Fundamental solutions and their short time estimates for jump-diffusions on manifolds

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1. SDE with jumps on manifold

M: Orientable, paracompact manifold of dimension d.

$$d\xi_t = V_0(\xi_t, t)dt + \sum_{j=1}^{m} V_j(\xi_t, t) \circ dW^j(t)$$

$$+ \lim_{\epsilon \to 0} \int_{U_\epsilon(z_0)^c} (\phi_{t,z}(\xi_{t-}) - \xi_{t-}) N(dtdz),$$

where $U_\epsilon(z_0)$ is an ϵ-neighborhood of z_0.
Here,

- $W(t) = (W^1(t), \ldots, W^m(t))$ is a standard Brownian motion.
- $N(dtdz)$: Poisson random measue on M with center $z_0 \in M$ and Lévy measure ν.
- We assume that the Lévy measure has a weak drift and satisfies an order condition of exponent $0 < \alpha < 2$.

Case $M = \mathbb{R}^d$: $z_0 = 0$: origin
$M = G$ (Lie group) $z_0 = e$: unit

- $V_j(t); j = 0, \ldots, m$: smooth vector fields on M.
- $\phi_{t,z} : M \to M$: diffeomorphisms.
  smooth with respect to $z \in M$, $\quad \phi_{t,z_0} = I$.

Solution is a jump-diffusion with generator

$$A(t)f = V_0(t)f + \frac{1}{2}\sum_{j=1}^{m} V_j(t)^2 f$$

$$+ \lim_{\epsilon \to 0} \int_{U_\epsilon(z_0)^c} \{f(\phi_{t,z}(x)) - f(x)\}\nu(dz).$$

We assume Hörmander condition for coefficients.

Object:

1. Transition probabilities $P(s, x; t, E)$ have smooth densities $p(s, x; t, y)$ with respect to a volume element $d\cancel{x}$.
   y

2. Short time estimate:

a) For any $0 < \beta < 2 - \alpha$, exponent of order cond.

$$|D_y^j p(s,x;t,y)| \le c_j (t-s)^{-(j+d)/\beta}$$

holds for any x, y.

b) Off diagonal estimate.

If $\bar{\phi}^\gamma(U^c) \cap \bar{V} = \phi$,

$$|D_y^j p(s,x;t,y)| \le c_j (t-s)^{\gamma-(j+d)/\beta}.$$

for any $x \in U^c, y \in V$.

3. It is smooth with respect to x and satisfies

$$(A(s)_x + \frac{\partial}{\partial s}) p(s,x;t,y) = 0.$$

4. It is a fundamental solution for the backward Cauchy problem:

$$(A(s)_x + \frac{\partial}{\partial s}) u(s,x) = 0,$$

$$\lim_{s \uparrow T_0} u(s,x) = f(x).$$

Method: Several steps.

1. Bounded jumps (Support of the Lévy measure is compact)
   Case of $M = \mathbb{R}^d$. 1,3 and 4 are known.
   Malliavin calculus for Wiener-Poisson space.
   Picard, Ishikawa-K, K.
   2 is known for pure jump process.
   Picard, Ishikawa.

2. Unbounded jumps (Support of the Lévy measure is non-compact)
   Malliavin calculus is not used.
   Use method of perturbation.

3. Localization.
   Smooth densities for a killed process.
   Let τ be the leaving time from $D \subset \mathbb{R}^d$:
   Set

τ^x

$$q(s,x;t,y) = p(s,x;t,y) - E[p(\tau, \xi_\tau; t, y) 1_{\tau<t}]$$

It is the density of the killed process at time τ.

For y-smoothness, we use short time estimate. For x-smoothness, we use the dual process. Chuchy problem on bounded domain D with Dirichlet boundary condition.

4. Patchwork for the smooth density on manifold. Piece together smooth densities on local charts.

The density is C^j if $y \notin \bar{\Phi}^\gamma(D^c)$ where $\gamma > \frac{j+d}{\beta}$

2. A perturbation method

1. SDE truncating jumps bigger than δ:
Jump coefficents are

$$\phi^{\delta}_{t,z} = \phi_{t,z}, \quad \text{if } |z| \le \delta, \\ = \text{identity}, \quad \text{if } |z| > \delta$$

Associated solution ξ^{δ}_t.
$p^{\delta}(s, x; t, y)$: smooth density of the law of ξ^{δ}_t.

2. We adjoin jumps bigger than δ as a perturbation.

For $\mathbf{u} = ((t_1, z_1), ..., (t_n, z_n))$, where $s < t_1 < \cdots t_n < t$, we define

$$p^{\mathbf{u}}(s, x; t, y) = \int p^{\delta}(s, x; t_1, y_1) p^{\delta}(t_1, \phi_{t_1, z_1}(y_1); t_2, y_2) \cdots p^{\delta}(t_n, \phi_{t_n, z_n}(y_n); t, y) dy_1 \cdots dy_n$$

Set

$$p(s, x; t, y) := E[\int p^u(s, x; t, y) N^\delta(du)].$$

where $N^\delta = N1_{|u|>\delta}$.

Then $p(s, x; t, y)$ is the density. Jacobian $\phi_{t,z}^{-1}$

3. If

$$\int_{U_\delta(z_0)^c} |J\phi_{t,z}^{-1}(x)| \nu(dz)$$

is locally bounded for some $\delta > 0$, the density is bounded continuous.

4. If

$$\int_{U_\delta(z_0)^c} (|D_x^j \phi_{t,z}(x)|^p + |D_x^j \phi_{t,z}^{-1}(x)|^p) \nu(dz)$$

are locally bounded for all j and $p > 1$, the density is C^∞.

Further two short time estimtes are valid for $p(s, x; t, y)$.

Application: Lévy process on Lie group.

Generator is given by

$$Af(x) = V_0 f(x) + \frac{1}{2}\sum_{j=1}^{d} V_j^2 f(x) + \lim_{\epsilon \to 0}\int_{U_\epsilon(e)^c} \{f(\phi_z(x)) - f(x)\}\nu(dz).$$

- $V_0, \ldots, V_d$ are left invariant vector fields; $\phi_z(x) := xz$ is the right translation.
- $\tilde{V}_j, j = 1, \ldots, d$ are also left invariant vector fields.
- Hörmander condition:
  The Lie algebra generated by left invariant vector fields
  $$\{V_0 + \frac{\partial}{\partial t}, V_1, \ldots, V_d, \tilde{V}_1, \ldots, \tilde{V}_d\}$$
  coincides with $\mathcal{G} \oplus \{\frac{\partial}{\partial t}\}$.

- Ad: $G \longrightarrow GL(\mathcal{G})$: Adjoint representation.
  Ad(z): Differential at e of the inner automorphism $x \longrightarrow zxz^{-1}$.
- If
$$\int_{U_\delta(e)^c} |\det \mathrm{Ad}(z)| \nu(dz) < \infty$$
  for some $\delta > 0$, the law has a continuous density with respect to the Haar measure.
- If G is unimodular, $|\det \mathrm{Ad}(z)| = 1$. The law has a continuous density.

  Examples of unimodular Lie groups:
  Abelian Lie groups,
  compact Lie groups,
  semisimple Lie groups,
  connected nilpotent Lie groups.
- If
$$\int_{U_\delta(e)^c} |\mathrm{Ad}(z)|^p \nu(dz) < \infty$$
  for some $\delta > 0$, the law has a C^∞-density.

3. Dual process

Integrals by volume element

dx: volume element of M.
(positive differential d-form of C^∞-class)

- $\phi^* dx$: pullback of dx by the diffeomorphism ϕ.
  $\exists$ positive or negative C^∞-function $J\phi$ on M such that $\phi^* dx = J\phi dx$.
- Formula for change of variable:
$$\int_M (f \circ \phi^{-1}) g dx = \int_M f(g \circ \phi)|J\phi| dx$$
- $L_V dx$; Lie derivative of dx. Then
$$L_V dx = \mathrm{div}\, V dx$$

Theorem 0.1 *Let $A(t)^*$ be a dual (adjoint) of $A(t)$ with respect to dx.*

1. It is written as

$$A(t)^*g = V_0^*(t)g + \frac{1}{2}\sum_{j=1}^{m} V_j(t)^2 g + c^*(x,t)g$$

$$+ \lim_{\epsilon\to 0}\int_{U_\epsilon(z_0)^c}\{g\circ\phi_{t,z}^{-1} - g\}|J\phi_{t,z}^{-1}|\nu(dz),$$

2.

$$V_0^*(t) = -V_0(t) + \sum_{j=1}^{m} \mathit{div} V_j(t)\cdot V_j(t),$$

$$c^*(x,t) = -\mathit{div} V_0(t)$$
$$+\frac{1}{2}\sum_{j=1}^{m}\{\mathit{div}\ V_j(t)^2 + V_j(t)(\mathit{div}\ V_j(t))\}$$
$$+ \lim_{\epsilon\to 0}\int_{U_\epsilon(z_0)^c}(|J\phi_{t,z}^{-1}(x)| - 1)\nu(dz).$$

Dual process

$\xi_{s,t}(x)$: Solution of SDE starting from x at time s.

$\xi_{s,t} : M \to M$: Diffeo.

$\eta_{t,s}; M \to M$: Inverse map of $\xi_{s,t}$: Backward jump-diffusion.

By change of variables,

$$\int_M f(\xi_{s,t}(x))g(x)dx = \int_M f(x)g(\eta_{t,s}(x))|J\eta_{t,s}(x)|dx$$

1. Define

$$= \int p(s,y:t,x)\, g(y)\, dy$$

$$P^*_{t,s}g(x) := E[g(\eta_{t,s}(x))|J\eta_{t,s}(x)|].$$

   Then, is is the dual of $P_{s,t}$:

$$\int_M P_{s,t}f(x) \cdot g(x)dx = \int_M f(x) \cdot P^*_{t,s}g(x)dx$$

2. $\{A(t)^*\}$ is the generator of $\{P^*_{t,s}\}$.
3. $|J\eta_{t,s}(x)|$ is a composite of Girsanov transformation and Feynman-Kac transformation.
4. We can apply the Malliavin calculus to the dual process $\eta_{t,s}$.

4. Smooth densities

For $z_0 \in M$, there exists a local chart

$$(U(z_0), z = (z^1, ..., z^d))$$

such that $z(z_0) = 0, z_*(z_0) = I$, where z_* is the differential of z.
For $\epsilon > 0$, we set

$$U_\epsilon(z_0) = \{z \in M; \sum_{i=1}^{m} |z^i|^2 < \epsilon\}.$$

We assume

1. <u>Weak drift</u>

$$\exists \lim_{\epsilon \to 0} \int_{U_\epsilon(z_0)^c \cap U_1(z_0)} z^i \nu(dz)$$

for any $i = 1, ..., d$.

2. <u>Order condition</u>: $\exists\ 0 < \alpha < 2$ such that

$$\varphi(\epsilon) := \int_{U_\epsilon(z_0)} (\sum_{i=1}^{d} |z^i|^2) \nu(dz) \geq c\epsilon^\alpha.$$

For $\rho > 0$, set

$$B_\rho = \varphi(\rho)^{-1}\Big(\int_{U_\rho(z_0)} z^i z^j \nu(dz)\Big).$$

B: a symmetric matrix such that $B \le B_\rho$ for $0 < \rho < \rho_0$.

(τ^{ij}): symmetric square root of B.

Define

$$\tilde{V}_j(t) = \sum_i \tau^{ji} \cdot \partial_{z^i}\phi_{t,z}(x)|_{z=z_0}, \quad j = 1, \dots, m$$

Let

$$\Sigma_0 = \{V_j(t), \tilde{V}_j(t);\ j = 1, \dots, m\}.$$

For $k = 1, 2, \dots$, set

$$\begin{aligned}\Sigma_k = \{&[V_0(t) + \frac{\partial}{\partial t}, V(t)],\\ &[V_j(t), V(t)], [\tilde{V}_j(t), V(t)];\\ &j = 1, \dots, m, V(t) \in \Sigma_{k-1}\},\end{aligned}$$

where [,] denotes the Lie bracket.

3. Hörmander Condition:

$\cup_{k=0}^{\infty}\Sigma_k$ spans the tangent space $T_x(M)$ for all x and t.

Theorem 0.2 *The weithted laws*

$$P(s, x; t, E) :=$$
$$E[\exp\{\int_s^t c(\xi_{s,u}(x), u)\,du\} 1_E(\xi_{s,t}(x))]$$

have measurable densities.

In particular, if the support of the Lévy measure is compact, these are (rapidly decreasing) C^∞-densities.

Theorem 0.3 *If*

$$\int_{U_\delta(z_0)^c} |J\phi_{t,z}^{-1}(x)| \nu(dz)$$

is locally bounded, the densities are bounded continuous.

1. If

$$\int_{U_\delta(z_0)^c} |D_x^i \phi_{t,z}(x)|^p \nu(dz),$$
$$\int_{U_\delta(z_0)^c} |D_x^i \phi_{t,z}^{-1}(x)|^p \nu(dz)$$

are locally bounded for any i, p, then the densities are bounded C^∞.

2. *It is also a bounded* C^∞*-function of* x *for any* $s < t, y$.

3. *For any* s, x, y, $p(s, x; t, y)$ *is differentiable with respect to* $t \in (s, T)$, *and satisfies* *Kolmogorov's forward equation*

$$\frac{\partial}{\partial t} p(s, x; t, y) = (A(t)_y^* + c(y, t)) p(s, x; t, y).$$

4. *For any* t, x, y, $p(s, x; t, y)$ *is differentiable with respect to* $s \in (0, t)$, *and satisfies* *Kolmogorov's backward equation*

$$\frac{\partial}{\partial s} p(s, x; t, y) = -(A(s)_x + c(x, s)) p(s, x; t, y).$$

5. Short time asymptotics

Theorem 0.4 *Let β be a positive number satisfying $\beta < 2-\alpha$. Let $j \in \mathbb{N}$.*
Then for any compact subset K of M, $\exists\ c_j > 0$ such that

$$\sup_{x,y\in K} |D_y^j p(s,x;t,y)| \le c_j (t-s)^{-(j+d)/\beta}.$$

Off diagonal estimate

For a subset U of M, we set

$$\bar{\phi}(U) = \overline{\cup \phi_{t,z}(U)},$$

where $\cup$ is taken for all $(t,z) \in \mathbb{T} \times \mathrm{Supp}(\nu)$. We set

$$\bar{\phi}^n(U) = \bar{\phi}(\bar{\phi}^{n-1}(U)).$$

Let U, V: relatively compact open subsets of M such that $\bar{V} \subset U$,
K: a compact subset.
γ: positive integer such that $\gamma > (j+d)/\beta$.

Theorem 0.5 *If $\bar{\phi}^\gamma(U^c) \cap \bar{V} = \phi$,*

$$\sup_{x\in U^c\cap K, y\in V} |D_y^j p(s,x;t,y)| \le c_j'(t-s)^{\gamma-(j+d)/\beta}.$$

6. Cauchy problems and their fundamental solutions

Let $0 < T_0 < T$. Consider the Cauchy problem:

$$\begin{cases} (A(s)_x + c(x,s) + \frac{\partial}{\partial s})u(x,s) = 0, \ \ 0 < s < T_0 \\ \lim_{s \to T_0} u(x,s) = f(x).(\text{term. cond}) \end{cases}$$

1) If the jump-diffusion associated with the generator $A(t)$ is conservative, i.e., $P_{s,t}1(x) = 1$ holds for any s, t, x, the Cauchy problem should have a unique solution.

2) If the jump-diffusion is not conservative, we need

<u>stochastic Dirichlet boundary condition</u>

For any s, x,

$$\exists \lim_{t \uparrow \zeta^{s,x}} u(\xi_{s,t}(x), t) = 0, \quad \text{if } \zeta^{s,x} < T_0, \ \text{a.s. } P.$$

Theorem 0.6 *1)(Conservative case)*

1. *For any bounded continuous function* f *on* M, *the above Cauchy problem has a unique solution.*

2. *The density function* $p(s, x; t, y)$ *is the fundamental solution of the Cauchy problem, i.e., the solution* $u(x, s)$ *of the Cauchy problem is written as*

$$u(x, s) = \int_M p(s, x; T_0, y) f(y) dy.$$

3. *Feynman-Kac formula.*

$$u(x, s) = E[\exp\{\int_s^{T_0} c(\xi_{s,u}(x), u) du\} f(\xi_{s,T_0}(x))]$$

2) (Non conservative case)

1. *For any bounded continuous function* f, *solutions of the Cauchy problem with the stochastic Dirichlet boundary condition exist uniquely.*
2. *The density* $p(s, x; t, y)$ *is its fundamental solution.*
3. *Further, we have the Feynman-Kac formula.*

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