Note Title 16/07/2013

Stochastic Reaction Diffusion equations driven by Lévy type noise

Z. Brzeźniak (York)

Wrocław 17/07/2013

Note Title 16/07/2013

joint work with

E Hausenblas (Leoben)

and P. Razafimandimby
(Leoben)

Note Title 16/07/2013

Let me begin with two examples based on a joint work with J. Zabczyk

Example 1. Assume that

$$1 < \alpha < 2$$

Put $\beta = \frac{\alpha}{2}$ and let

$Z = Z_\beta$ be a subordinator process s. th.

$$\mathbb{E}\, e^{-r Z_\beta(t)} = e^{-t\psi(r)}$$

$$\psi(r) = r^\beta$$

Note Title 16/07/2013

$H = L^2(D), \quad D \subset \mathbb{R}^d$

Let $W(t) := \sum_{j=1}^{\infty} w_j(t) e_j$

$\{e_j\}_{j=1}^{\infty}$ ONB of H

(cylindrical Wiener process)

$L_\alpha(t) := W(Z_{\alpha/2}(t))$

Cylindrical α-stable Lèvy process

$0 < \gamma \leq 2$

Thm 1. If $0 \leq \delta < \frac{\gamma}{\alpha} - \frac{d}{2}$

then the O-U process generated by

$$\begin{cases} dX = -(-\Delta)^{\gamma/2} X \, dt + dL_\alpha \\ X = 0 \text{ on } \partial D \end{cases}$$

takes values in $C_0^\delta(D)$

In particular, if $d = 2, \gamma = 2$ and $\alpha < 2$, then $X(t) \in C_0^\delta(D)$ for $\delta < \frac{2}{\alpha} - 1$.

Example 2. Let $E = C(S^1)$ or $E = W^{\theta,q}(S^1)$, i.e.

$$\infty > \int_{S^1}\int_{S^1} \frac{|u(y)-u(x)|^q}{|y-x|^{\theta q+1}}\, ds\, dt.$$

Let $L = (L(t))_{t\geq 0}$ be α-stable $(1<\alpha<2)$ $\mathbb{R}$-valued process

Thm 2. If $f \in W^{\theta,q}$, $q > \alpha$ then the O-U process

$$dX = AX\,dt + f\,dL$$

where A generates the group of rotations, is $W^{\theta,q}$-valued càdlàg

Thm 3. $\exists f \in C(S^1)$:
the mild solution to (2)
doesn't take values in $C(S^1)$.

Remark. If $(S(t))_{t\in\mathbb{R}}$
is the rotation group generated
by L the a mild solution
satisfies

$$X(t) = \int_0^t S(t-r) f \, dL(r)$$

$$= S(t) \int_0^t S(-r) f \, dL(r)$$

Proof of Th. 3 uses
Kwapień, Marcus, Rosiński (2006)

Model problem

$$du = \Delta u - u^3 + g(u)\,dL$$
$$t>0, \quad x \in D$$

$$u(t,\cdot)\big|_{\partial D} = 0$$

$$u(0) = u_0$$

L – Lèvy process

For simplicity 1-dim with Lèvy measure

$$\nu(dz) = \frac{c}{|z|^{1+\alpha}} e^{-|z|}\,dz$$

tempered α-stable
$1 < \alpha < 2$

Put $X = C_0(D)$

$A = \Delta$

Then A is a generator of a C_0 analytic semigroup of contractions on X denoted by $S(t)$, $t \geq 0$.

A mild solution is

$$u(t) = S(t)u_0 + \int_0^t S(t-r) f(u(r))\,dr + \int_0^t S(t-r) g(u(r))\,dL$$

Let η be the Poisson random measure associated to L

and $\tilde{\eta}$ - corresponding compensated PRM

We only consider integrals w.r.t. $\tilde{\eta}$ and put

$$\int_0^t S(t-r)\, g(u(r))\, dL$$

$$= \int_0^t \int_Z S(t-r)\, g(u(r))\, z\, \tilde{\eta}(dz, dr)$$

$$Z = \mathbb{R}$$

Put

$$G(u, z) = g(u)\, z$$

$$G(u, \cdot) \in L^p(Z, \nu, X) \ ?$$

$C_0(D)$ – not good for integration, but

$$E = W_0^{\theta,r}(D)$$

or $H_0^{\theta,r}(D)$, $r > \alpha$

are good!

Fact: E is a martingale type p (if $r \geq p$) Banach space and

(B + Haus. 2009)

$$E\left|\int_0^T \int_Z \xi(s,z)\,\tilde{\eta}(dz,ds)\right|_E^p$$

$$\leq C\, E \int_0^T \int_Z |\xi(s,z)|^p \nu(dz)\, ds$$

Moreover, if $1 \leq q \leq p$,

$$\mathbb{E} \sup_{t \in [0,T]} \left| \int_0^t \int_Z \xi(s,z) \tilde{\eta}(dz\, ds) \right|_E^q$$

$$\leq C\, \mathbb{E} \left(\int_0^T \int_Z |\xi(s,z)|^p \nu(dz) ds \right)^{q/p}$$

Finally, if

$$u(t) = \int_0^t \int_Z S(t-r)\, \xi(r,z)\, \tilde{\eta}(dz, dr)$$

and

$$D_A(\theta, p) := (E, D(A))_{\theta, p}$$

real interpolation

then

$$\mathbb{E}\int_0^T |u(t)|^p_{D_A(\theta+\frac{1}{p},p)}\,dt$$

$$\leq C\,\mathbb{E}\int_0^T\int_Z |\xi(t,z)|^p_{D_A(\theta,p)}\,\nu(dz)\,dt$$

Roughly speaking:

u has better regularity by order $\frac{1}{p}$

Remark: Gaussian case: $\frac{1}{2}$

Weaker form: if E_0 is martingale type p Banach space and $-A$ a generator of an analytic semigroup on E_0 then for $\alpha < \frac{1}{p}$

$$E \int_0^T |A^\alpha u(t)|^p \, dt \leq C \int_0^T \int_Z |\xi(t,z)|^p \nu(dz) dt$$

$$u(t) = \int_0^t \int_Z e^{-(t-s)A} \xi(s,z) \tilde{\eta}(dz, ds)$$

Another useful result is

Thm (B+ Hausenblas and Zhu)
if additionally the semigroup is of contraction type and norm on E_0 sufficiently smooth then $u(t)$, is E_0 - càdlàg and

$$\mathbb{E} \sup_{0 \le t \le \tau} |u(t)|^p \le C \mathbb{E} \int_0^T \int_Z |\xi(s,z)|^p \nu(dz) \, ds$$

Recall model problem

$$du = (\Delta u - u^3)dt + g(u)\,dL$$

Put $\quad X = C_0(D)$

$E_0 = L^r(D)$

$E = H_0^{k,r}(D)$ or $W^{k,r}(D)$

$$A = -\Delta \qquad F(u) = -u^3$$

$-A$ generates a C_0 analytic contraction semigroup on X, E_0

Also $\quad -A + F$ is "dissipative" i.e.

$$\langle -Ax + F(x+y), z \rangle \leq a(|y|) - k|x|$$

$$x \in D(A), y \in X, z \in \partial|x|$$

$$a(r) := k_0(1 + r^q)$$

Function g is only continuous

Want to prove existence of a martingale solution, i.e. $\exists$

$$(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F}, \eta, u)$$

where η is a PRM with the same characteristics as the one associated to L

$u = (u(t))$ is E-valued progressively meas. process càdlàg in E_0:

$$u(t) = e^{-tA} u_0 + \int_0^t e^{-(t-s)A} F(u(s))ds$$

$$+ \int_0^t \int_Z e^{-(t-s)A} G(u(s), z) \tilde{\eta}(dz, ds)$$

Strategy. Approximate F by (F_n) satisfying uniformly the "dissipativity" condition and bounded. Get u_n:

$$u_n(t) = \underbrace{e^{-tA} u_0}_{= x(t)} + \underbrace{\int_0^t e^{-(t-s)A} F_n(u_n(s))\,ds}_{= Z_n(t)} + \underbrace{\int_0^t \int_Z e^{-(t-s)A} G(u_n(s))\,\tilde{\eta}(dz\,ds)}_{=: v_n(t)}$$

$$Z_n(t) = \int_0^t e^{-(t-s)A} F_n(z_n(s) + v_n(s) + X(s))\,ds$$

Uniform "dissipativity"
+ boundedness of G $\Rightarrow$
(via Da Prato Lemma)
uniform estimates on $\{z_n\}$

Then we proceed as in the case when F is bounded.

Main tools. For $M > 0$

$\mathcal{B}_M(E_0) :=$ E_0-valued p.m. processes $\xi(t,z)$ s.th.

$$\mathbb{E}\int_Z |\xi(t,z)|^p_{E_0} \, \nu(dz) \le M^p$$

For $\xi \in \mathcal{B}_M(E_0)$ put $u = \Lambda(\xi)$

$$u(t) = \int_0^t \int_Z e^{-(t-s)A} \xi(s,z) \tilde{\eta}(dz,ds)$$

i.e.

$$\begin{cases} du = -Au\,dt + \int_Z \xi(t,z)\tilde{\eta}(dz,dt) \\ u(0) = 0 \end{cases}$$

Claim 1. If $\rho < \frac{1}{p}$

$$\mathbb{E} \int_0^T |A^\rho u(t)|_{E_0}^p \, dt \leq C_1 M^p T$$

Claim 2

$$\mathbb{E} \sup_{0 \leq t \leq T} |A^{-1+\rho} u(t)|_{E_0}^p \leq C_2 M^p$$

and u is càdlàg in "$D(A^{-1+\rho})$" extrapolation space

Proof

$$A^{\rho-1} u(t) = \int_0^t A^\rho u(s) \, ds$$

$$+ A^{\rho-1} \int_0^t \int_Z \xi(s,z) \, \tilde{\eta}(dz, ds)$$

Claim 3. If $\rho + \alpha < 1/p$

$$\mathbb{E} \, |A^{\rho} u|^{p}_{W^{\alpha,p}(0,T;E_0)} \leq \frac{C}{T} M^{p}$$

Claim 4. Laws of $\{ \Lambda(\xi) : \xi \in B_M(E_0) \}$ are tight on $L^p(0,T; D(A^{\rho}))$ if $\rho < 1/p$.

Claim 5 Laws of $\{ \Lambda(\xi) : \xi \in B_M(E_0) \}$ are tight on $D(0,T; D(A^{-1+\rho}))$

Proof. Use earlier representation
+

Lemma (Ethier + Kurtz)

If $\{x_n\}$ is a sequence of Y-valued càdlàg processes:

(a) $\forall \varepsilon > 0 \ \exists K \subset Y$ compact:

$$\mathbb{P}_n \left(x_n(t) \in K \ \forall t \in [0,T] \right) \geq 1 - \varepsilon$$

$\forall n$

(b) $\exists c, \gamma, r > 0 : \forall \theta < T, t \leq T - \theta$

$$\mathbb{E}_n \sup_{t \leq s \leq t+\theta} |x_n(t) - x_n(s)|^r \leq C \theta^{\gamma}$$

then Laws of $\{x_n\}$ are tight on $D(0,T;Y)$.

Back to our problem

$$du = -Au\,dt + F(u)\,dt + \int_Z G(u(s), z)\,\tilde{\eta}(dz, ds)$$

We assume that

$$F : X \to X$$

$$G : X \to L^p(Z, \nu)$$

are bounded

We find an approximating sequence $\{u_n\}$:

$$u_n(t) = e^{-tA} u_0 + \int_0^t e^{-(t-s)A} F(\hat{u}_n(s))\,ds + \int_0^t \int_Z e^{-(t-s)A} G(\hat{u}_n(s), z)\,\tilde{\eta}(dz\,ds)$$

By previous argument we show that laws of $\{u_n\}$ are tight on

$$L^p(0,T;E) \cap \mathcal{D}(0,T;B)$$

$$E = D(A^\rho)$$

$$B = D(A^{\rho-1})$$

Apply Prokhorov Theorem to pairs (u_n, η)

Then apply our version of Skorokhod Theorem to infer that

$\exists\ (\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})$

$\exists\ \left[L^p(0,T;E) \cap D(0,T;E) \right]$

$\times\ M_{\mathbb{N}}(Z \times [0,T]) = X_T$

r.v. $(\bar{u}_n, \bar{\eta}_n)$ s.th.

$(\bar{u}_n, \bar{\eta}_n) \longrightarrow (u_*, \eta_*)$

$\bar{\mathbb{P}}$ - a.s. in X_T

and $\bar{\eta}_n = \eta_*$

And long & tedious argument $\Rightarrow$ (u_*, η_*) is a martingale solution.

Thank you!