&= \int_0^\infty \frac{d}{dx}\left(\frac{1}{\Gamma(2-\alpha)}\frac{d}{dx}\int_x^\infty p(y,t)(y-x)^{1-\alpha}\,dy\right)\,dx \\
&= -\int_0^\infty \frac{d}{dx} D_{-x}^{\alpha-1} p(x,t)\,dx \\
&= D_{-x}^{\alpha-1} p(0,t).
\end{aligned}
$$

The boundary condition keeps all probability mass in $[0,\infty)$.

Lévy-like Markov processes

Patie and Simon (2012) show that Z_t has backward generator

$$Af(x) = f'(0)\frac{x^{1-\alpha}}{\Gamma(2-\alpha)} + \int_0^x f''(x-y)\frac{y^{1-\alpha}}{\Gamma(2-\alpha)}\,dy.$$

Integrate by parts to get the pseudo-differential operator form

$$Af(x) = b(x)f'(x) + \int \Big[f(x+y) - f(x) - yf'(x)\Big]\,\phi(x,dy).$$

The drift coefficient $b(x) = x^{1-\alpha}/\Gamma(2-\alpha)$ blows up as $x \downarrow 0$.

The jump intensity

$$\phi(x,dy) = \frac{\alpha(\alpha-1)}{\Gamma(2-\alpha)}|y|^{-1-\alpha}dyI(-x<y<0) + \frac{\alpha-1}{\Gamma(2-\alpha)}\,x^{-\alpha}\delta_{-x}(dy)$$

keeps negative jumps inside the state space $[0,\infty)$.

Compare Y_t Lévy measure $\phi(dy) = \dfrac{\alpha(\alpha-1)}{\Gamma(2-\alpha)}|y|^{-1-\alpha}dyI(y<0)$.

Reflected Brownian motion

We show Z_{t+s} given $Z_s = x$ has a smooth density $y \mapsto p(x,y,t)$.

Reflected Brownian motion transition density solves the diffusion equation with a reflecting boundary condition

$$\frac{\partial}{\partial y} p(x,y,t)\bigg|_{y=0+} := \lim_{h\to 0+} \frac{p(x,y+h,t) - p(x,y,t)}{h}\bigg|_{y=0} = 0$$

In the stable case, the reflecting boundary condition is

$$D_{-y}^{\alpha-1} p(x,y,t)\bigg|_{y=0+} := \lim_{h\to 0+} \frac{1}{h^{\alpha-1}} \sum_{k=0}^{\infty} w_k^{\alpha-1} p(x,y+kh,t)\bigg|_{y=0} = 0$$

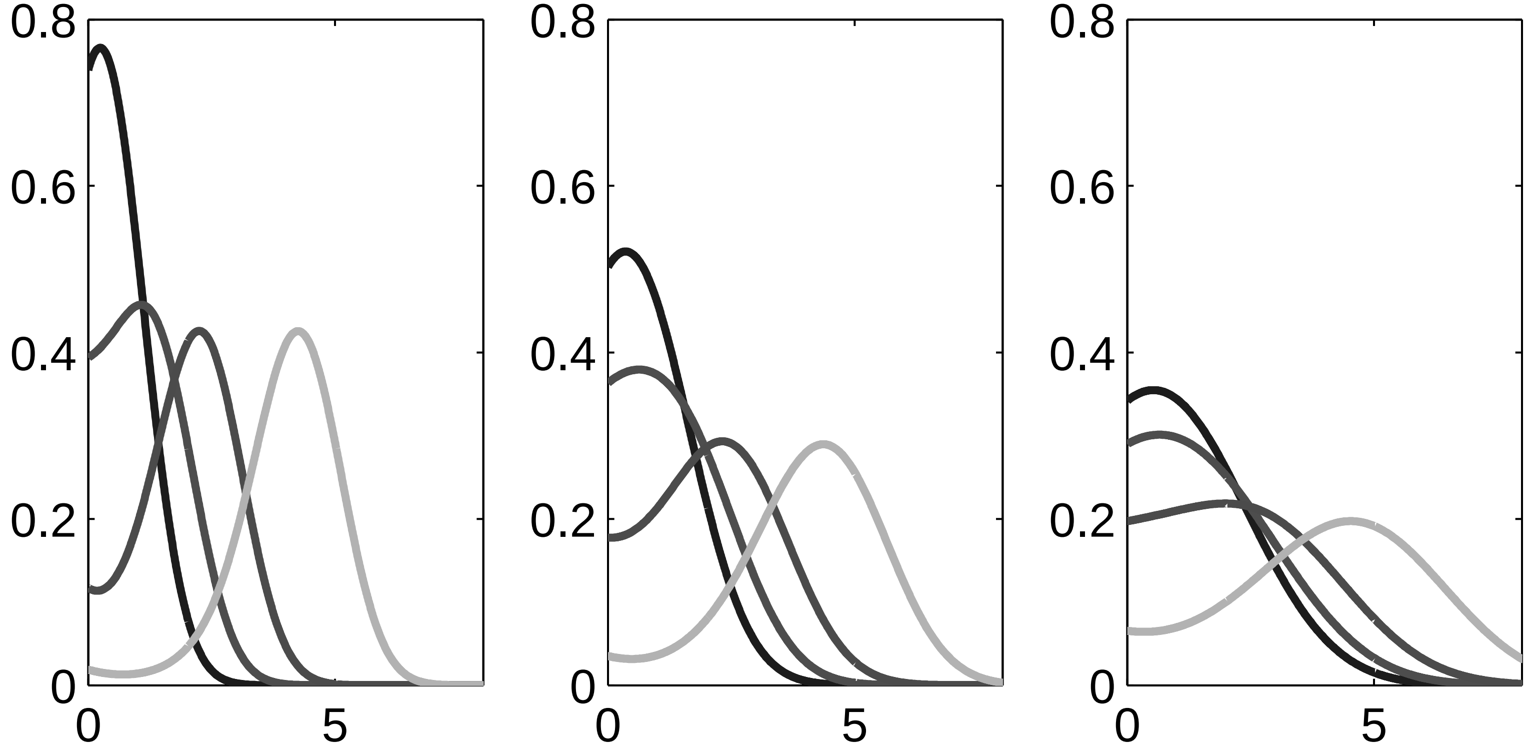
where

$$w_k^{\alpha} := (-1)^k \frac{\Gamma(\alpha+1)}{k!\Gamma(\alpha-k+1)}$$

When $\alpha = 2$, $w_0^{\alpha-1} = 1$, $w_1^{\alpha-1} = -1$, and $w_k^{\alpha-1} = 0$ for $k > 1$.

Reflected stable transition density

Transition densities $y \mapsto p(x, y, t)$ with index $\alpha = 1.8$ and initial state $x = 0, 1, 2, 4$ (left to right) at times $t = 0.5$ (left), $t = 1$ (middle), and $t = 2$ (right). See Appendix for MATLAB code.



Fractional Cauchy problems

Suppose $T_t f(x) = \mathbb{E}[f(X_t)|X_0 = x]$ is a Feller semigroup

Then $p(x,t) = \mathbb{E}[f(X_{Z_t})|X_0 = x]$ solves the fractional Cauchy problem

$$\partial_t^\beta p(x,t) = Lp(x,t); \quad p(x,0) = f(x)$$

for any $f \in \mathrm{D}(L)$, when $\beta = 1/\alpha$ for some $\alpha \in (1,2)$.

Proof: $D_t = \inf\{r > 0; Y_r > t\}$ is a β-stable subordinator whose inverse $E_t = \inf\{u > 0; D_u > t\}$ is also the supremum process (inverse of the inverse) of Y_t.

The result is known for the inverse stable subordinator E_t.

But Prop VI.3 in Bertoin (1996) implies $E_t = Z_t$ in distribution.

Open problems

- General reflected stable processes (w/ drift)

- Compute forward generator

- Compute transition density

- Understand fractional boundary conditions

- Particle tracking codes

References

1. W. Arendt, C.J.K. Batty, M. Hieber and F. Neubrander (2011) *Vector-valued Laplace Transforms and Cauchy Problems*. 2nd ed., Birkhäuser, Basel.

2. B. Baeumer and M.M. Meerschaert (2001) Stochastic solutions for fractional Cauchy problems. *Fract. Calc. Appl. Anal.* **4**, 481–500.

3. B. Baeumer, M.M. Meerschaert and E. Nane (2009) Space-time duality for fractional diffusion. *J. Appl. Prob.*, **46**, 1100–1115.

4. B. Baeumer, M. Kovcs, M.M. Meerschaert, R.L. Schilling, and P. Straka (2013) Reflected stable subordinators and their governing equations. Preprint at `www.stt.msu.edu/users/mcubed/duality-process.pdf`

5. V. Bernyk, R.C. Dalang and G. Peskir (2011) Predicting the ultimate supremum of a stable Lévy process with no negative jumps. *Ann. Probab.*, **39**, 2385–2423.

6. J. Bertoin (1996) *Lévy Processes*. Cambridge University Press, Cambridge.

7. K. Itô and H. P. McKean (1963) Brownian motions on a half line. *Illinois J. Math.* **7**, 181–231.

8. N. Jacob (1996) *Pseudo-Differential Operators and Markov Processes Vol. 1*. Imperial College Press, London.

9. M.M. Meerschaert and H.-P. Scheffler (2004) Limit theorems for continuous time random walks with infinite mean waiting times. *J. Applied Probab.* **41**, 623–638.

10. P. Patie and T. Simon (2012) Intertwining certain fractional derivatives. *Potential Anal.* **36**, 569–587.

11. R.L. Schilling (1998) Growth and Hölder conditions for the sample paths of Feller processes. *Probab. Theor. Rel. Fields* **112**, 565–611.

Duality and adjoints

For a Markov process Z_t with transition density $p(x, y, t)$, let

$$\begin{aligned}
u(t,x) &= T_t^* u_0(x) = \int p(x,y,t) u_0(y) dy \\
v(t,y) &= T_t v_0(x) = \int p(x,y,t) v_0(x) dx \\
H(s,t) &= \int v(s,y) u(t,y) dy \\
&= \int \int p(x,y,s) v_0(x) dx \int p(y,z,t) u_0(z) dz \, dy \\
&= \int \int p(x,z,s+t) v_0(x) u_0(z) dx dz
\end{aligned}$$

Then $\partial H / \partial s = \partial H / \partial t$ so

$$\begin{aligned}
\int Av(s,y) u(t,y) dy &= \int \frac{\partial}{\partial s} v(s,y) u(t,y) dy \\
&= \int v(s,y) \frac{\partial}{\partial t} u(t,y) dy = \int v(s,y) A^* u(t,y) dy
\end{aligned}$$

R code for reflected stable sample path

```
# Plot stable Y_t with characteristic function exp(t(ik)^a)
# and the reflected stable process Z_t=Y_t-inf{Y_u:0<=u<=t}
#
# You need to install the fBasics package on your R platform.
# Try Packages > Load package to see if fBasics is available.
# If not then use Packages > Install package(s)
 library(fBasics)
 t=seq(1:1000)
 a=1.3
 g=(abs(cos(pi*a/2)))^(1/a)
 y=rstable(t,alpha=a,beta=-1.0,gamma=g,delta=0.0,pm=1)
 Y=cumsum(y)
 Z=Y-cummin(Y)
 plot(t,Y,type="l",ylim=c(min(Y),max(Z)))
 lines(t,Z,lwd=2)
```

MATLAB code for reflected stable pdf

```
%%% Matlab script to compute p(x,y,t)
%% enter variables
    alpha=1.2; ymax=12; N=1200; t=[0,.5,1,2]; x=2;
%% initialise parameters
    h=ymax/N; y=(h:h:ymax)';
    u0=zeros(N,1);u0(floor(x/h)+1)=1/h; % initial condition
%% Make Grunwald matrix
    w=ones(1,N+1);
    for k=1:N
      w(k+1)=w(k)*(k-alpha-1)/k;
    end
    w=w/h^alpha;
    M=spdiags(repmat(w,N,1),-1:1:N-1,N,N); %enter w's along diagonals
    M(1,:)=-cumsum(w(1:N))'; %change first row for BC
%% Solve ODE system
    [~,p]=ode113(@(t,u) M*u,t,u0);
```