Law of the time to absorption at zero of a (not-necessarily) symmetric stable Lévy process

Alexey Kuznetsov[1] Andreas E. Kyprianou[2]
Juan-Carlos Pardo[3] Alex Watson[2]

[1]York University, Toronto, Canada.
[2]Unversity of Bath, UK.
[3]CIMAT, Mexico.

Stable processes

Definition I

A Lévy process X is called (strictly) α-stable if it satisfies the scaling property

$$\left(cX_{c^{-\alpha}t}\right)_{t\geq 0}\Big|_{\mathrm{P}_x} \overset{d}{=} X|_{\mathrm{P}_{cx}}, \quad c > 0.$$

Necessarily $\alpha \in (0,2]$. [$\alpha = 2 \to$ BM, exclude this.]
The quantity $\rho = \mathrm{P}_0(X_t \geq 0)$ will frequently appear as will $\hat{\rho} = 1 - \rho$.

Stable processes

Definition I

A Lévy process X is called (strictly) α-stable if it satisfies the scaling property

$$\left(cX_{c^{-\alpha}t}\right)_{t\geq 0}\Big|_{\mathrm{P}_x} \stackrel{d}{=} X|_{\mathrm{P}_{cx}}, \quad c>0.$$

Necessarily $\alpha \in (0,2]$. [$\alpha = 2 \to$ BM, exclude this.]
The quantity $\rho = \mathrm{P}_0(X_t \geq 0)$ will frequently appear as will $\hat{\rho} = 1-\rho$.

Definition II

Let α, ρ be admissible parameters, X the Lévy process with Lévy density

$$c_+ x^{-(\alpha+1)} \mathbb{1}_{(x>0)} + c_- \left|x\right|^{-(\alpha+1)} \mathbb{1}_{(x<0)}, \qquad x \in \mathbb{R},$$

no Gaussian part.

Stable processes

Additional notes:

- X does not have one-sided jumps,
- We assume that $\alpha \in (1, 2)$, in which case X is point-recurrent.

Problem: statement

The problem

Let

$$T_0 = \inf\{t > 0 : X_t = 0\}$$

be the first hitting time of $\{0\}$.

Can we find an explicit expression for

$$p(t)\mathrm{d}t := \mathrm{P}_1(T_0 \in \mathrm{d}t)?$$

Problem: history

- G. Peskir (2008) The law of the hitting times to points by a stable Lévy process with no negative jumps. Electron. Commun. Probab., 13, 653–659.
- K. Yano, Y. Yano, and M. Yor. (2009) On the laws of first hitting times of points for one-dimensional symmetric stable Lévy processes. In *Séminaire de Probabilités XLII*, volume 1979 of *Lecture Notes in Math.*, pages 187–227. Springer, Berlin.
- F. Cordero. (2010) *On the excursion theory for the symmetric stable Lévy processes with index* $\alpha \in]1,2]$ *and some applications*. PhD thesis, Université Pierre et Marie Curie – Paris VI, 2010.

Positive, self-similar Markov processes

α-pssMp

$[0,\infty)$-valued Markov process,
equipped with initial measures P_x, $x > 0$,
with 0 an absorbing state,
satisfying the scaling property

$$(cX_{c^{-\alpha}t})_{t\geq 0}\Big|_{P_x} \overset{d}{=} X|_{P_{cx}}, \qquad x, c > 0$$

Lamperti transform

$(X, \mathrm{P}_x)_{x>0}$ pssMp	$\leftrightarrow$	$(\xi, \mathbb{P}_y)_{y\in\mathbb{R}}$ killed Lévy
$X_t = \exp(\xi_{S(t)})$,		$\xi_s = \log(X_{T(s)})$,
S a random time-change		T a random time-change

Lamperti transform

$(X, \mathrm{P}_x)_{x>0}$ pssMp $\leftrightarrow$ $(\xi, \mathbb{P}_y)_{y\in\mathbb{R}}$ killed Lévy

$$X_t = \exp(\xi_{S(t)}),$$

S a random time-change

$$\xi_s = \log(X_{T(s)}),$$

T a random time-change

$$\left.\begin{array}{c} X \text{ never hits zero} \\ X \text{ hits zero continuously} \\ X \text{ hits zero by a jump} \end{array}\right\} \leftrightarrow \left\{\begin{array}{c} \xi \to \infty \text{ or } \xi \text{ oscillates} \\ \xi \to -\infty \\ \xi \text{ is killed} \end{array}\right.$$

Example 1

Example 1

Let X be a stable process, and define

$$X_t^* = X_t \mathbb{1}_{(t<\tau_0^-)}, \qquad t \geq 0,$$

where

$$\tau_0^- = \inf\{t > 0 : X_t < 0\}.$$

Example 1

Let X be a stable process, and define

$$X_t^* = X_t \mathbb{1}_{(t<\tau_0^-)}, \qquad t \geq 0,$$

where

$$\tau_0^- = \inf\{t > 0 : X_t < 0\}.$$

Then X^* is a pssMp, with Lamperti transform ξ^*.

Example 1

Let X be a stable process, and define

$$X_t^* = X_t \mathbb{1}_{(t<\tau_0^-)}, \qquad t \geq 0,$$

where

$$\tau_0^- = \inf\{t > 0 : X_t < 0\}.$$

Then X^* is a pssMp, with Lamperti transform ξ^*.
ξ^* has Lévy density

$$c_+ \frac{e^x}{(e^x - 1)^{\alpha+1}} \mathbb{1}_{(x>0)} + c_- \frac{e^x}{(1 - e^x)^{\alpha+1}} \mathbb{1}_{(x<0)},$$

and is killed at rate $c_-/\alpha = \frac{\Gamma(\alpha)}{\Gamma(\alpha\hat{\rho})\Gamma(1-\alpha\hat{\rho})}$.

Example 2

Let X be a **symmetric** α-stable process with $\alpha \in (1, 2)$, and define

$$R_t = |X_t| \mathbb{1}_{(t < T_0)}, \qquad t \geq 0.$$

Example 2

Let X be a **symmetric** α-stable process with $\alpha \in (1,2)$, and define

$$R_t = |X_t| \mathbb{1}_{(t<T_0)}, \qquad t \geq 0.$$

Then R is a pssMp with Lamperti-transform $\xi = \xi^{\mathsf{L}} \oplus \xi^{\mathsf{C}}$, such that

(i) The Lévy process ξ^{L} has characteristic exponent

$$\Psi^*(\theta) - c_-/\alpha, \qquad \theta \in \mathbb{R},$$

where Ψ^* is the characteristic exponent of the process ξ^*.

(ii) The process ξ^{C} is a compound Poisson process whose jumps occur at rate c_-/α, whose Lévy density is

$$\pi^{\mathsf{C}}(y) = c_- \frac{e^y}{(1+e^y)^{\alpha+1}}, \qquad y \in \mathbb{R}.$$

Example 2

Let X be an α-stable process with $\alpha \in (1,2)$, and define

$$R_t = |X_t| \mathbb{1}_{(t<T_0)}, \qquad t \geq 0.$$

Then R is a pssMp with Lamperti-transform $\xi = \xi^{\mathsf{L}} \oplus \xi^{\mathsf{C}}$, such that

(i)

$$\Psi(\theta) = 2^{\alpha} \frac{\Gamma(\alpha/2 - \mathrm{i}\theta/2)}{\Gamma(-\mathrm{i}\theta/2)} \frac{\Gamma(1/2 + \mathrm{i}\theta/2)}{\Gamma((1-\alpha)/2 + \mathrm{i}\theta/2)}, \qquad \theta \in \mathbb{R}.$$

(ii) For later convenience we also note $\psi(z) := \log \mathbb{E} e^{-z\alpha\xi_1}$ is given by

$$\psi(z) = -2^{\alpha} \frac{\Gamma(1/2 - \alpha z/2)}{\Gamma(1/2 - \alpha(1+z)/2)} \frac{\Gamma(\alpha(1+z)/2)}{\Gamma(\alpha z/2)}, \qquad \operatorname{Re} z \in (-1, 1/\alpha).$$

Standard theory for pssMp

(i) (T_0, P_1) has the same law as $(I(\alpha\xi), \mathbb{P})$, where

$$I(\alpha\xi) = \int_0^\infty e^{\alpha\xi_t}\,\mathrm{d}t$$

Standard theory for pssMp

(i) (T_0, P_1) has the same law as $(I(\alpha\xi), \mathbb{P})$, where

$$I(\alpha\xi) = \int_0^\infty e^{\alpha\xi_t}\,\mathrm{d}t$$

(ii) If $\mathcal{M}(s) := \mathbb{E}[I(\alpha\xi)^{s-1}]$, $s \in \mathbb{C}$, then when the right hand side is well defined,

$$\mathcal{M}(s+1) = -\frac{s}{\psi(-s)}\mathcal{M}(s),$$

Standard theory for pssMp

(i) (T_0, P_1) has the same law as $(I(\alpha\xi), \mathbb{P})$, where

$$I(\alpha\xi) = \int_0^\infty e^{\alpha\xi_t}\,\mathrm{d}t$$

(ii) If $\mathcal{M}(s) := \mathbb{E}[I(\alpha\xi)^{s-1}]$, $s \in \mathbb{C}$, then when the right hand side is well defined,

$$\mathcal{M}(s+1) = -\frac{s}{\psi(-s)}\mathcal{M}(s),$$

(iii) Because of the explicit form of ψ, we can guess (and then prove) that

$$\mathrm{E}_1[T_0^{s-1}] = \sin(\pi/\alpha)\frac{\cos\left(\frac{\pi\alpha}{2}(s-1)\right)}{\sin\left(\pi\left(s-1+\frac{1}{\alpha}\right)\right)}\frac{\Gamma(1+\alpha-\alpha s)}{\Gamma(2-s)},$$

for $\mathrm{Re}\, s \in \left(-\frac{1}{\alpha}, 2-\frac{1}{\alpha}\right)$.

Markov additive processes (MAPs)

Let E be a finite state space and $(\mathscr{G}_t)_{t\geq 0}$ a standard filtration. A càdlàg process (ξ, J) in $\mathbb{R}\times E$ with law $\mathbb{P}$ is called a *Markov additive process (MAP)* with respect to $(\mathscr{G}_t)_{t\geq 0}$ if $(J(t))_{t\geq 0}$ is a continuous-time, irreducible Markov chain in E, and the following property is satisfied, for any $i \in E$, $s, t \geq 0$:

Given $\{J(t) = i\}$, the pair $(\xi(t+s) - \xi(t), J(t+s))$ is independent of $\mathscr{G}_t$, and has the same distribution as $(\xi(s) - \xi(0), J(s))$ given $\{J(0) = i\}$.

Pathwise description of a MAP

The pair (ξ, J) is a Markov additive process if and only if, for each $i, j \in E$, there exist a sequence of iid Lévy processes $(\xi_i^n)_{n\geq 0}$ and a sequence of iid random variables $(U_{ij}^n)_{n\geq 0}$, independent of the chain J, such that if $T_0 = 0$ and $(T_n)_{n\geq 1}$ are the jump times of J, the process ξ has the representation

$$\xi(t) = \mathbb{1}_{(n>0)}(\xi(T_n-) + U^n_{J(T_n-),J(T_n)}) + \xi^n_{J(T_n)}(t - T_n),$$

for $t \in [T_n, T_{n+1})$, $n \geq 0$.

rssMps, MAPs, Lamperti-Kiu (Chaumont, Panti, Rivero)

- Take the statespace of the MAP to be $E = \{1, 2\}$.

rssMps, MAPs, Lamperti-Kiu (Chaumont, Panti, Rivero)

- Take the statespace of the MAP to be $E = \{1, 2\}$.
- Let

$$X_t = x \exp\{\xi(\tau(t)) + \mathrm{i}\pi(J(\tau(t)) + 1) \qquad 0 \leq t < T_0, \}$$

where

$$\tau(t) = \inf\left\{s > 0 : \int_0^s \exp(\alpha\xi(u))\mathrm{d}u > t|x|^{-\alpha}\right\}$$

and

$$T_0 = |x|^{-\alpha} \int_0^\infty \mathrm{e}^{\alpha\xi(u)}\mathrm{d}u.$$

rssMps, MAPs, Lamperti-Kiu (Chaumont, Panti, Rivero)

- Take the statespace of the MAP to be $E = \{1, 2\}$.
- Let

$$X_t = x \exp\{\xi(\tau(t)) + \mathrm{i}\pi(J(\tau(t)) + 1) \qquad 0 \leq t < T_0, \}$$

where

$$\tau(t) = \inf\left\{s > 0 : \int_0^s \exp(\alpha\xi(u))\mathrm{d}u > t|x|^{-\alpha}\right\}$$

and

$$T_0 = |x|^{-\alpha} \int_0^\infty \mathrm{e}^{\alpha\xi(u)}\mathrm{d}u.$$

- Then X_t is a real-valued self-similar Markov process in the sense that the law of $(cX_{tc^{-\alpha}} : t \geq 0)$ under P_x is P_{cx}.

rssMps, MAPs, Lamperti-Kiu (Chaumont, Panti, Rivero)

- Take the statespace of the MAP to be $E = \{1, 2\}$.
- Let

$$X_t = x \exp\{\xi(\tau(t)) + i\pi(J(\tau(t)) + 1) \qquad 0 \le t < T_0, \}$$

where

$$\tau(t) = \inf\left\{s > 0 : \int_0^s \exp(\alpha\xi(u))du > t|x|^{-\alpha}\right\}$$

and

$$T_0 = |x|^{-\alpha}\int_0^\infty e^{\alpha\xi(u)}du.$$

- Then X_t is a real-valued self-similar Markov process in the sense that the law of $(cX_{tc^{-\alpha}} : t \ge 0)$ under P_x is P_{cx}.
- The converse (within a special class of rssMps) is also true.

Characteristics of a MAP

- Denote the transition rate matrix of the chain J by $Q = (q_{ij})_{i,j \in E}$.

Characteristics of a MAP

- Denote the transition rate matrix of the chain J by $Q = (q_{ij})_{i,j \in E}$.
- For each $i \in E$, the Laplace exponent of the Lévy process ξ_i will be written ψ_i (when it exists).

Characteristics of a MAP

- Denote the transition rate matrix of the chain J by $Q = (q_{ij})_{i,j \in E}$.
- For each $i \in E$, the Laplace exponent of the Lévy process ξ_i will be written ψ_i (when it exists).
- For each pair of $i, j \in E$, define the Laplace transform $G_{ij}(z) = \mathbb{E}(e^{zU_{ij}})$ of the jump distribution U_{ij} (when it exists).

Characteristics of a MAP

- Denote the transition rate matrix of the chain J by $Q = (q_{ij})_{i,j \in E}$.
- For each $i \in E$, the Laplace exponent of the Lévy process ξ_i will be written ψ_i (when it exists).
- For each pair of $i, j \in E$, define the Laplace transform $G_{ij}(z) = \mathbb{E}(e^{zU_{ij}})$ of the jump distribution U_{ij} (when it exists).
- Write $G(z)$ for the $N \times N$ matrix whose (i, j)th element is $G_{ij}(z)$.

Characteristics of a MAP

- Denote the transition rate matrix of the chain J by $Q = (q_{ij})_{i,j \in E}$.
- For each $i \in E$, the Laplace exponent of the Lévy process ξ_i will be written ψ_i (when it exists).
- For each pair of $i, j \in E$, define the Laplace transform $G_{ij}(z) = \mathbb{E}(e^{zU_{ij}})$ of the jump distribution U_{ij} (when it exists).
- Write $G(z)$ for the $N \times N$ matrix whose (i,j)th element is $G_{ij}(z)$.
- Let

$$F(z) = \mathrm{diag}(\psi_1(z), \ldots, \psi_N(z)) + Q \circ G(z), \qquad (1)$$

(when it exists), where $\circ$ indicates elementwise multiplication.

Characteristics of a MAP

- Denote the transition rate matrix of the chain J by $Q = (q_{ij})_{i,j \in E}$.
- For each $i \in E$, the Laplace exponent of the Lévy process ξ_i will be written ψ_i (when it exists).
- For each pair of $i, j \in E$, define the Laplace transform $G_{ij}(z) = \mathbb{E}(e^{zU_{ij}})$ of the jump distribution U_{ij} (when it exists).
- Write $G(z)$ for the $N \times N$ matrix whose (i,j)th element is $G_{ij}(z)$.
- Let

$$F(z) = \mathrm{diag}(\psi_1(z), \dots, \psi_N(z)) + Q \circ G(z), \qquad (1)$$

(when it exists), where $\circ$ indicates elementwise multiplication.
- The matrix exponent of the MAP (ξ, J) is given by

$$\mathbb{E}_i(e^{z\xi(t)}; J(t) = j) = \left(e^{F(z)t}\right)_{i,j}, \qquad i, j \in E,$$

(when it exists).

An α-stable process is a rssMp

- An α-stable process is a rssMp. Remarkably (thanks to work of Chaumont, Panti and Rivero) we can compute precisely its matrix exponent explicitly

An α-stable process is a rssMp

- An α-stable process is a rssMp. Remarkably (thanks to work of Chaumont, Panti and Rivero) we can compute precisely its matrix exponent explicitly
- Denote the underlying MAP (ξ, J), we prefer to give the matrix exponent of $(-\alpha\xi, J)$ as follows:

$$F(z) = \begin{pmatrix} -\dfrac{\Gamma(\alpha(1+z))\Gamma(1-\alpha z)}{\Gamma(\alpha\hat{\rho}+\alpha z)\Gamma(1-\alpha\hat{\rho}-\alpha z)} & \dfrac{\Gamma(\alpha(1+z))\Gamma(1-\alpha z)}{\Gamma(\alpha\hat{\rho})\Gamma(1-\alpha\hat{\rho})} \\ \dfrac{\Gamma(\alpha(1+z))\Gamma(1-\alpha z)}{\Gamma(\alpha\rho)\Gamma(1-\alpha\rho)} & -\dfrac{\Gamma(\alpha(1+z))\Gamma(1-\alpha z)}{\Gamma(\alpha\rho+\alpha z)\Gamma(1-\alpha\rho-\alpha z)} \end{pmatrix}$$

for $\mathrm{Re}\, z \in (-1, 1/\alpha)$.

Cramér condition for a MAP

Proposition

(i) Suppose that $z \in \mathbb{C}$ is such that $F(z)$ is defined. Then, the matrix $F(z)$ has a real simple eigenvalue $\kappa(z)$, which is larger than the real part of all its other eigenvalues.
(ii) Suppose that F is defined in some open interval D of $\mathbb{R}$. Then, the leading eigenvalue κ of F is smooth and convex on D.

Cramér condition for a MAP

Proposition

(i) Suppose that $z \in \mathbb{C}$ is such that $F(z)$ is defined. Then, the matrix $F(z)$ has a real simple eigenvalue $\kappa(z)$, which is larger than the real part of all its other eigenvalues.
(ii) Suppose that F is defined in some open interval D of $\mathbb{R}$. Then, the leading eigenvalue κ of F is smooth and convex on D.

Assumption (Cramér condition for a MAP)

There exists $z_0 < 0$ such that $F(s)$ exists on $(z_0, 0)$, and some $\theta \in (0, -z_0)$, called the Cramér number, such that $\kappa(-\theta) = 0$.

Cramér condition for a MAP

Proposition

(i) Suppose that $z \in \mathbb{C}$ is such that $F(z)$ is defined. Then, the matrix $F(z)$ has a real simple eigenvalue $\kappa(z)$, which is larger than the real part of all its other eigenvalues.
(ii) Suppose that F is defined in some open interval D of $\mathbb{R}$. Then, the leading eigenvalue κ of F is smooth and convex on D.

Assumption (Cramér condition for a MAP)

There exists $z_0 < 0$ such that $F(s)$ exists on $(z_0, 0)$, and some $\theta \in (0, -z_0)$, called the Cramér number, such that $\kappa(-\theta) = 0$.

Note that this dictates "$\kappa'(0) > 0$" which ensures that $\lim_{t\uparrow\infty} \xi_t/t = \kappa'(0) > 0$.

Integrated exponential MAPs

- For a MAP ξ, let

$$I(-\xi) = \int_0^\infty \exp(-\xi(t))\,\mathrm{d}t.$$

Integrated exponential MAPs

- For a MAP ξ, let

$$I(-\xi) = \int_0^\infty \exp(-\xi(t))\,\mathrm{d}t.$$

- One way to characterise the law of $I(-\xi)$ is via its Mellin transform, which we write as $\mathcal{M}(s)$. This is the vector in $\mathbb{R}^N$ whose ith element is given by

$$\mathcal{M}_i(s) = \mathbb{E}_i[I(-\xi)^{s-1}], \qquad i \in E.$$

Vector-valued functional equation

Proposition

Suppose that ξ satisfies the Cramér condition with Cramér number $\theta \in (0,1)$. Then, $\mathcal{M}(s)$ is finite and analytic when $\operatorname{Re} s \in (0, 1+\theta)$, *and we have the following vector-valued functional equation:*

$$\mathcal{M}(s+1) = -s(F(-s))^{-1}\mathcal{M}(s), \text{ for } s \in (0,\theta).$$

Back to the case of an α-stable process, $\alpha \in (1,2)$

- Suffices to consider the case that the stable process starts from $|x| = 1$.

Back to the case of an α-stable process, $\alpha \in (1, 2)$

- Suffices to consider the case that the stable process starts from $|x| = 1$.
- Recall that $T_0 = \int_0^\infty \exp\{-(-\alpha\xi(u))\}\mathrm{d}u$ and that $E = \{1, 2\}$

Back to the case of an α-stable process, $\alpha \in (1,2)$

- Suffices to consider the case that the stable process starts from $|x| = 1$.
- Recall that $T_0 = \int_0^\infty \exp\{-(-\alpha\xi(u))\}\mathrm{d}u$ and that $E = \{1,2\}$
- It is obvious (using asymmetry) that $\mathbb{E}_1(T_0^{s-1})$ is the same expression as $\mathbb{E}_2(T_0^{s-1})$ modulo interchanging the roles of ρ and $\hat{\rho}$.

Back to the case of an α-stable process, $\alpha \in (1, 2)$

- Suffices to consider the case that the stable process starts from $|x| = 1$.
- Recall that $T_0 = \int_0^\infty \exp\{-(-\alpha\xi(u))\}du$ and that $E = \{1, 2\}$
- It is obvious (using asymmetry) that $\mathbb{E}_1(T_0^{s-1})$ is the same expression as $\mathbb{E}_2(T_0^{s-1})$ modulo interchanging the roles of ρ and $\hat{\rho}$.
- Easy to check that $\kappa(1/\alpha - 1) = 0$, i.e. $\theta = 1 - 1/\alpha < 1$.

Back to the case of an α-stable process, $\alpha \in (1,2)$

- Suffices to consider the case that the stable process starts from $|x| = 1$.
- Recall that $T_0 = \int_0^\infty \exp\{-(-\alpha\xi(u))\}\mathrm{d}u$ and that $E = \{1,2\}$
- It is obvious (using asymmetry) that $\mathbb{E}_1(T_0^{s-1})$ is the same expression as $\mathbb{E}_2(T_0^{s-1})$ modulo interchanging the roles of ρ and $\hat{\rho}$.
- Easy to check that $\kappa(1/\alpha - 1) = 0$, i.e. $\theta = 1 - 1/\alpha < 1$.
- Guess a solution to the vector-valued functional equation and then verify uniqueness

Theorem

For $-1/\alpha < \mathrm{Re}(s) < 2 - 1/\alpha$ *we have*

$$\mathbb{E}_1[T_0^{s-1}] = \frac{\sin\left(\frac{\pi}{\alpha}\right)}{\sin(\pi\hat{\rho})} \frac{\sin\left(\pi\hat{\rho}(1-\alpha+\alpha s)\right)}{\sin\left(\frac{\pi}{\alpha}(1-\alpha+\alpha s)\right)} \frac{\Gamma(1+\alpha-\alpha s)}{\Gamma(2-s)}.$$

Inversion (rational $\alpha \in (1,2)$): $p(t) = \mathrm{dP}_1(T_0 \le t)/\mathrm{d}t$

If $\alpha = m/n$ (where m and n are coprime natural numbers) then for all $t > 0$ we have

$$p(t) = \frac{\sin\left(\frac{\pi}{\alpha}\right)}{\pi \sin(\pi\hat\rho)} \sum_{\substack{k\ge 1 \\ k\ne -1 \ (\mathrm{mod}\ m)}} \sin(\pi\hat\rho(k+1)) \frac{\sin\left(\frac{\pi}{\alpha}k\right)}{\sin\left(\frac{\pi}{\alpha}(k+1)\right)} \frac{\Gamma\left(\frac{k}{\alpha}+1\right)}{k!} (-1)^{k-1} t^{-1-\frac{k}{\alpha}}$$

$$- \frac{\sin\left(\frac{\pi}{\alpha}\right)^2}{\pi \sin(\pi\hat\rho)} \sum_{\substack{k\ge 1 \\ k\ne 0 \ (\mathrm{mod}\ n)}} \frac{\sin(\pi\alpha\hat\rho k)}{\sin(\pi\alpha k)} \frac{\Gamma\left(k-\frac{1}{\alpha}\right)}{\Gamma(\alpha k - 1)} t^{-k-1+\frac{1}{\alpha}}$$

$$- \frac{\sin\left(\frac{\pi}{\alpha}\right)^2}{\pi^2 \alpha \sin(\pi\hat\rho)} \sum_{k\ge 1} (-1)^{km} \frac{\Gamma\left(kn-\frac{1}{\alpha}\right)}{(km-2)!} R_k(t) t^{-kn-1+\frac{1}{\alpha}},$$

where

$$R_k(t) := \pi\alpha\hat\rho \cos(\pi\hat\rho k m) - \sin(\pi\hat\rho k m)\left[\pi \cot\left(\frac{\pi}{\alpha}\right) - \psi\left(kn - \frac{1}{\alpha}\right) + \alpha\psi(km-1) + \ln(t)\right].$$

The three series converge uniformly for $t \in [\varepsilon, \infty)$ and any $\varepsilon > 0$.

Inversion (almost every irrational $\alpha \in (1,2)$)

Define $||x|| = \min_{n\in\mathbb{Z}} |x-n|$, and

$$\mathcal{L} = \mathbb{R}\backslash(\mathbb{Q} \cup \{x \in \mathbb{R} : \lim_{n\to\infty} \frac{1}{n} \ln ||nx|| = 0\}).$$

If $\alpha \notin \mathcal{L} \cup \mathbb{Q}$ then

$$p(t) = \frac{\sin\left(\frac{\pi}{\alpha}\right)}{\pi \sin(\pi\hat{\rho})} \sum_{k\geq 1} \sin(\pi\hat{\rho}(k+1)) \frac{\sin\left(\frac{\pi}{\alpha}k\right)}{\sin\left(\frac{\pi}{\alpha}(k+1)\right)} \frac{\Gamma\left(\frac{k}{\alpha}+1\right)}{k!} (-1)^{k-1} t^{-1-\frac{k}{\alpha}}$$
$$- \frac{\sin\left(\frac{\pi}{\alpha}\right)^2}{\pi \sin(\pi\hat{\rho})} \sum_{k\geq 1} \frac{\sin(\pi\alpha\hat{\rho}k)}{\sin(\pi\alpha k)} \frac{\Gamma\left(k-\frac{1}{\alpha}\right)}{\Gamma(\alpha k - 1)} t^{-k-1+\frac{1}{\alpha}}.$$

The two series in the right-hand side of the above formula converge uniformly for $t \in [\varepsilon, \infty)$ and any $\varepsilon > 0$.

Conditioning to avoid zero (Chaumont, Panti, Rivero)

Let X be an α stable process with $\alpha \in (1,2)$ and let h the function

$$h(x) = -\Gamma(1-\alpha)\frac{\sin(\pi\alpha\hat{\rho})}{\pi}x^{\alpha-1}, \qquad x > 0,$$

and the same expression with $\hat{\rho}$ replaced by ρ when $x < 0$.

- The function h is invariant for the stable process killed on hitting 0, that is,

$$\mathrm{E}_x[h(X_t), t < T_0] = h(x), \qquad t > 0, x \neq 0. \qquad (2)$$

Therefore, we may define a family of measures $P_x^{\updownarrow}$ by

$$\mathrm{P}_x^{\updownarrow}(\Lambda) = \frac{1}{h(x)}\mathrm{E}_x[h(X_t)\mathbb{1}_\Lambda, t < T_0], \qquad x \neq 0, \Lambda \in \mathcal{F}_t,$$

for any $t \geq 0$.

Conditioning to avoid zero (Chaumont, Panti, Rivero)

Let X be an α stable process with $\alpha \in (1,2)$ and let h the function

$$h(x) = -\Gamma(1-\alpha)\frac{\sin(\pi\alpha\hat{\rho})}{\pi}x^{\alpha-1}, \qquad x > 0,$$

and the same expression with $\hat{\rho}$ replaced by ρ when $x < 0$.

- The function h can be represented as

$$h(x) = \lim_{q\downarrow 0}\frac{\mathrm{P}_x(T_0 > \mathbf{e}_q)}{n(\zeta > \mathbf{e}_q)}, \qquad x \neq 0,$$

  where $\mathbf{e}_q$ is an independent exponentially distributed random variable with parameter q. Furthermore, for any stopping time T and $\Lambda \in \mathcal{F}_T$, and any $x \neq 0$,

$$\lim_{q\downarrow 0}\mathrm{P}_x(\Lambda, T < \mathbf{e}_q | T_0 > \mathbf{e}_q) = P_x^{\updownarrow}(\Lambda).$$

Another representation of $P^{\updownarrow}$

- $P_x(T_0 > t) = P_1(T_0 > x^{-\alpha} t)$, for $x > 0$, $t \geq 0$.

Another representation of $\mathrm{P}^{\updownarrow}$

- $\mathrm{P}_x(T_0 > t) = \mathrm{P}_1(T_0 > x^{-\alpha} t)$, for $x > 0,\ t \geq 0$.
- The density of T_0

$$p(t) = -\frac{\sin^2(\pi/\alpha)}{\pi \sin(\pi\bar{\rho})} \frac{\sin(\pi\alpha\rho)}{\sin(\pi\alpha)} \frac{\Gamma(1-1/\alpha)}{\Gamma(\alpha-1)} t^{1/\alpha-2} + O(t^{-1/\alpha-1}).$$

Another representation of $\mathrm{P}^{\updownarrow}$

- $\mathrm{P}_x(T_0 > t) = \mathrm{P}_1(T_0 > x^{-\alpha} t)$, for $x > 0$, $t \geq 0$.
- The density of T_0

$$p(t) = -\frac{\sin^2(\pi/\alpha)}{\pi \sin(\pi\bar{\rho})} \frac{\sin(\pi\alpha\rho)}{\sin(\pi\alpha)} \frac{\Gamma(1-1/\alpha)}{\Gamma(\alpha-1)} t^{1/\alpha-2} + O(t^{-1/\alpha-1}).$$

- Stable (inverse) local time at zero:

$$n(\zeta \in \mathrm{d}t) = \frac{\alpha-1}{\Gamma(1/\alpha)} \frac{\sin(\pi/\alpha)}{\cos(\pi(\rho-1/2))} t^{1/\alpha-2}\, \mathrm{d}t, \qquad t \geq 0.$$

Another representation of $\mathrm{P}^{\updownarrow}$

- $\mathrm{P}_x(T_0 > t) = \mathrm{P}_1(T_0 > x^{-\alpha} t)$, for $x > 0$, $t \geq 0$.
- The density of T_0

$$p(t) = -\frac{\sin^2(\pi/\alpha)}{\pi \sin(\pi\bar{\rho})} \frac{\sin(\pi\alpha\rho)}{\sin(\pi\alpha)} \frac{\Gamma(1-1/\alpha)}{\Gamma(\alpha-1)} t^{1/\alpha-2} + O(t^{-1/\alpha-1}).$$

- Stable (inverse) local time at zero:

$$n(\zeta \in \mathrm{d}t) = \frac{\alpha-1}{\Gamma(1/\alpha)} \frac{\sin(\pi/\alpha)}{\cos(\pi(\rho-1/2))} t^{1/\alpha-2}\,\mathrm{d}t, \qquad t \geq 0.$$

- Verify directly

$$h(x) = \lim_{s\to\infty} \frac{\mathrm{P}_x(T_0 > s)}{n(\zeta > s)}.$$

Another representation of $\mathrm{P}^{\updownarrow}$

- For any a.s. finite stopping time T and $\Lambda \in \mathcal{F}_T$,

$$\begin{aligned}
&\mathrm{P}_x(\Lambda | T_0 > T + s)\\
&= \mathrm{E}_x\left[\frac{\mathrm{P}_x(\mathbf{1}_\Lambda, T_0 > T + s | \mathcal{F}_T)}{\mathrm{P}_x(T_0 > T + s)}\right]\\
&= \mathrm{E}_x\left[\mathbf{1}_\Lambda \mathbf{1}_{(T_0 > T)} \frac{\mathrm{P}_{X_T}(T_0 > s)}{\mathrm{P}_x(T_0 > T + s)}\right]\\
&= \mathrm{E}_x\left[\mathbf{1}_\Lambda \mathbf{1}_{(T_0 > T)} \frac{h(X_T)}{h(x)} \frac{\mathrm{P}_{X_T}(T_0 > s)}{h(X_T) n(\zeta > s)} \frac{n(\zeta > s)}{n(\zeta > T + s)} \frac{h(x) n(\zeta > T + s)}{\mathrm{P}_x(T_0 > T + s)}\right].
\end{aligned}$$

- For any a.s. stopping time T, $\Lambda \in \mathcal{F}_T$,

$$\mathrm{P}_x^{\updownarrow}(\Lambda) = \lim_{s \to \infty} \mathrm{P}_x(\Lambda | T_0 > T + s).$$